

DIMORPHIC TRIMAT SEQUENCES: ALGEBRAIC PROPERTIES AND DEGENERATE EXTENSIONS

ENGİN ESER¹, BAHAR KULOĞLU^{2*}, ENGİN OZKAN³

Manuscript received: 17.12.2025; Accepted paper: 19.06.2026;

Published online: 30.09.2026.

Abstract. In this paper, we introduce and investigate a new class of Fermat-type integer sequences and polynomials, called the Trimat numbers and Trimat polynomials, inspired by the recurrence structure of the Tribonacci sequence. Although motivated by a third-order sequence, the Trimat sequence satisfies a second-order recurrence relation, leading to several elegant identities and structural properties. We establish Binet-type formulas, ordinary and exponential generating functions, and derive a comprehensive collection of classical identities including the Catalan, Cassini, d'Ocagne, Honsberger, and Vajda identities. In addition, we explore matrix representations of the Trimat sequence and its polynomials and analyze determinant invariances associated with these matrices. Transformative structures such as the Hankel, Catalan, and binomial transforms are applied to the Trimat sequence to further reveal its internal algebraic and combinatorial behavior. Motivated by recent developments on degenerate special numbers, we also introduce a new dimorphic type of Fermat numbers and establish their connections with the degenerate Stirling polynomials of the second kind. These results broaden the framework of Fermat-related number sequences and provide new perspectives for both theoretical and applied research.

Keywords: Mersenne numbers; Fermat numbers; Tribonacci numbers; Catalan transform; Hankel transform; binomial transform; Stirling numbers; dimorphic Fermat.

Mathematics Subject Classification: 11C08; 11C20; 11S05; 15A04; 15B99.

1. INTRODUCTION

This survey focuses on polynomials of positive integer sequences, especially Fermat integer sequences, which are subject to certain recurrence relations. Of all integer sequences, the Fibonacci and Lucas sequences are undoubtedly the most popular. Several leading researchers, including Hoggatt and Koshy [1,2], have extensively analyzed Fibonacci sequences and polynomials derived from sequences of integers. Their leaflet introduces the intriguing properties of Fibonacci and Lucas numbers, allowing pupils to see how many mathematical generalizations can arise from simple concepts. Many researchers have also examined innovative formulas relating Mersenne and Fermat numbers to the generalized class of Fibonacci numbers [3,4]. We give several identities related to these sequences and

¹ Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, 24002 Erzincan, Türkiye. E-mail: engineser1978@gmail.com.

² Sivas University of Science and Technology, Faculty of Engineering and Natural Sciences, Department of Engineering Basic Sciences, 58000 Sivas, Türkiye.

³ Marmara University, Faculty of Science, Department of Mathematics, 34722 Kadıköy, İstanbul, Türkiye. E-mail: engin.ozkan@marmara.edu.tr.

* Corresponding author: bkuloglu@sivas.edu.tr; bahar_kuloglu@hotmail.com.

polynomials [5]. They derived these new formulas by examining the relationships between Fermat and generalized Fibonacci polynomials. Their approach included power series representations and inversion techniques.

In this study, a novel Fermat polynomial sequence that depends on second-order recursive relations is discussed. We search for established [6] basic identities for these polynomials and highlight notable properties of them. We also present the generating functions and Binet's formula that depend on these polynomials. The survey also investigates matrix representations of this Fermat polynomial sequence and focuses especially on the determinants of these matrices. The Hankel transform, also called the Fourier–Bessel transform and attributed to Herman Hankel, appears frequently in combinatorial analysis through the calculation of Hankel determinants.

Barry [7] conducted research on the Hankel, Catalan, and binomial transformations of these sequences and polynomials and obtained important insights. This approach provides new insights into the interconnections within these polynomials. Moreover, many different transformations have been discussed with the help of polynomial sequences of integer sequences.

Fermat numbers, which satisfy $F_n = 2^n + 1$, named after Pierre de Fermat (1601-1665), the first to present them. Tsang, Theorey, and Stein investigated $2^{2^n} + 1$ of form numbers [8]. (sequence A000215 in the OEIS) The primality of numbers in this form has always been attractive to researchers. Although the existence of infinite prime numbers in this form has not been resolved, their work has shed light on the primality and divisibility of these numbers. Kuloğlu [9] presented a new class of numbers and polynomials, the Gauss-Fermat numbers and polynomials. This research examined the properties of these numbers, their connections with other number sequences, and introduced several new identities. Furthermore, the author introduced a new mathematical structure: The Gauss-Fermat numbers and polynomials.

Some researchers investigated the connection between Fibonacci numbers and Fermat numbers (and Mersenne) [3,10]. In this study, we examine Fermat numbers and a special version of Fermat numbers. Hoggatt was the first to present the Tribonacci number sequence, and various studies have been carried out on these numbers [1,11]. Inspired by Tribonacci numbers, we present a novel set of Fermat numbers and polynomials, which we call Trimat (tri-Fermat) numbers and polynomials. In this survey, we examine properties and the derivation of various identities of Trimat polynomials. We denote this number sequence by R_n and the polynomials of this number sequence by $\{R_n(x)\}$. We polish the features of our new number sequence using past techniques. Firstly, we give Fermat numbers. Fermat numbers are shown by F_n and have the second-order recurrence relation (1). Some Fermat numbers are 2,3,5,9,17,33,65, 129, ..., and

$$F_{n+2} = 3F_{n+1} - 2F_n \quad (1)$$

with the initial conditions $F_0 = 2, F_1 = 3$. The characteristic equation of the Fermat sequence is as follows,

$$r^2 - 3r + 2 = 0$$

and the roots are $r_1 = 2$ and $r_2 = 1$. Using the roots, we find the Binet-like formula of the Fermat sequence as $2^n + 1$.

The interest in integer sequences, which began with the discovery of the Fibonacci sequence and has been generalized in many ways, continues to increase as these sequences

find applications in other scientific fields. Many properties of sequences are derived by relating them to concepts such as matrices, polynomials, quaternions, and octonions [12,13]. One such sequence is the Fermat sequence.

In this study, we extend existing research on Fermat numbers by introducing a new type of polynomial derived from these numbers. Trimat polynomials exhibit many intriguing properties and identities, including well-known formulations such as Binet's formula and the related generating functions. These findings enrich the mathematical understanding of trimat numbers and their associated polynomials by providing a deeper understanding of their structure and behavior.

We also examine the matrix representations of Fermat polynomials, highlighting the importance of the determinants of these matrices. By examining the constancy of these determinants, we introduce important propositions that lead to a deeper understanding of the algebraic properties of Trimat polynomials. This matrix approach suggests an alternative perspective on how these polynomials can be analyzed and manipulated in various mathematical contexts.

Alongside these fundamental results, we also investigate the application of various transformations to Trimat sequences and polynomials. We focus particularly on the Hankel and Catalan transformations, which play an important role in the study of sequences and polynomials. These transformations allow us to explore relationships between Trimat polynomials and other well-known mathematical structures, opening potential avenues for further research in both theoretical and applied mathematics.

Another aim of this paper is to introduce the dimorphic new Fermat numbers as a degenerate analog of the classical Mersenne numbers and to explore several of their applications related to the degenerate second kind Stirling polynomials. The novelty of the study lies in presenting the dimorphic new Fermat numbers for the first time and demonstrating their connections with the degenerate second kind Stirling polynomials.

For $\lambda \neq 0, \lambda \in \mathbb{R}$ the degenerate exponentials are defined by [14-20] according to equation (2):

$$e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{s=0}^{\infty} (x)_{s,\lambda} \frac{t^s}{s!} \quad (2)$$

where $(x)_{0,\lambda} = 1, (x)_{s,\lambda} = x(x - \lambda)(x - 2\lambda) \dots (x - (s - 1)\lambda), (s \geq 1)$.

Also, we see equation (3)

$$\lim_{\lambda \rightarrow 0} e_{\lambda}^x(t) = e^{xt}, \lim_{\lambda \rightarrow 0} (x)_{s,\lambda} = x^s, (s \geq 0) \quad (3)$$

The second-kind Stirling numbers are classically defined through [14];

$$x^s = \sum_{j=0}^s S_2(s, j)(x)_j, s \geq 0.$$

For second-kind Stirling numbers, equation (4) holds [14]:

$$\frac{1}{s!} (e^t - 1)^s = \sum_{m=s}^{\infty} S_2(m, s) \frac{t^m}{m!}, s \geq 0. \quad (4)$$

There has been substantial interest in developing various degenerate versions of numerous special numbers and polynomials, using a wide range of techniques such as generating functions, combinatorial methods, ppp-adic analysis, umbral calculus, differential equations, probability theory, operator theory, analytic number theory, and quantum physics. These investigations produced many noteworthy results of both arithmetical and combinatorial significance (see [14-24] and the references therein).

Hoggart [1] defined the Tribonacci sequence recurrence as follows

$$N_{n+3} = N_n + N_{n+1} + N_{n+2},$$

with $N_0 = 0, N_1 = 1$ and $N_2 = 1$.

Also, Feng introduced more identities of the Tribonacci sequence [25]. As we see above, the Tribonacci number sequence has a third-order recurrence relation.

2. NEW FERMAT NUMBERS: TRIMAT NUMBERS

Inspired by the Tribonacci number sequence, we examine the Trimat number sequence. We give some important identities

Definition 1. *Defining Trimat numbers as R_n , and Fermat numbers as F_n , thus*

$$\begin{aligned} R_0 &= F_0 + F_1 + F_2 \\ R_1 &= F_1 + F_2 + F_3 \\ R_2 &= F_2 + F_3 + F_4 \\ &\vdots \\ R_n &= F_n + F_{n+1} + F_{n+2}. \end{aligned}$$

Using the recurrence relation above, we get the sequence of Trimat numbers as follows,

$$10, 17, 31, 59, 115, 227, 451, 899, 1795, \dots$$

The recurrence of this number sequence is given by equation (5)

$$R_{n+2} = 3R_{n+1} - 2R_n. \quad (5)$$

with the initial value $R_0 = 10, R_1 = 17$.

Unlike the Tribonacci number sequence, the Trimat number sequence has a second-order recurrence relation. Characteristic equation of $\{R_n\}$ as follows

$$x^2 - 3x + 2 = 0.$$

The roots of the equation are $\alpha = 2, \beta = 1$.

Theorem 2. *The Binet-like formula of $\{R_n\}$ is given as follows,*

$$R_n = 7 \cdot 2^n + 3$$

Proof:

$$R_n = c_1 \alpha^n + c_2 \beta^n$$

using the first two terms,

$$\begin{aligned} 10 &= c_1 + c_2 \\ 17 &= c_1 + 2c_2 \end{aligned}$$

we obtain

$$R_n = 7 \cdot 2^n + 3 \quad (6)$$

as desired.

Theorem 3. *The generating function of $\{R_n\}$ as follows,*

$$R_n(t) = \frac{10 - 13t}{1 - 3t + 2t^2} \quad (7)$$

Proof:

$$\begin{aligned} R_n(t) &= R_0 + R_1 t + R_2 t^2 + \cdots + R_m t^m + \cdots \\ -3t R_n(t) &= -3R_0 t - 3R_1 t^2 - 3R_2 t^3 + \cdots + -3R_m t^{m+1} + \cdots \\ 2t^2 R_n(t) &= 2R_0 t^2 + 2R_1 t^3 + 2R_2 t^4 + \cdots + 2R_m t^{m+2} + \cdots \end{aligned}$$

Summing both sides of the equations,

$$\begin{aligned} &(1 - 3t + 2t^2)R_n(t) \\ &= R_0 + (R_1 - 3R_0)t + (R_2 - 3R_1 + 2R_0)t^2 + \cdots + (R_m - 3R_{m-1} + 2R_{m-2})t^m + \cdots \end{aligned}$$

Using (1) and the initial conditions, we get

$$\begin{aligned} (1 - 3t + 2t^2)\mathcal{T}_n &= R_0 + (R_1 - 3R_0)t \\ R_n &= \frac{R_0 + (R_1 - 3R_0)t}{1 - 3t + 2t^2} \end{aligned}$$

Using (1), we get

$$R_n(t) = \frac{10 - 13t}{1 - 3t + 2t^2}$$

as desired.

Theorem 4. *Exponential generating function of $\{\mathcal{T}_n\}$ as follows,*

$$E\mathcal{T}_n(t) = 7e^{2t} - 3e^t$$

Proof: We know that

$$e^t = \sum_{u=0}^{\infty} \frac{t^u}{u!}.$$

To find the exponential generating function of $\{R_n\}$, we use the Binet formula as follows

$$\begin{aligned} ER_n(t) &= \sum_{n=0}^{\infty} R_n \frac{t^n}{n!} \\ ER_n(t) &= \sum_{n=0}^{\infty} (7 \cdot 2^n + 3) \frac{t^n}{n!} \\ ER_n(t) &= 7 \sum_{n=0}^{\infty} 2^n \frac{t^n}{n!} + 3 \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{aligned}$$

we get

$$ER_n(t) = 7e^{2t} + 3e^t.$$

as desired.

Theorem 5. (Catalan-like identity) *Catalan's identity relates to sums of binomial coefficients and is commonly used in combinatorics. It expresses a relationship between two sums of binomial coefficients, leading to simplifications in various problems.*

For $n \geq r$,

$$R_{n-r}R_{n+r} - R_n^2 = -21 \cdot 2^n(2^r - 2^{-r} - 2) \quad (8)$$

Proof: Using (5)

$$\begin{aligned} R_{n-r}R_{n+r} - R_n^2 &= (7 \cdot 2^{n-r} + 3)(7 \cdot 2^{n+r} + 3) - (7 \cdot 2^n + 3)^2 \\ &= 21 \cdot 2^{n-r} + 21 \cdot 2^{n+r} - 42 \cdot 2^n = 21 \cdot 2^n(2^{-r} + 2^r - 2) \end{aligned}$$

as desired.

Corollary 6. (Cassini-like identity) taking $r = 1$ in the Catalan identity, it yields the Cassini identity as follows

$$R_{n-1}R_{n+1} - R_n^2 = 21(2)^{n-1}$$

Theorem 7. (D'ocagne-like Identity) *This identity relates to binomial coefficients and is used in number theory and combinatorics. It is an identity involving sums and products of binomial coefficients and can be used to derive other mathematical identities.*

Let n and m be any integers.

$$R_{m+s}R_{n+r} - R_{m+r}R_{n+s} = 21(2^s - 2^r)(2^m - 2^n) \quad (9)$$

Proof:

$$\begin{aligned} (7 \cdot 2^{m+s} + 3)(7 \cdot 2^{n+r} + 3) &- (7 \cdot 2^{m+r} + 3)(7 \cdot 2^{n+s} + 3) \\ &= 21 \cdot 2^{m+s} + 21 \cdot 2^{n+r} - 21 \cdot 2^{m+r} - 21 \cdot 2^{n+s} \end{aligned}$$

$$\begin{aligned}
 &= 21 \cdot 2^m(2^s - 2^r) + 21 \cdot 2^n(2^r - 2^s) \\
 &= 21(2^s - 2^r)(2^m - 2^n)
 \end{aligned}$$

as desired.

Theorem 8. (Honsberger-like Identity) *Honsberger's identity is a formula involving sums of products of binomial coefficients and has applications in combinatorial mathematics. It is useful for simplifying complex binomial expressions and finding closed forms for sums.*

Let n and m be any integers.

$$R_{m-1}R_n + R_mR_{n+1} = 245 \cdot 2^{m+n-2} - 63 \cdot 2^{m-1} - 63 \cdot 2^n + 18 \quad (10)$$

Proof:

$$\begin{aligned}
 &= (7 \cdot 2^{m-1} + 3) \cdot (7 \cdot 2^n + 3) + (7 \cdot 2^m + 3)(7 \cdot 2^{n+1} + 3) \\
 &= 49 \cdot 2^{m+n-1} + 21 \cdot 2^{m-1} + 21 \cdot 2^n + 49 \cdot 2^{m+n+1} + 21 \cdot 2^m + 21 \cdot 2^{n+1} + 18 \\
 &= 245 \cdot 2^{m+n-1} + 63 \cdot 2^{m-1} + 63 \cdot 2^n + 18.
 \end{aligned}$$

Theorem 9. (Vajda-like Identity) *Vajda's identity is related to sequences and sums involving binomial coefficients, often used in number theory. It provides a powerful tool for transforming and simplifying expressions in combinatorial contexts.*

Let n and m be any integers.

$$R_{n+r}R_{n+s} + R_nR_{n+r+s} = 98 \cdot 2^{2n+r+s} - 21 \cdot 2^n(2^r + 2^s + 2^{r+s} + 1) + 18$$

Proof:

$$\begin{aligned}
 R_{n+r}R_{n+s} + R_nR_{n+r+s} &= (7 \cdot 2^{n+r} + 3)(7 \cdot 2^{n+s} + 3) + (7 \cdot 2^n + 3)(7 \cdot 2^{n+r+s} + 3) \\
 &= 49 \cdot 2^{2n+r+s} + 21 \cdot 2^{n+r} + 21 \cdot 2^{n+s} + 49 \cdot 2^{2n+r+s} + 21 \cdot 2^n + 21 \cdot 2^{n+r+s} \\
 &\quad + 18 = 98 \cdot 2^{2n+r+s} + 21 \cdot 2^n(2^r + 2^s + 2^{r+s} + 1) + 18
 \end{aligned}$$

as desired.

2.1. HANKEL TRANSFORM OF TRIMAT SEQUENCE

The Hankel transform, first introduced by mathematician Herman Hankel, is an integral transform used in many areas, including coding and combinatorial computations. Fourier-Bessel transform is also called the Hankel transform. The Hankel determinant calculus originates in combinatorial analysis. Barry [7] studied the Hankel transform, the Catalan transform, and the binomial transform of sequences of integers and polynomials and found quite remarkable results. Keeping these studies in mind, we apply the Hankel transform to Trimat polynomials.

This is done by finding the explicit form for the coefficients in the second-term recurrence relations. Trimat integer sequence is $\{10, 17, 31, 59, 115, 227, 451, 899, 1795, \dots\}$. The Hankel matrix of $\{R_n\}$ is an infinite matrix as

$$R = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & \dots \\ b_2 & b_3 & b_4 & b_5 & \dots \\ b_3 & b_4 & b_5 & b_6 & \dots \\ b_4 & b_5 & b_6 & b_7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with elements b_i which are Trimat numbers. The Hankel matrix of R_n of order n is the upper-left $n \times n$ submatrix of R , and t_n , the Hankel determinant of R_n , is the determinant of the corresponding Hankel matrix, $t_n = \det(R_n)$. Cvetkovic et al. [26] calculated the Catalan numbers, the Hankel transform of Fibonacci numbers. Inspired by that study, we present the first four Hankel matrices of Trimat sequences as follows:

$$R_1 = [10], R_2 = \begin{bmatrix} 10 & 17 \\ 17 & 31 \end{bmatrix}, R_3 = \begin{bmatrix} 10 & 17 & 31 \\ 17 & 31 & 59 \\ 31 & 59 & 115 \end{bmatrix}, R_4 = \begin{bmatrix} 10 & 17 & 31 & 59 \\ 17 & 31 & 59 & 115 \\ 31 & 59 & 115 & 227 \\ 59 & 115 & 227 & 451 \end{bmatrix},$$

The first four terms of the Hankel transform of Trimat integer sequences are $t_1 = 10, t_2 = 21, t_3 = 0, t_4 = 0$. It can be calculated that the other terms of this sequence equal 0. Ehrenbourg [27] calculated the Hankel determinant of exponential polynomials. Peart and Woan [28] have studied generating functions via Hankel and Stieltjes matrices. Using these results in [27] and [28], we calculated the Hankel determinant of Fermat numbers and their polynomials.

2.2. CATALAN AND BINOMIAL TRANSFORMATION OF TRIMAT NUMBERS

Catalan numbers, C_n are given by

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (11)$$

We can also write the above equation as follows

$$C_n = \frac{(2n)!}{(n+1)!n!} \quad (12)$$

The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132, 429, Also, one can easily see the following recurrence relation for C_n

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}.$$

The ordinary generating function (o.g.f.) of $C(x)$ as follows:

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

$$c(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

Barry first studied the Catalan transform [7], and many studies have examined this transformation. Also, Özkan and Taştan investigated the Catalan transform of k -Jacobsthal sequence [29]. We define the Catalan transform of $\{CR_n\}$ as

$$CR_n = \sum_{i=0}^n \frac{i}{2n-i} \binom{2n-i}{n-i} R_i \quad (13)$$

for $n \geq 1$ with $CR_0 = 10$.

The first few terms of CR_n as follow:

$$CR_1 = \sum_{i=0}^1 \frac{i}{2-i} \binom{2-i}{1-i} R_i = 17$$

$$CR_2 = \sum_{i=0}^2 \frac{i}{4-i} \binom{4-i}{2-i} R_i = 48$$

$$CR_3 = \sum_{i=0}^3 \frac{i}{6-i} \binom{6-i}{3-i} R_i = 155$$

$$CR_4 = \sum_{i=0}^4 \frac{i}{8-i} \binom{8-i}{4-i} R_i = 532$$

The binomial transform can be given as follows

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k.$$

The matrix representation of the Binomial transform is as follows:

$$\mathcal{A} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Bernstein and Solane [30] studied the transformation on integer sequences. We define the Binomial transform of Trimat numbers BR_n , as

$$BR_n = \sum_{j=0}^n \binom{n}{j} R_j \quad (14)$$

for $n \geq 0$ with $BR_0 = 10, BR_1 = 27$.

First few terms of BR_n as follows:

$$\begin{aligned} BR_2 &= \sum_{j=0}^2 \binom{2}{j} R_j = 75, \\ BR_3 &= \sum_{j=0}^3 \binom{3}{j} R_j = 213, \\ BR_4 &= \sum_{j=0}^4 \binom{4}{j} R_j = 615 \\ BR_5 &= \sum_{j=0}^5 \binom{5}{j} R_j = 1797. \end{aligned}$$

The Binomial transform of Trimat sequences is as follows:

$$\begin{bmatrix} BR_0 \\ BR_1 \\ BR_2 \\ BR_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 27 \\ 75 \\ 213 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 17 \\ 31 \\ 59 \\ \vdots \end{bmatrix}.$$

The Catalan transform of $(R_n)_{n \geq 0}$ is defined to be the sequence whose o.g.f. is $A(xc(x))$. Also, the binomial transform of the Catalan transform can be obtained as follows:

$$BC(a_n) = \sum_{j=1}^n \sum_{i=1}^j \frac{i}{2j-i} \binom{n}{j} \binom{2j-i}{j-i} a_n.$$

3. TRIMAT POLYNOMIALS

Tribonacci polynomial sequence [1] is defined by the recurrence as follows

$$N_{n+3}(x) = x^2 N_n(x) + x N_{n+1}(x) + N_{n+2} \quad (15)$$

with $N_0(x) = 0, N_1(x) = 1$ and $N_2(x) = 1$.

Fermat polynomials [4] are defined by

$$F_{n+2}(x) = 3xF_{n+1}(x) - 2F_n(x) \quad (16)$$

$n \geq 0$, with the initial conditions $F_0(x) = 2, F_1(x) = 3$. The characteristic equation of the Fermat sequence is as follows,

$$r^2(x) - 3xr(x) + 2 = 0$$

and the roots are

$$\alpha(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}, \beta(x) = (x) = \frac{3x - \sqrt{9x^2 - 8}}{2}$$

The Tribonacci sequence is a set of integers that begins with zero, one, and one. This sequence is A000073. Each number in this sequence is the sum of the three preceding numbers.

Definition 10. *Inspired by the Tribonacci number sequence, we examine the Trimat polynomial sequence. We give some important identities. We define Trimat polynomials as $R_n(x)$, and Fermat polynomials as $F_n(x)$. Thus*

$$\begin{aligned} R_0(x) &= F_0(x) + F_1(x) + F_2(x) \\ R_1(x) &= F_1(x) + F_2(x) + F_3(x) \\ R_2(x) &= F_2(x) + F_3(x) + F_4(x) \\ &\vdots \\ R_n(x) &= F_n(x) + F_{n+1}(x) + F_{n+2}(x) \end{aligned}$$

Taking $x = 1$ we get Trimat numbers. The recurrence of this number sequence is as follows

$$R_{n+2}(x) = 3xR_{n+1}(x) - 2R_n(x).$$

with the initial values

$$R_0(x) = 10, R_1(x) = 17.$$

Unlike the Tribonacci number sequence, Trimat polynomials have a second-order recurrence relation. The first few terms of $\{R_n(x)\}$

$$\begin{aligned} R_2(x) &= 51x - 20 \\ R_3(x) &= 153x^2 - 60x - 34 \\ R_4(x) &= 459x^3 - 180x^2 - 204x + 40 \\ R_5(x) &= 1377x^4 - 540x^3 - 918x^2 + 240x + 68 \end{aligned}$$

Taking $x = 1$ we get Trimat numbers.

Theorem 11. *The Binet-like formula of $\{R_n(x)\}$ is given as follows,*

$$R_n(x) = \frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^n(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^n(x) \quad (17)$$

Proof:

$$R_n(x) = c_1 \alpha^n(x) + c_2 \beta^n(x)$$

where

$$\alpha(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}, \beta(x) = (x) = \frac{3x - \sqrt{9x^2 - 8}}{2}.$$

Taking $x=1$, we get $\alpha=2, \beta=1$. Using the first two terms,

$$\begin{aligned} 10 &= c_1 + c_2 \\ 17 &= c_1 \alpha(x) + c_2 \beta(x) \end{aligned}$$

One can easily find the c_1, c_2 as follows

$$\begin{aligned} c_1 &= \frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \\ c_2 &= \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \end{aligned}$$

as desired.

Theorem 12. *The generating function of $R_n(x)$ as follows,*

$$R_n(x) = \frac{10 + (17 - 30x)t}{1 - 3xt + 2t^2}. \quad (18)$$

Proof:

$$\begin{aligned} R_n(x) &= R_0(x) + R_1(x)t + R_2(x)t^2 + \dots + R_m(x)t^m + \dots \\ -3txR_n(x) &= -3txR_0(x) - 3t^2xR_1(x) - 3t^3xR_2(x) + \dots - 3t^{m+1}xR_m(x) + \dots \\ 2t^2R_n(x) &= 2t^2R_0(x) + 2t^3R_1(x) + 2t^4R_2(x) + \dots + 2t^{m+2}R_m(x) + \dots \end{aligned}$$

Summing both sides of the equations,

$$\begin{aligned} &(1 - 3xt + 2t^2)R_n(x) \\ &= R_0(x) + (R_1(x) - 3xR_0(x))t + (R_2(x) - 3xR_1(x) + 2R_0(x))t^2 + \dots \\ &\quad + (R_m(x) - 3xR_{m-1}(x) + 2R_{m-2}(x))t^m + \dots \end{aligned}$$

Using equation (1) and initial conditions, we get

$$(1 - 3xt + 2t^2)R_n(x) = R_0(x) + (R_1(x) - 3xR_0(x))t$$

$$R_n(x) = \frac{R_0(x) + (R_1(x) - 3xR_0(x))t}{1 - 3xt + 2t^2}$$

Using initial values in equation (1), we get

$$R_n(x) = \frac{10 + (17 - 30x)t}{1 - 3xt + 2t^2}.$$

Theorem 13. Exponential generating function of $\{R_n(x)\}$ as follows,

$$ER_n(t) = \frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} e^{\alpha^n(x)t} + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} e^{\beta^n(x)t}. \quad (19)$$

Proof: We know that $e^t = \sum_{u=0}^{\infty} \frac{t^u}{u!}$.

To find the exponential generating function of $\{R_n\}$, we get

$$\begin{aligned} ER_n(t) &= \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!} \\ ER_n(t) &= \sum_{n=0}^{\infty} \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^n(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^n(x) \right) \frac{t^n}{n!} \\ ER_n(t) &= \frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \sum_{n=0}^{\infty} \alpha^n(x) \frac{t^n}{n!} + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \sum_{n=0}^{\infty} \beta^n(x) \frac{t^n}{n!} \end{aligned}$$

we get

$$ER_n(t) = \frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} e^{\alpha^n(x)t} + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} e^{\beta^n(x)t}.$$

as desired.

Theorem 14. Catalan-like identity of $\{R_n\}$ as follows, for $n \geq r$,

$$R_{n-r}(x)R_{n+r}(x) - R_n^2(x) = \left(\frac{-489 + 510x}{(\alpha(x) - \beta(x))^2} \right) 2^{n-r} (\alpha^r(x) - \beta^r(x))^2. \quad (20)$$

Proof: Using the Binet-like formula of Trimat polynomials, we get the following equality

$$\begin{aligned}
R_{n-r}(x)R_{n+r}(x) - R_n^2(x) &= \\
&= \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{n-r}(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{n-r}(x) \right) \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{n+r}(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{n+r}(x) \right) \\
&\quad - \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^n(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^n(x) \right)^2 \\
&= \left(\frac{-489 + 510x}{(\alpha(x) - \beta(x))^2} \right) \alpha^n(x) \beta^n(x) (\alpha^{-r}(x) \beta^r(x) + \alpha^r(x) \beta^{-r}(x) - 2) \\
&= \left(\frac{-489 + 510x}{(\alpha(x) - \beta(x))^2} \right) 2^n \cdot \frac{(\alpha^r(x) - \beta^r(x))^2}{2^r} \\
&= \left(\frac{-489 + 510x}{(\alpha(x) - \beta(x))^2} \right) 2^{n-r} (\alpha^r(x) - \beta^r(x))^2.
\end{aligned}$$

as desired.

Corollary 15. Taking $r = 1$ in the Catalan-like identity, we get the Cassini-like identity as follows,

$$R_{n-1}(x)R_{n+1}(x) - R_n^2(x) = \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^{n-1}. \quad (21)$$

Theorem 16. *D'ocagne-like Identity of $\{R_n(x)\}$ as follows,*

$$R_{m+s}(x)R_{n+r}(x) - R_{m+r}(x)R_{n+s}(x) = \left(\frac{-489 + 510x}{9x^2 - 8} \right) (2^s - 2^r)(2^m - 2^n). \quad (22)$$

Proof:

$$\begin{aligned}
&\left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{m+s}(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{m+s}(x) \right) \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{n+r}(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{n+r}(x) \right) \\
&- \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{m+r}(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{m+r}(x) \right) \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{n+s}(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{n+s}(x) \right) \\
&= \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^{m+s} + \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^{n+r} - \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^{m+r} \\
&\quad - \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^{n+s}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^m (2^s - 2^r) - \left(\frac{-489 + 510x}{9x^2 - 8} \right) 2^n (2^s - 2^r) \\
&= \left(\frac{-489 + 510x}{9x^2 - 8} \right) (2^s - 2^r) (2^m - 2^n).
\end{aligned}$$

as desired.

Theorem 17. Let n and m be any integers. Honsberger-like Identity of $\{R_n(x)\}$ as follows

$$\begin{aligned}
&R_{m-1}(x)R_n(x) + R_m(x)R_{n+1}(x) \\
&= \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \right)^2 \beta^{m+n}(x)(\beta^{-1}(x) + \beta(x)) \\
&+ \left(\frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \right)^2 \alpha^{m+n}(x)(\alpha^{-1}(x) + \alpha(x)) \\
&- \frac{(10\alpha(x) - 17)(-10\beta(x) + 17)}{(\alpha(x) - \beta(x))^2} \alpha^m(x)\beta^n(x)(\alpha^{-1}(x) + \beta(x)) \\
&- \frac{(10\alpha(x) - 17)(-10\beta(x) + 17)}{(\alpha(x) - \beta(x))^2} \alpha^n(x)\beta^m(x)(\beta^{-1}(x) + \alpha(x)).
\end{aligned} \tag{23}$$

Proof: From the Binet-like formula of new Fermat numbers, we get the following equality:

$$\begin{aligned}
&R_{m-1}(x)R_n(x) + R_m(x)R_{n+1}(x) = \\
&\left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{m-1}(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{m-1}(x) \right) \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^n(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^n(x) \right) \\
&+ \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^m(x) + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^m(x) \right) \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \alpha^{n+1}(x) \right. \\
&\quad \left. + \frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \beta^{n+1}(x) \right) \\
&= \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \right)^2 \beta^{m+n}(x)(\beta^{-1}(x) + \beta(x)) \\
&+ \left(\frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \right)^2 \alpha^{m+n}(x)(\alpha^{-1}(x) + \alpha(x)) \\
&- \frac{(10\alpha(x) - 17)(-10\beta(x) + 17)}{(\alpha(x) - \beta(x))^2} \alpha^m(x)\beta^n(x)(\alpha^{-1}(x) + \beta(x)) \\
&- \frac{(10\alpha(x) - 17)(-10\beta(x) + 17)}{(\alpha(x) - \beta(x))^2} \alpha^n(x)\beta^m(x)(\beta^{-1}(x) + \alpha(x)).
\end{aligned}$$

Theorem 18. Let n and m be any integers. Vajda-like Identity of $\{R_n(x)\}$ as follows

$$\begin{aligned}
 R_{n+r}(x)R_{n+s}(x) + R_n(x)R_{n+r+s}(x) &= 2 \cdot \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \right)^2 \alpha^{2n+r+s}(x) \\
 &+ 2 \cdot \left(\frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \right)^2 \beta^{2n+r+s}(x) \\
 &+ 2^n \left(\frac{10\alpha(x) - 17}{\alpha(x) - \beta(x)} \right) \left(\frac{-10\beta(x) + 17}{\alpha(x) - \beta(x)} \right) (\alpha^r(x) \\
 &+ \beta^r(x))(\alpha^s(x) + \beta^s(x)).
 \end{aligned} \tag{24}$$

Proof:

$$\begin{aligned}
 R_{n+r}(x)R_{n+s}(x) + R_n(x)R_{n+r+s}(x) &= \\
 \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \alpha^{n+r}(x) + \frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \beta^{n+r}(x) \right) &\left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \alpha^{n+s}(x) + \frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \beta^{n+s}(x) \right) + \\
 \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \alpha^n(x) + \frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \beta^n(x) \right) &\left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \alpha^{n+r+s}(x) + \frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \beta^{n+r+s}(x) \right) = \\
 2 \cdot \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \right)^2 \alpha^{2n+r+s}(x) + 2 \cdot \left(\frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \right)^2 &\beta^{2n+r+s}(x) + \\
 \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \right) \left(\frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \right) (\alpha^r(x) + \beta^r(x)) &(\alpha^n(x)\beta^{n+s}(x) + \alpha^{n+s}(x)\beta^n(x)) = \\
 2 \cdot \left(\frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \right)^2 \alpha^{2n+r+s}(x) + 2 \cdot \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \right)^2 &\beta^{2n+r+s}(x) + \\
 2^n \left(\frac{10\alpha(x)-17}{\alpha(x)-\beta(x)} \right) \left(\frac{-10\beta(x)+17}{\alpha(x)-\beta(x)} \right) (\alpha^r(x) + \beta^r(x)) &(\alpha^s(x) + \beta^s(x)).
 \end{aligned}$$

as desired.

4. DIMORPHIC NEW FERMAT NUMBERS

In this section, we define the dimorphic new type of Fermat numbers as follows:

$$R_{k,\lambda} = 7(2)_{k,\lambda} + 3(1)_{k,\lambda}, \lambda \in \mathbb{R} \tag{25}$$

Also, we can see

$$\lim_{\lambda \rightarrow 0} R_{k,\lambda} = R_k$$

where R_k , k .th Trimat sequences.

Also,

$$e_\lambda^{x+y}(t) = (1+\lambda t)^{\frac{x+y}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda} + \frac{y}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t)^{\frac{y}{\lambda}} = e_\lambda^x(t) e_\lambda^y(t).$$

Recent studies have increasingly focused on degenerate extensions of classical combinatorial and algebraic structures. In this context, Dişkaya [31] introduced Fibonacci-based generalizations of degenerate Stirling numbers, while Kuloğlu [32] investigated structural aspects through a generalized q -shifted-factorial Vandermonde-type matrix framework.

Theorem 19. For $m \geq 0$ we have

$$T_{m,\lambda}(x) = 7(x+1)_{m,\lambda} - \sum_{k=0}^{m-1} \binom{m}{k} T_{k,\lambda}(x) \frac{R_{m-k+1,\lambda}}{(m-k+1)}$$

Proof: We observe that

$$\begin{aligned} 7e_\lambda^2(t) + 3e_\lambda(t) &= \sum_{k=0}^{\infty} 7(2)_{k,\lambda} \frac{t^k}{k!} + \sum_{k=0}^{\infty} 3(1)_{k,\lambda} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} (7(2)_{k,\lambda} + 3(1)_{k,\lambda}) \frac{t^k}{k!} \end{aligned} \quad (26)$$

From Equations (25) and (26), we have

$$7e_\lambda^2(t) + 3e_\lambda(t) = \sum_{k=0}^{\infty} R_{k,\lambda} \frac{t^k}{k!} \quad (27)$$

Thus, by Equations (24) and noting that $R_{0,\lambda} = 10$, we see that

$$\frac{1}{t} (7e_\lambda^2(t) + 3e_\lambda(t)) = \frac{1}{t} \sum_{k=1}^{\infty} R_{k,\lambda} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{R_{k+1,\lambda}}{(k+1)} \frac{t^k}{k!} \quad (28)$$

From equation (28), we get equation (29)

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{R_{k+1,\lambda}}{(k+1)} \frac{t^k}{k!} &= \frac{1}{t} (7e_\lambda^2(t) + 3e_\lambda(t)) = \frac{e_\lambda(t)}{t} (7e_\lambda(t) + 3) \\ &= \frac{e_\lambda^{x+1}(t)}{te_\lambda^x(t)} (7e_\lambda(t) + 3) \end{aligned} \quad (29)$$

Now, we observe that

$$\begin{aligned} e_\lambda^{x+1}(t) &= \frac{te_\lambda^x(t)}{(7e_\lambda(t) + 3)} \frac{e_\lambda^{x+1}(t)}{te_\lambda^x(t)} (7e_\lambda(t) + 3) \\ &= \frac{1}{7} \frac{te_\lambda^x(t)}{(e_\lambda(t) + 3/7)} \sum_{s=0}^{\infty} \frac{R_{s+1,\lambda}}{(s+1)} \frac{t^s}{s!} \end{aligned}$$

$$= \frac{1}{7} \sum_{k=0}^{\infty} T_{k,\lambda}(x) \frac{t^k}{k!} \sum_{s=0}^{\infty} \frac{R_{s+1,\lambda}}{(s+1)} \frac{t^s}{s!}$$

where

$$T_{k,\lambda}(x) = k! \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{7}{3}\right)^n \binom{\frac{x+n}{\lambda}}{k-1} \lambda^{k-1}.$$

From the definition of the Cauchy product, we can write the following equation:

$$e_{\lambda}^{x+1}(t) = \frac{1}{7} \sum_{m=0}^{\infty} \sum_{k=0}^m \binom{m}{k} T_{k,\lambda}(x) \frac{R_{m-k+1,\lambda}}{(m-k+1)} \frac{t^m}{m!} \quad (30)$$

On the other hand, by equation (2), we know that

$$e_{\lambda}^{x+1}(t) = \sum_{k=0}^{\infty} (x+1)_{k,\lambda} \frac{t^k}{k!}$$

From equations (30) and (2), we write

$$\begin{aligned} (x+1)_{m,\lambda} &= \sum_{k=0}^m \binom{m}{k} T_{k,\lambda}(x) \frac{R_{m-k+1,\lambda}}{(m-k+1)} \\ &= \frac{1}{7} \left(T_{m,\lambda}(x) + \sum_{k=0}^{m-1} \binom{m}{k} T_{k,\lambda}(x) \frac{R_{m-k+1,\lambda}}{(m-k+1)} \right) \end{aligned} \quad (31)$$

Therefore, by equation (31), we reach the desired conclusion.

Theorem 20. For $k \in \mathbb{N}$, we have

$$\frac{R_{m-k+1,\lambda}}{(m-k+1)} = \sum_{n=0}^l \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} \right)^l.$$

Proof: From equation (4), we know that [20]

$$\begin{aligned} \frac{1}{k!} (e^t - 1)^k &= \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, k \geq 0. \\ \frac{e_{\lambda}(t) + 3/7}{te_{\lambda}^x(t)} &= \frac{1}{\frac{te_{\lambda}^x(t)}{e_{\lambda}(t) + 3/7}} = \frac{1}{\sum_{k=0}^{\infty} T_{k,\lambda} \frac{t^k}{k!}} \\ &= \sum_{s=0}^{\infty} (-1)^s \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right)^s \\ &= \sum_{s=0}^{\infty} (-1)^s s! \frac{1}{s!} \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right)^s \end{aligned} \quad (32)$$

$$\begin{aligned}
&= \sum_{s=0}^{\infty} (-1)^s s! \frac{1}{s!} \left(e^{\ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right)} \right)^s \\
&= \sum_{s=0}^{\infty} (-1)^s s! \frac{1}{s!} \sum_{n=s}^{\infty} S_2(n, s) \frac{\left(\ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right) \right)^n}{n!}
\end{aligned}$$

On the other hand, by equation (29), we write

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{R_{k+1,\lambda}}{(k+1)k!} \frac{t^k}{k!} = \frac{e_{\lambda}^{x+1}(t)}{te_{\lambda}^x(t)} (7e_{\lambda}(t) + 3) \\
&e_{\lambda}^{x+1}(t) \left(\sum_{n=0}^{\infty} \sum_{s=0}^n (-1)^s s! S_2(n, s) \frac{\left(\ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right) \right)^n}{n!} \right) \\
&= \sum_{l=0}^{\infty} (x+1)_{l,\lambda} \frac{t^l}{l!} \left(\sum_{n=0}^{\infty} \sum_{s=0}^n (-1)^s s! S_2(n, s) \frac{\left(\ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right) \right)^n}{n!} \right) \quad (33) \\
&= \sum_{l=0}^{\infty} \sum_{n=0}^l \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \frac{\left(\ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} - 1 \right) \right)^l}{l!} t^l
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of equation (33), we obtain the desired conclusion.

Theorem 21. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
&-S_2(n, 0) \ln x^l \\
&= -\frac{R_{k+1,\lambda}}{k+1} \\
&+ \sum_{s=1}^l (-1)^s s! S_2(n, s) \ln x^l + \sum_{n=0}^{l-1} \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \ln x^l.
\end{aligned}$$

Proof: From Theorem 20, we know that

$$\frac{R_{k+1,\lambda}}{(k+1)} = \sum_{n=0}^l \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \ln \left(\sum_{i=0}^{\infty} T_{i,\lambda} \frac{t^i}{i!} \right)^l.$$

If we expand this expression for the $n = l$ th term, we obtain the following equality:

$$\begin{aligned}
&\sum_{s=0}^l (-1)^s s! S_2(n, s) \ln x^l + \sum_{n=0}^{l-1} \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \ln x^l \\
&= S_2(n, 0) \ln x^l + \sum_{s=1}^l (-1)^s s! S_2(n, s) \ln x^l + \sum_{n=0}^{l-1} \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n,\lambda} (-1)^s s! S_2(n, s) \ln x^l
\end{aligned}$$

we have

$$S_2(n, 0) \ln x^l = -\frac{R_{k+1, \lambda}}{(k+1)} + \sum_{s=1}^l (-1)^s s! S_2(n, s) \ln x^l \\ + \sum_{n=0}^{l-1} \sum_{s=0}^n \binom{l}{n} (x+1)_{l-n, \lambda} (-1)^s s! S_2(n, s) \ln x^l \quad (34)$$

Therefore, by equation (34), we obtain the desired conclusion.

5. CONCLUSIONS

In this study, we introduced the Trimat numbers and Trimat polynomials as a new Fermat-type structure derived from a second-order recurrence relation. We established their essential analytical characteristics, including Binet formulas, ordinary and exponential generating functions, and several classical identities of combinatorial significance. The matrix representations of these sequences and polynomials revealed determinant-based invariants, providing an alternative algebraic viewpoint for studying their internal structure.

We also examined transformative properties of Trimat sequences, particularly through their Hankel, Catalan, and binomial transforms, which uncovered further connections with well-known combinatorial sequences. Moreover, we defined a new dimorphic class of Fermat numbers and demonstrated their relationships with the degenerate Stirling polynomials of the second kind, enriching the existing theory of degenerate special numbers.

Future investigations may include computational approaches, efficient algorithms for generating Trimat polynomials, and potential applications of these structures in areas such as cryptography, coding theory, combinatorics, and differential equations. The framework developed here not only extends classical Fermat and Mersenne-related theories but also opens promising directions for subsequent mathematical research.

REFERENCES

- [1] Hoggatt Jr., V.E., Bicknell, M., Generalized Fibonacci Polynomials, *Fibonacci Quarterly*, **11**(5), 457, 1973. <https://doi.org/10.1080/00150517.1973.12430785>
- [2] Koshy, T., Fibonacci and Lucas Numbers with Applications, *A Wiley Interscience publication*, New York, U.S.A., 2001.
- [3] Abd-Elhameed, W.M., Youssri, Y.H., Solutions of the connection problems between Fermat and generalized Fibonacci polynomials, *JP Journal of Algebra, Number Theory and Applications*, **38**(4), 349, 2016. <https://doi.org/10.17654/NT038040349>
- [4] Abd-Elhameed, W.M., Alqubori, O.M., Alsafri, N.M.A., Amin, A.K., Atta, A.G., A Matrix Approach by Convolved Fermat Polynomials for Solving the Fractional Burgers' Equation, *Mathematics*, **13**(7), 1135, 2025. <https://doi.org/10.3390/math13071135>
- [5] Özkan, E., Kuloğlu, B., On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials, *Asian-European Journal of Mathematics*, **14**(06), 2150100, 2021. <https://doi.org/10.1142/S179355712150100X>

- [6] Kuloğlu, B., Eser, E., Özkan, E., Exploring novel Fermat polynomials and their transformative properties, *Journal of Information & Optimization Sciences*, 47(3), 1011, 2026. <https://doi.org/10.47974/JIOS-1811>
- [7] Barry, P., A Catalan Transform and Related Transformations on Integer Sequences, *Journal of Integer Sequences*, 8, 1, 2005.
- [8] Tsang, C., Theorey, M.N., Stein, W., Fermat numbers, *University of Washington*, 2010. <https://wstein.org/edu/2010/414/projects/tsang.pdf>
- [9] Kuloğlu, B., Gaussian Fermat Numbers, Polynomials and Their Associated Transforms, *Journal of Science and Arts*, 24(3), 573, 2024. <https://doi.org/10.46939/J.Sci.Arts-24.3-a11>
- [10] Boussayoud, A., Chelgham, M., Boughaba, S., On Some Identities and Generating Functions for Mersenne Numbers and Polynomials, *Turkish Journal of Analysis and Number Theory*, 6(3), 93, 2018. <https://doi.org/10.12691/tjant-6-3-5>
- [11] Scott, A., Delaney, T., Hoggatt Jr., V.E., The Tribonacci Sequence, *The Fibonacci Quarterly*, 15(3), 193, 1977. <https://doi.org/10.1080/00150517.1977.12430431>
- [12] Uysal, M., Kumari, M., Kuloğlu, B., Prasad, K., Özkan, E., On the Hyperbolic k-Mersenne and k-Mersenne-Lucas Octonions, *Kragujevac Journal of Mathematics*, 49(5), 765, 2025. <https://doi.org/10.46793/KgJMat2505.765U>
- [13] Eser, E., Kuloğlu, B., Özkan, E., On the Mersenne and Mersenne-Lucas hybrinomial quaternions, *Bulletin of the Transilvania University of Brasov Series III, Mathematics and Computer Science*, 3(65), 129, 2023. https://webbut.unitbv.ro/index.php/Series_III/article/download/4991/3867/9654
- [14] Kim, D. S., Kim, T., A Note on a New Type of Degenerate Bernoulli Numbers, *Russian Journal of Mathematical Physics*, 27(2), 227, 2020. <https://doi.org/10.1134/S1061920820020090>
- [15] Kim, B. M., Kim, Y., Park, J.-W., On the reciprocal degenerate Lah-Bell polynomials and numbers, *Advanced Studies in Contemporary Mathematics (Kyungshang)*, 32(1), 63, 2022. <http://dx.doi.org/10.17777/ascm2022.32.1.63>
- [16] Kim, D. S., Kim, T., Degenerate Sheffer sequences and λ -Sheffer sequences, *J. Math. Anal. Appl.*, 493(1), 124521, 2021. <https://doi.org/10.1016/j.jmaa.2020.124521>
- [17] Kim, H.K., The incomplete and complete r-central Lah-Bell polynomials, *Proceedings of the Jangjeon Mathematical Society*, 25(1), 95, 2022. <http://doi.org/10.17777/pjms.2022.25.1.011>
- [18] Kim, T., Kim, D.S., Kwon, J., Lee, H., Some identities involving degenerate r-Stirling numbers, *Proceedings of the Jangjeon Mathematical Society*, 25(2), 245, 2022. <http://dx.doi.org/10.17777/pjms2022.25.2.245>
- [19] Kwon, J., Kim, T., Kim, D. S., Kim, H. Y., Some identities for degenerate complete and incomplete r-Bell polynomials, *Journal of Inequalities and Applications*, 23, 1, 2020. <https://doi.org/10.1186/s13660-020-2298-x>
- [20] Ma, M., Lim, D., Some identities on the fully degenerate Bell polynomials of the second kind, *Advanced Studies in Contemporary Mathematics (Kyungshang)*, 30(1), 145, 2020. <http://dx.doi.org/10.17777/ascm2020.30.1.145>
- [21] Kim, T., Kim, D.S., Dimorphic Mersenne Numbers and Their Applications, *Advanced Studies in Contemporary Mathematics*, 32(4), 435, 2022. <http://dx.doi.org/10.17777/ascm2022.32.4.435>
- [22] Kim, T., Degenerate complete Bell polynomials and numbers, *Proceedings of the Jangjeon Mathematical Society*, 20(4), 533, 2017. <http://dx.doi.org/10.17777/pjms2017.20.4.533>

- [23] Kim, T., Kim, D.S., On some degenerate differential and degenerate difference operators, *Russian Journal of Mathematical Physics*, **29**(1), 37, 2022. <https://doi.org/10.1134/S1061920822010046>
- [24] Kim, T., Kim, D.S., Degenerate Laplace transform and degenerate gamma function, *Russian Journal of Mathematical Physics*, **24**(2), 241, 2017. <https://doi.org/10.1134/S1061920817020091>
- [25] Feng, J., More identities on the Tribonacci numbers, *Ars Combinatoria*, **100**, 73, 2011.
- [26] Cvetkovic, A., Rajkovic, P., Ivkovic, M., Catalan Numbers, the Hankel Transform, and Fibonacci Numbers, *Journal of Integer Sequences*, **5**(1), 1, 2002.
- [27] Ehrenborg, R., The Hankel Determinant of Exponential Polynomials, *The American Mathematical Monthly*, **107**(6), 557, 2000. <https://doi.org/10.1155/2014/989640>
- [28] Peart, P., Woan, W. J., Generating Functions via Hankel and Stieltjes Matrices, *Journal of Integer Sequences*, **3**, 00.2.1, 2000.
- [29] Tastan, M., Özkan, E., Catalan Transform of the k-Jacobsthal Sequence, *Electronic Journal of Mathematical Analysis and Applications*, **8**(2), 70, 2020. <http://math-frac.org/Journals/EJMAA/>
- [30] Bernstein, M., Sloane, N.J.A., Some Canonical Sequences of Integers, *Linear Algebra Appl.*, **226/228**, 57, 1995.
- [31] Dişkaya, O., Fibonacci-based generalizations of degenerate Stirling numbers, *The Ramanujan Journal*, **68**(2), 51, 2025. <https://doi.org/10.1007/s11139-025-01198-0>
- [32] Kuloğlu, B., Spectral and structural properties of a q-shifted-factorial Vandermonde-type matrix, *Computational and Applied Mathematics*, **46**, 24, 2027. <https://doi.org/10.1007/s40314-026-03886-z>