

DIFFERENTIAL TRANSFORM METHOD: A POWERFUL TOOL FOR NUMERICAL SOLUTIONS OF NONLINEAR ODES

GHULAM FARID¹, MUHAMMAD SARMAD IFTIKHAR², MADIHA GHAMKHAR¹,
FAIZA FIAZ¹, ALEEZA ISHFAQ¹

Manuscript received: 27.05.2025; Accepted paper: 02.04.2026;

Published online: 30.09.2026.

Abstract. In this paper, mathematical models of some well-known physical phenomena are described in terms of homogeneous nonlinear Initial Value Problems (IVPs). The working principle of the Differential Transform Method (DTM) is presented, and solutions of the IVPs are approximated using DTM. The DTM also approximates solutions of some non-homogeneous IVPs. The accuracy of the method is observed from the infinitesimal values of residual errors at different points of the domain. Computational work is carried out using MATHEMATICA.

Keywords: DTM; transforms; differential equation; nonlinear; numerical; IVPs.

Mathematics Subject Classification: 35A22; 35C10; 41A58; 34A25.

1. INTRODUCTION

Numerous phenomena in science and engineering that involve dynamic or evolving processes can be effectively modeled using Differential Equations (DEs). These equations are typically classified based on their order and degree, and further characterized as linear or nonlinear, homogeneous or nonhomogeneous, and as having constant or variable coefficients. In many cases, especially for nonlinear or variable coefficient equations, obtaining exact analytical solutions is either extremely difficult or entirely infeasible. Consequently, significant attention has been directed toward the development and application of numerical and semi-analytical methods to construct approximate solutions to such problems. A wide array of these methods is available in the literature, offering practical and efficient tools for solving complex DEs arising in real-world applications.

The study of nonlinear Ordinary Differential Equations (ODEs) plays a fundamental role in the mathematical modeling of natural phenomena, engineering systems, and applied sciences. Nonlinear ODEs frequently arise in disciplines such as fluid dynamics, thermal processes, population dynamics, electrical engineering, and biological systems. Due to the inherent complexity and nonlinear nature of such equations, obtaining exact or closed-form solutions is often either challenging or impossible. As a result, a wide range of analytical, semi-analytical, and numerical techniques have been developed over the years to obtain approximate solutions of initial value problems associated with nonlinear ODEs.

¹ University of Agriculture Faisalabad, Faculty of Sciences, Department of Mathematics and Statistics, 38900 Faisalabad, Pakistan. E-mail: ghulamfarid@uaf.edu.pk; madiha.ghamkhar@uaf.edu.pk; faizafiaz26@gmail.com; aleeza0107ishfaq@gmail.com.

² University of Agriculture Faisalabad, Faculty of Agriculture, Department of Plant Breeding and Genetics, 38900 Faisalabad, Pakistan. E-mail: m.sarmad@uaf.edu.pk.

Analytical methods aim to derive exact expressions for the solutions of DEs. While they provide deep theoretical insight and understanding, their application is typically limited to simple linear or weakly nonlinear problems. The scope of such methods diminishes rapidly as the complexity of the system increases. Consequently, researchers have turned to semi-analytical and numerical methods, which, while yielding approximate solutions, offer much broader applicability and practical relevance for real-world problems.

Semi-analytical methods serve as a bridge between purely analytical techniques and fully numerical approaches. These methods often generate solutions in the form of rapidly converging series, enabling high accuracy with relatively low computational cost. Among the most prominent semi-analytical methods is the Adomian Decomposition Method (ADM), which expresses the nonlinear terms of an equation in terms of Adomian polynomials and constructs the solution as an infinite series. This method has been widely applied due to its simplicity and efficiency in solving a broad class of nonlinear problems.

Another notable technique is the Homotopy Perturbation Method (HPM), which blends classical Perturbation Methods with the concept of homotopy from topology. The HPM constructs a family of solutions parameterized by an embedding parameter that continuously deforms a simple problem into the original complex one. The Variational Iteration Method (VIM), on the other hand, utilizes variational theory and Lagrange multipliers to generate correction functionals that lead to successive approximations of the solution. VIM is particularly effective for strongly nonlinear DEs and has been recognized for its straightforward implementation and rapid convergence.

Other semi-analytical methods include the Rationalized Haar Function Method (RHF), which uses Haar wavelets to approximate functions over a domain. This method is well-suited for handling functions with high gradients or discontinuities and offers localized approximations with minimal computational effort. The Galerkin Method (GM), commonly employed in finite element analysis, projects the original DE onto a finite-dimensional subspace using basis functions. GM is known for its theoretical robustness and adaptability across a wide range of problems in engineering and science.

In addition to semi-analytical techniques, fully numerical methods continue to be a cornerstone in the field of DEs. These methods discretize the problem domain and replace the DE with algebraic equations that can be solved using numerical algorithms. Among the most frequently used numerical methods is the Finite Difference Method (FDM), which approximates derivatives by finite differences on a uniform or non-uniform grid. FDM is conceptually simple and widely used, though care must be taken to ensure stability and convergence of the numerical solution. The Shooting Method (SM) is another widely applied approach, especially for boundary value problems (BVPs), in which the problem is converted into an IVP. The solution is obtained iteratively by adjusting the unknown initial conditions until the boundary conditions are satisfied. The Predictor-Corrector Method (PCM) offers a compromise between computational efficiency and accuracy, using a prediction step to estimate the next value and a correction step to refine it. These numerical methods are generally robust but may become computationally expensive or sensitive to step size and discretization parameters when dealing with stiff or highly nonlinear problems.

Among the recent and effective semi-analytical methods, the DTM has gained considerable attention. Originally introduced by Zhou in the mid-1980s, DTM is based on the Taylor series expansion and provides a systematic way to derive recursive relations for the solution coefficients. Unlike traditional Taylor methods, DTM does not require symbolic differentiation or integration and is particularly advantageous for solving IVPs. The key strength of DTM lies in its ability to convert DEs into simple algebraic recurrence relations, which can then be solved iteratively. The resulting solution is expressed as a finite power series, often sufficient for practical purposes and easily extendable if higher accuracy is

needed. The method is notable for its ease of implementation and accuracy, especially when applied to nonlinear or time-dependent differential equations [1]. For the study of numerical, analytical, and semi-analytical methods for finding solutions of DEs, we refer to Gerald and Wheatley [2], Rao [3], Griffiths and Higham [4], and Jain et al. [5].

He introduced the HPM, a novel analytical technique for solving nonlinear DEs; by integrating homotopy concepts from topology with classical perturbation methods, HPM offered a straightforward approach that did not require small parameters, making it broadly applicable to various nonlinear problems; the method demonstrated rapid convergence and effectiveness, as illustrated through several examples, positioning it as a valuable tool in the analytical solution of nonlinear equations [6]. Ertürk and Momani applied the DTM to fifth-order BVPs, deriving convergent series solutions with easily computable components; the comparative analysis with existing methods confirmed DTM's reliability and efficiency in handling higher-order BVPs [7]. Jafari et al. applied DTM to multidimensional partial differential equations, deriving analytical solutions for both linear and nonlinear cases; the study confirmed DTM's reliability and efficiency in handling complex multidimensional problems [8]. Chen and Dai employed DTM to solve Richards' equation, a nonlinear partial differential equation modeling water movement in unsaturated soils; the method provided approximate analytical solutions, demonstrating DTM's applicability to complex hydrological models [9].

Patel and Dhodiya utilized DTM to solve both linear and nonlinear diffusion equations; the results aligned well with exact solutions, confirming DTM's accuracy and efficiency in addressing diffusion problems [10]. Wang and Li applied DTM to linear systems of first-order fuzzy differential equations, demonstrating its effectiveness in obtaining analytical solutions and reinforcing its utility in fuzzy mathematical modelling [11]. Hassan and Ertürk present the application of the DTM to solve linear and nonlinear DEs; the authors demonstrated the method's effectiveness through illustrative examples, yielding accurate and rapidly converging series solutions; the study highlighted DTM's computational efficiency compared to traditional techniques [12]. Oke established the convergence criteria for DTM when applied to ordinary differential equations; by compiling and proving relevant theorems, the study demonstrated that if a solution can be expressed as a Taylor series, DTM can effectively reconstruct it, thereby validating its theoretical foundation [13].

Kadkhoda et al. employed DTM to solve fuzzy DEs by considering strongly generalized differentiability; the method successfully obtained all possible solutions, showcasing DTM's capability in handling fuzzy systems [14]. Saqlain et al. compared the DTM and Elzaki Transform Method in solving third-order ODEs; the findings indicated that DTM offers efficient and accurate solutions with reduced computational effort, highlighting its advantage over alternative transform methods [15]. Madi and Abdelwahid extended DTM to two-dimensional forms for solving nonlinear partial differential equations with constant or variable coefficients; the results showcased DTM's effectiveness and simplicity in obtaining approximate and exact solutions, reducing computational workload compared to traditional methods [16]. For the study of DTM, we refer to Agarwal [17], Ince [18], Radhika et al. [19], Hatami et al. [20], and Patel and Dhodiya [21].

2. MATERIALS AND METHODS

Throughout the article, the notations C , W and N are used for the set of complex numbers, the set of non-negative integers, and the set of positive integers, respectively.

For $\alpha \in C$ and $m \in W$; the Pochhammer's symbol $(\alpha)_m$ is defined as

$$(\alpha)_m = \begin{cases} 1, & m = 0, \alpha \in C - \{0\}, \\ \prod_{j=0}^{m-1} (\alpha + j), & m \in N, \alpha \in C, \end{cases}$$

which for $\alpha = 1$ becomes the classical factorial function $m!$ initially defined as

$$m! = \begin{cases} \prod_{j=1}^m j, & m \in N, \\ 1, & m = 0. \end{cases}$$

The notations $\overset{m}{C}_j$ with $m, j \in W$ while $j \leq m$, $\{k_m\}_{m=1}^j$, $\overset{m}{C}_{(j_i)_{i=1}^n}$ with $n \in N$, $m, j_i \in W$ such that $j_i \leq m$ for each i , $u^{(k)}(v_0)$ with $k \in W$ while $u = u(v)$, $U_k(v_0)$ and $\frac{d^0}{dx^0}(u(v))$ are being used in the following sense:

$$\begin{aligned} \overset{m}{C}_j &= \frac{m!}{(m-j)!j!}, \\ \{k_m\}_{m=1}^j &= \{k_1, k_2, k_3, \dots, k_j\}, \\ \overset{m}{C}_{(k_m)_{m=1}^j} &= \frac{m!}{\left(m - \sum_{m=1}^j k_m\right)! \prod_{m=1}^j (k_m!)}, \\ u^{(k)}(v_0) &= \frac{d^k}{dv^k}(u(v)) \text{ at } v = v_0, \\ U_k(v_0) &= \frac{u^{(k)}(v_0)}{k!} \end{aligned} \tag{1}$$

and

$$\frac{d^0}{dx^0}(u(v)) = u(v),$$

respectively.

2.1. THE DIFFERENTIAL TRANSFORM METHOD

One may write a function $u = u(v)$, using the Maclaurin series, in the form

$$u(v) = \sum_{k=0}^{\infty} U_k(0) v^k. \tag{2}$$

One may define an operator O_k as $O_k(u(v)) = U_k(0)$. That is $u(v) \xrightarrow{O_k} U_k(0)$. For convenience, we denote $U_k(0)$ by simply U_k . So, another form of (2) becomes

$$u(v) = \sum_{k=0}^{\infty} U_k v^k. \quad (3)$$

Thus, the inverse operator O_k^{-1} follows the property $O_k^{-1}(U_k) = u(v)$ and is already defined in (3). Practically, instead of using the infinite series defined in (3), we use the finite series

$$u_n(v) = \sum_{k=0}^n U_k v^k \quad (4)$$

to approximate $u(v)$ for some $n \in N$.

3. RESULTS AND DISCUSSION

3.1. RESULTS

For given functions $f_1(v)$ and $f_2(v)$; we have

$$(f_1(v) f_2(v))^{(k)} = \sum_{m=0}^k C_m^k (f_1(v))^{(k-m)} (f_2(v))^{(m)}$$

for each $n \in N$. One may prove it (Leibniz Rule) by induction on k . Then, using induction on n , it follows by use of the above result that

$$\left(\prod_{j=1}^n f_j(v) \right)^{(k)} = \sum_{i_1=0}^k \sum_{i_2=0}^{k-i_1} \sum_{i_3=0}^{k-i_1-i_2} \dots \sum_{i_{n-1}=0}^{k-\sum_{m=1}^{n-2} i_m} C_{\{i_m\}_{m=1}^{n-1}}^k \left(\prod_{m=1}^{n-1} (f_m(v))^{(i_m)} \right) (f_n(v))^{(k-\sum_{m=1}^{n-1} i_m)}, \quad (5)$$

for any collection $\{f_j(v)\}_{j=1}^n$ of functions with $n \in N - \{1\}$. From (1), one may observe that

$$U_k = \alpha F_k \pm \beta G_k \text{ as } u(v) = \alpha f(v) \pm \beta g(v),$$

where α and β are constants,

$$U_k = F_{k+n} \prod_{j=1}^n (k+j) \text{ as } u(v) = f^{(n)}(v), \quad (6)$$

$$U_k = \sum_{j=0}^k F_{k-j} G_j \text{ as } u(v) = f(v)g(v),$$

$$U_k = \delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n \end{cases} \text{ as } u(v) = v^n, \quad (7)$$

$$U_k = a^k \frac{e^b}{k!} \text{ as } u(v) = e^{av+b}$$

and

$$U_k = \frac{1}{k!} \alpha^k \cos\left(\frac{k}{2}\pi + \beta\right) \text{ as } u(v) = \cos(\alpha v + \beta),$$

where α and β are constants. It follows from (5) and (1) that

$$U_k = \sum_{i_1=0}^k \sum_{i_2=0}^{k-i_1} \sum_{i_3=0}^{k-i_1-i_2} \cdots \sum_{i_{n-1}=0}^{k-\sum_{m=1}^{n-2} i_m} \left(\prod_{m=1}^{n-1} F_{i_m} \right) F_{k-\sum_{m=1}^{n-1} i_m}$$

as

$$u(v) = \prod_{j=1}^n f_j(v). \quad (8)$$

In particular:

(i) if $u(v) = f^2(v)$, we have

$$U_k = \sum_{j=0}^k F_{k-j} F_j, \quad (9)$$

(ii) if $u(v) = f^3(v)$, we have

$$U_k = \sum_{i=0}^k \sum_{j=0}^{k-i} F_i F_j F_{k-i-j} \quad (10)$$

and

(iii) if $u(v) = (f'(v))^2$, we have

$$U_k = \sum_{j=0}^k (k-j+1)(j+1) F_{k-j+1} F_{j+1}. \quad (11)$$

We refer to Chang and Chang [22], El-Shahed [23], Munganga et al. [24], and Mukhtarov et al. [25] for the study of the above-mentioned and similar properties of the differential operator.

For any $j \in W$; if $u^{(j)}(0) = c_j$, it follows from (3) that

$$U_j = \frac{c_j}{j!}. \quad (12)$$

A higher-order DE, with order $r \in N - \{1\}$, can be written in the general form as

$$g(v, u', u'', \dots, u^{(r)}) = 0 \quad (13)$$

defined over a domain D . Let u be a test solution evaluated at a point $v_0 \in D$. If

$$g(v_0, u'(v_0), u''(v_0), \dots, u^{(n)}(v_0)) = 0$$

then u is said to be an exact solution of the DE at $v = v_0$. However, if this condition is not satisfied exactly, the infinitesimal value of $g(v_0, u'(v_0), u''(v_0), \dots, u^{(n)}(v_0))$ serves as an

indicator of the accuracy of the approximate solution at the point v_0 . The closer this value is to zero, the more accurate the solution is at that point. This quantity is referred to as the **residual error** or *satisfaction error* of the DE at $v = v_0$, and one may denote it by $E_r(v_0)$. That is;

$$E_r(v_0) = g(v_0, u'(v_0), u''(v_0), \dots, u^{(n)}(v_0)).$$

In case the closed-form solution of some nonlinear DEs is not available, one may approximate their solutions using any technique like the DTM, and the accuracy of the proposed method may be observed by finding the residual errors at different points of the domain.

3.2. DISCUSSION

Firstly, just to refer, we review the formation of DEs for nonlinear springs (hard and soft) and later we approximate the solutions of the DEs.

In classical mechanics, the motion of a spring-mass system is governed by Newton's Second Law. That is;

$$m \frac{d^2 u}{dt^2} = F_r,$$

where m is the mass attached to the spring, $u(t)$ is the displacement from the equilibrium position at time t and F_r is the restoring force acting on the mass due to the spring.

In the ideal linear case, the restoring force follows Hooke's Law. That is;

$$F_r = -ku,$$

where $k > 0$ is the spring constant. Thus, we have

$$m \frac{d^2 u}{dt^2} + ku = 0.$$

This is a second-order linear ordinary differential equation (ODE) with sinusoidal solutions. It describes simple harmonic motion, assuming force is linearly proportional to displacement.

But in real-world systems, especially at large displacements, the spring's behavior becomes nonlinear. A more accurate model includes a cubic term in the restoring force:

$$F_r = -ku - \alpha u^3.$$

This leads to the nonlinear second-order ODE

$$m \frac{d^2 u}{dt^2} + ku + \alpha u^3 = 0.$$

This is known as the Duffing equation, a standard model for nonlinear oscillators. Since the force in a spring should be symmetric about equilibrium (that is $F(-u) = -F(u)$), the cubic term u^3 satisfies this property, but the square term u^2 does not. Experiments show that real springs often stiffen or soften at large displacements. The cubic term is the simplest way to model this while preserving symmetry and zero force at equilibrium. Springs are classified as Hard and Soft according to their stiffness.

In the case of a hard (soft) spring, as the spring is stretched, the force increases more (less) rapidly; the spring becomes stiffer (softer), and thus the governing equations become

$$m \frac{d^2 u}{dt^2} + ku + \alpha u^3 = 0, \alpha > 0 \quad (14)$$

and

$$m \frac{d^2 u}{dt^2} + ku + \alpha u^3 = 0, \alpha < 0, \quad (15)$$

causing faster (slower) oscillations for larger amplitudes, respectively; see Fig. 1. One may study this topic in Zill [26].

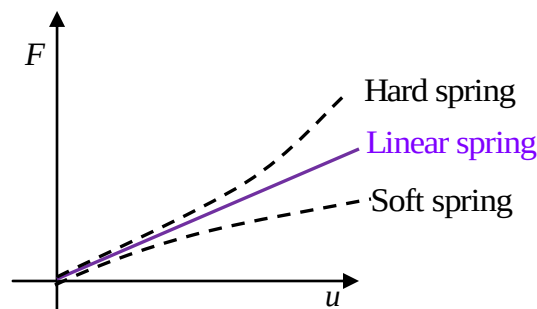


Figure 1. Hard and Soft Springs.

We approximate solutions of these two nonlinear DEs with the DTM and ensure the accuracy of the proposed method by observing infinitesimal values of the residual errors.

Problem 1. Consider a particular case of (14) with $m = k = \alpha = 1$ (hard spring). That is;

$$\frac{d^2 u}{dt^2} + u + u^3 = 0 \quad (16)$$

with ICs $u(0) = 3$ and $u'(0) = -2$.

Solution: Comparing (16) with (13), we have $r = 2$ and

$$g = g(t) = \frac{d^2 u}{dt^2} + u + u^3. \quad (17)$$

Here $(c_0, c_1) = (3, -2)$, see (12). Let $u = u_n(v)$ be the solution of (16). Then by (4), we have

$$u = \sum_{k=0}^n U_k t^k. \quad (18)$$

It follows from (12) with use of the ICs that $(U_0, U_1) = (3, -2)$.

Applying the operator O_k , with the collaboration of (6) and (10), the relation (16) leads to the recurrence relation

$$U_{k+2} = -\frac{\sum_{i=0}^k \sum_{j=0}^{k-i} U_i U_j U_{k-i-j}}{(k+1)(k+2)}.$$

For instance, if we use $n = 9$, it follows from the above recurrence relation that

$$(U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9) = \left(-15, \frac{28}{3}, 32, -\frac{119}{3}, -\frac{481}{6}, \frac{9031}{63}, \frac{29951}{168}, -\frac{38303}{81}\right).$$

Eventually, (18) becomes

$$u(t) = 3 - 2t - 15t^2 + \frac{28}{3}t^3 + 32t^4 - \frac{119}{3}t^5 - \frac{481}{6}t^6 + \frac{9031}{63}t^7 + \frac{29951}{168}t^8 - \frac{38303}{81}t^9. \quad (19)$$

The responses of the DE (16) at points of the domain $D = [0, 0.2]$ with step size $h = 0.02$, say, are found from (16). Moreover, $u(t)$ expressed in (19) is substituted into the DE, and the values of $g(t)$ are calculated at the same points of the domain, which are so far called the residual errors at the individual points of the domain. The results are summarized in Table 1.

Table 1. Solution of the DE of Problem 1 through DTM with Residual Errors (hard spring).

t	$u(t)$ DTM (5 d.p.)	Residual Errors $E_r(t)$ (4 d.p.)
0	3	0.
0.02	2.95408	$6.8533 \cdot 10^{-10}$
0.04	2.89667	$1.5409 \cdot 10^{-7}$
0.06	2.8284	$3.4054 \cdot 10^{-6}$
0.08	2.74994	$2.8667 \cdot 10^{-5}$
0.10	2.66207	$1.3971 \cdot 10^{-4}$
0.12	2.56559	$4.7083 \cdot 10^{-4}$
0.14	2.46133	$1.1868 \cdot 10^{-3}$
0.16	2.35013	$2.2603 \cdot 10^{-3}$
0.18	2.23278	$2.8918 \cdot 10^{-3}$
0.2	2.11009	$3.4756 \cdot 10^{-4}$

The residual errors of the method are shown in Fig. 2.

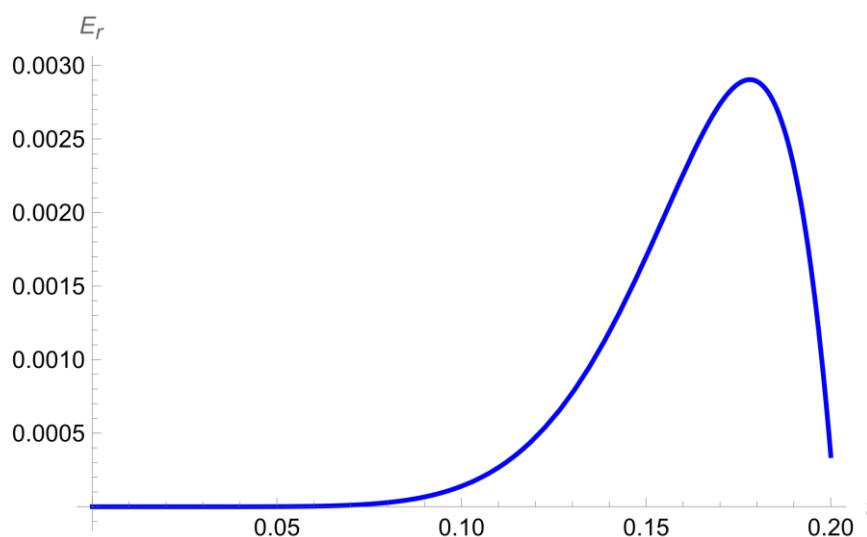


Figure 2. Residual error of DTM for Problem 1.

The infinitesimal values of the residual errors, particularly near $t=0$, show the accuracy of the proposed method. The residual errors can be minimized by taking the high values of n . All the calculations involved are carried out through MATHEMATICA.

Problem 2. Consider a particular case of (15) with $m=k=1$ and $\alpha=-1$ (soft spring). That is;

$$\frac{d^2u}{dt^2} + u - u^3 = 0 \quad (20)$$

with ICs $u(0)=3$ and $u'(0)=0$.

Solution: In this case, the DTM solution, for $n=9$, is

$$u(t) = 3 + 12t^2 + 26t^4 + \frac{986t^6}{15} + \frac{33949t^8}{210}.$$

The solutions are found using the same procedure analogous to the above problem, and the results are summarized in Table 2.

Table 2. Solution of the DE of Problem 2 through DTM with Residual Errors (soft spring).

t	$u(t)$ DTM (5 d.p.)	Residual Errors $E_r(t)$ (4 d.p.)
0	3.	0.
0.02	3.0048	$-9.1559 \cdot 10^{-10}$
0.04	3.01927	$-2.3533 \cdot 10^{-7}$
0.06	3.04354	$-6.0715 \cdot 10^{-6}$
0.08	3.07788	$-6.1218 \cdot 10^{-5}$
0.10	3.12267	$-3.6935 \cdot 10^{-4}$
0.12	3.17839	$-1.6121 \cdot 10^{-3}$

t	$u(t)$ DTM (5 d.p.)	Residual Errors $E_r(t)$ (4 d.p.)
0.14	3.24571	$-5.633 \cdot 10^{-3}$
0.16	3.32541	$-1.674 \cdot 10^{-2}$
0.18	3.41851	$-4.3996 \cdot 10^{-2}$
0.2	3.52622	$-1.0503 \cdot 10^{-1}$

The residual errors of the method are shown in Fig. 3.

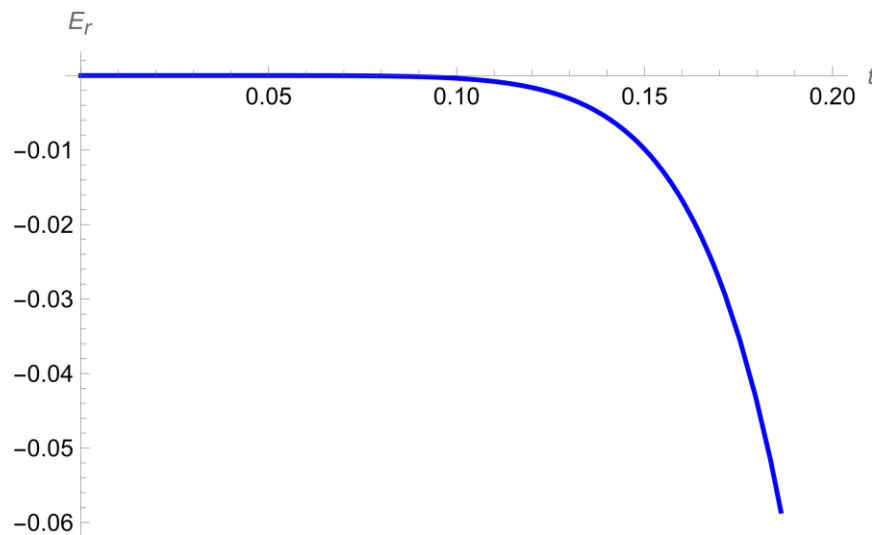


Figure 3. Residual error of THE DTM for Problem 2.

Here, the infinitesimal values of the residual errors can also be observed near the origin, which reflect the accuracy of the proposed method for this case too.

Likewise, we consider some nonhomogeneous problems corresponding to Problem 1 and Problem 2 to observe the accuracy of the proposed method:

Problem 3. Consider the nonhomogeneous DE related to the case of a hard spring

$$\frac{d^2 u}{dt^2} + u + u^3 = 2t^5 \quad (21)$$

with ICs $u(0) = 3$ and $u'(0) = -2$.

Comparing (21) with (13), we have $r = 2$ and

$$g = g(t) = \frac{d^2 u}{dt^2} + u + u^3 - 2t^5. \quad (22)$$

It is obvious from (12) that $(c_0, c_1) = (3, -2)$. Let $u = u_n(t)$ be a solution of (21). Then by (4), we have

$$u = \sum_{k=0}^n U_k t^k. \quad (23)$$

It follows by (11) with use of the ICs that $(U_0, U_1) = (3, -2)$.

Applying the operator O_k , with the collaboration of (6), (7) and (10), the relation (21) leads to the recurrence relation

$$U_{k+2} = \frac{2\delta(k-5) - U_k - \sum_{i=0}^k \sum_{j=0}^{k-i} U_i U_j U_{k-i-j}}{(k+1)(k+2)}.$$

For instance, if we use $n = 9$, it follows from the above recurrence relation that

$$(U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9) \\ = (-15, \frac{28}{3}, 32, -\frac{119}{3}, -\frac{481}{6}, \frac{9034}{63}, \frac{29951}{168}, -\frac{76609}{162}).$$

Eventually, (23) becomes

$$u(t) = 3 - 2t - 15t^2 + \frac{28}{3}t^3 + 32t^4 - \frac{119}{3}t^5 - \frac{481}{6}t^6 + \frac{9034}{63}t^7 + \frac{29951}{168}t^8 - \frac{76609}{162}t^9. \quad (24)$$

The responses of the DE (21) at points of the domain $D = [0, 0.2]$ with step size $h = 0.02$, say, are found from (24). Moreover, $u(t)$ expressed in (24) is substituted into the DE, and the values of $g(t)$ are calculated at the same points of the domain, which are so far called the residual errors at the individual points of the domain. The results are summarized in Table-3.

Table 3. Solution of DE of Problem 3 through DTM with Residual Errors (hard spring).

t	$u(t)$ DTM (5 d.p.)	Residual Errors $E_r(t)$ (4 d.p.)
0.	3.	0.
0.02	2.95408	$7.0853 \cdot 10^{-9}$
0.04	2.89667	$3.5887 \cdot 10^{-7}$
0.06	2.8284	$4.9602 \cdot 10^{-6}$
0.08	2.74994	$3.5216 \cdot 10^{-5}$
0.1	2.66207	$1.5968 \cdot 10^{-4}$
0.12	2.56559	$5.2047 \cdot 10^{-4}$
0.14	2.46133	$1.2939 \cdot 10^{-3}$
0.16	2.35013	$2.4686 \cdot 10^{-3}$
0.18	2.23278	$3.2661 \cdot 10^{-3}$
0.2	2.11009	$9.7911 \cdot 10^{-4}$

The residual errors of the method are shown in Fig. 4.

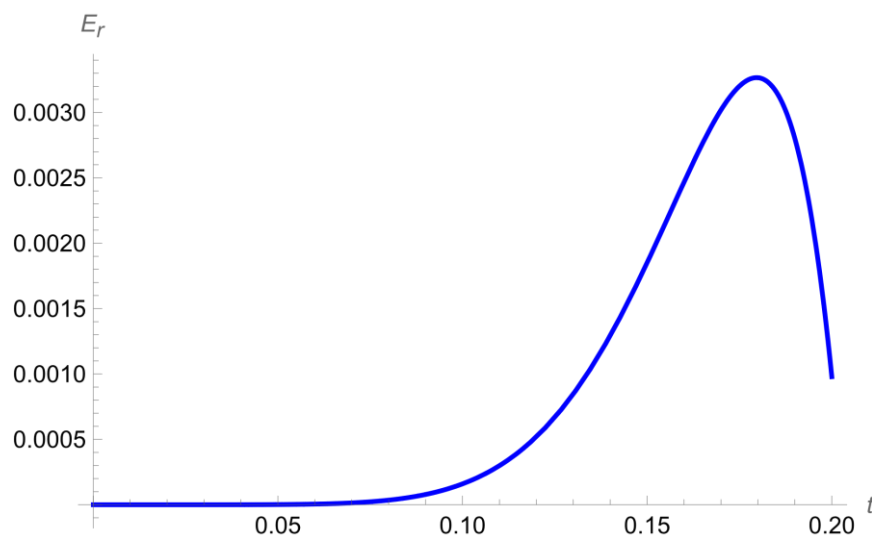


Figure 4. Residual error of DTM for Problem 3.

Again, the method seems accurate due to very small values of the residual errors, particularly near the origin.

Problem 4. Consider the nonhomogeneous DE related to the case of a soft spring

$$\frac{d^2u}{dt^2} + u - u^3 = 2t^5 \quad (25)$$

with ICs $u(0) = 3$ and $u'(0) = 0$.

In this case, the DTM solution, for $n = 9$, is

$$u(t) = 3 + 12t^2 + 26t^4 + \frac{986}{15}t^6 + \frac{1}{21}t^7 + \frac{33949}{210}t^8 + \frac{13}{756}t^9.$$

The solutions are found using the same procedure analogous to the above problem, and the results are summarized in Table 4.

Table 4. Solution of the DE of Problem 4 through DTM with Residual Errors (soft spring)

t	$u(t)$ DTM (5 d.p.)	Residual Errors $E_r(t)$ (4 d.p.)
0.	3.	0.
0.02	3.0048	$5.4844 \cdot 10^{-9}$
0.04	3.01927	$-3.0529 \cdot 10^{-8}$
0.06	3.04354	$-4.5164 \cdot 10^{-6}$
0.08	3.07788	$-5.4666 \cdot 10^{-5}$
0.1	3.12267	$-3.4936 \cdot 10^{-4}$
0.12	3.17839	$-1.5624 \cdot 10^{-3}$
0.14	3.24571	$-5.5257 \cdot 10^{-3}$
0.16	3.32541	$-1.6531 \cdot 10^{-2}$
0.18	3.41851	$-4.3621 \cdot 10^{-2}$
0.2	3.52622	$-1.044 \cdot 10^{-1}$

The residual errors of the method are shown in Fig. 5.

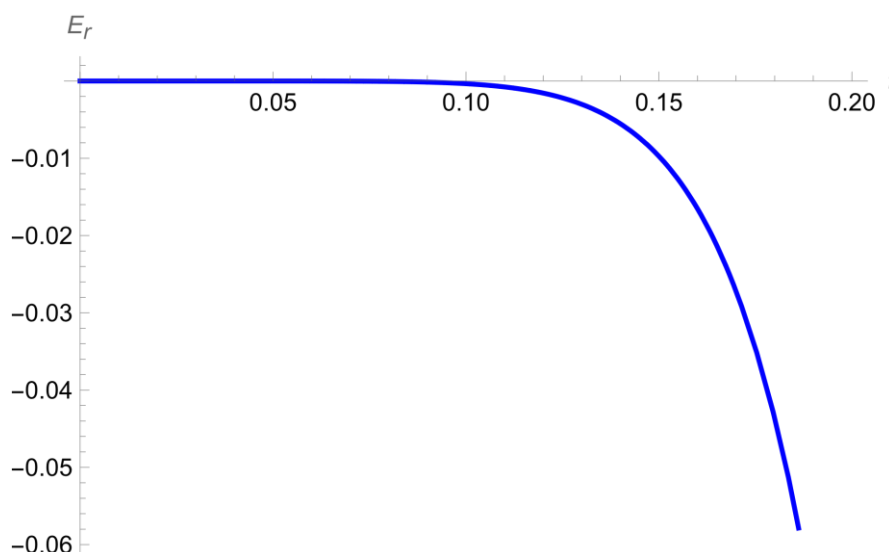


Figure 5. Residual error of DTM for Problem 4.

Again, the method appears accurate because the residual errors approach zero, particularly near the origin.

4. CONCLUSIONS

In this study, the fundamental principles of the Differential Transform Method are thoroughly examined. The transforms of various elementary and composite functions are derived, and the method is applied to solve several nonlinear initial value problems, encompassing both homogeneous and non-homogeneous ordinary differential equations. The obtained solutions are substituted back into the original equations to evaluate the residual errors. The resulting residual errors, which are found to be negligible across the considered domain, confirm the high accuracy and reliability of the DTM. Numerical results are presented in both tabular and graphical forms, and all computations are carried out using *MATHEMATICA*.

REFERENCES

- [1] Zhou, J.K., *Differential Transformation and Its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, 413, 1986.
- [2] Gerald, C. F., Wheatley, P. O., *Applied Numerical Analysis*, Pearson Education, Boston, 329, 2004.
- [3] Rao, S.S., *Engineering Optimization: Theory and Practice*, Wiley, Hoboken, 428, 2009.
- [4] Griffiths, D.F., Higham, D.J., *Numerical Methods for Ordinary Differential Equations: Initial Value Problems*, Springer, London, 135, 2010.
- [5] Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, New Age International Publishers, New Delhi, 432, 2012

- [6] He, J.H., *Homotopy Perturbation Method: A New Non-linear Analytical Technique*, Springer, Berlin, 73, 2003,
- [7] Ertürk, V.S., Momani, S., Differential transform technique for solving fifth-order boundary value problems, *Mathematical and Computational Applications*, **13**, 113, 2008. <https://doi.org/10.3390/mca13020113>
- [8] Jafari, H., Sadeghi, S., Biswas, A., The differential transform method for solving multidimensional partial differential equations, *Indian Journal of Science and Technology*, **5**, 1, 2012. <https://dx.doi.org/10.17485/ijst/2012/v5i2.10>
- [9] Chen, X., Dai, Y., Differential transform method for solving Richards' equation, *Applied Mathematics and Mechanics* (English Edition), **37**, 169, 2016.
- [10] Patel, Y., Dhodiya, J.M., Differential transform method to solve linear and nonlinear diffusion equation, *International Journal of Mathematics Trends and Technology*, **33**, 1, 2016. <https://doi.org/10.14445/22315373/IJMTT-V33P501>
- [11] Wang, L., Li, N., Differential Transform Method for Solving Linear System of First-Order Fuzzy Differential Equations, *International Conference on Oriental Thinking and Fuzzy Logic*, Springer, Cham, 235, 2016. https://doi.org/10.1007/978-3-319-30874-6_23
- [12] Abdel-Halim Hassan, I.H., Ertürk, V.S., Applying differential transformation method to the one-dimensional planar Bratu problem, *International Journal of Contemporary Mathematical Sciences*, **2**, 1493, 2007. <https://doi.org/10.12988/IJCMS.2007.07157>
- [13] Oke, A.S., Convergence of Differential Transform Method for Ordinary Differential Equations, *Journal of Advances in Mathematics and Computer Science*, **24**, 1, 2017. <https://doi.org/10.9734/JAMCS/2017/36489>
- [14] Kadhoda, N., Roushan, S.S., Jafari, H., Differential Transform Method: A tool for solving fuzzy differential equations, *International Journal of Applied and Computational Mathematics*, **4**, 33, 2018. <https://doi.org/10.1007/s40819-017-0471-9>
- [15] Saqlain, M., Irshad, W., Bashir, U., Murtaza, G., Numerical Solutions for Third Order Ordinary Differential Equation by Differential Transform Method & Elzaki Transform Method, *Scientific Inquiry and Review*, **2**, 1, 2018. <https://doi.org/10.32350/sir.24.01>
- [16] Madi, F.A.M., Abdelwahid, F., Using the Differential Transform Method to Solve Non-Linear Partial Differential Equations, *Journal of Advances in Mathematics and Computer Science*, **35**(8), 34, 2020. <https://doi.org/10.9734/jamcs/2020/v35i830311>
- [17] Agarwal, R.P., *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, New York, 68, 2000.
- [18] Ince, E.L., *Ordinary Differential Equations*, Courier Corporation, New York, 317, 2012.
- [19] Radhika, T.S.L., Iyengar, T.K.V., Rani, T.R., *Approximate Analytical Methods for Solving Ordinary Differential Equations*, CRC Press, Boca Raton, 65, 2014.
- [20] Hatami, M., Ganji, D.D., Sheikholeslami, M., *Differential Transformation Method for Mechanical Engineering Problems*, Academic Press, London, 55, 2016.
- [21] Patel, Y.F., Dhodiya, J.M., *Applications of Differential Transform to Real World Problems*, Chapman and Hall/CRC, Boca Raton, 32, 2022.
- [22] Chang, S.H., Chang, I.L., A New Algorithm for Calculating One-Dimensional Differential Transform of Nonlinear Functions, *Applied Mathematics and Computation*, **195**, 799, 2008. <https://doi.org/10.1016/j.amc.2007.05.026>
- [23] El-Shahed, M., Application of differential transform method to non-linear oscillatory systems, *Communications in Nonlinear Science and Numerical Simulation*, **13**(8), 1714, 2008. <https://doi.org/10.1016/j.cnsns.2007.03.005>
- [24] Munganga, J.M.W., Mwambakana, J.N., Maritz, R., Batubenge, T.A., Moremedi, G.M., Introduction of the differential transform method to solve differential equations at

- undergraduate level, *International Journal of Mathematical Education in Science and Technology*, **45**, 781, 2014. <https://doi.org/10.1080/0020739X.2013.877609>
- [25] Mukhtarov, O., Yucel, M., Aydemir, K., A new generalization of the differential transform method for solving boundary value problems, *Journal of New Results in Science*, **10**(2), 49, 2021. <https://izlik.org/JA48JK96AS>
- [26] Zill, D.G., *A First Course in Differential Equations with Modeling Applications (10th Edition)*, Cengage Learning, Boston, 178, 2013.