

NEW INEQUALITIES OF HUYGENS-TYPE INVOLVING HYPERBOLICUS TANGENT AND SINE FUNCTIONS

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Manuscript received: 11.07.2026; Accepted paper: 23.08.2026;

Published online: 30.09.2026.

Abstract. Based on the inequalities from (Hacet. J. Math. Stat. 52:1 (2023), 36–61), we consider analogous inequalities with hyperbolicus functions. We have proved four new inequalities and proposed two conjectures.

Keywords: Huygens' inequality; hyperbolicus trigonometric inequalities; power series.

Mathematics Subject Classification: 26D07; 41A44.

1. INTRODUCTION

Huygens' inequality states that

$$2 \frac{\sin x}{x} + \frac{\tan x}{x} > 3$$

for $0 < x < \frac{\pi}{2}$. Improvements of this inequality were considered in numerous papers [1-13]. In [13], the following Huygens-type inequalities were established.

Theorem 1.1. If $0 < |x| < \frac{\pi}{2}$, then

$$\frac{3}{20} x^3 \tan x < 2 \frac{\sin x}{x} + \frac{\tan x}{x} - 3 < \frac{16}{\pi^4} x^3 \tan x. \quad (1)$$

Furthermore, $\frac{3}{20}$ and $\frac{16}{\pi^4}$ are the best constants in (1).

Theorem 1.2. Let $0 < |x| < \frac{\pi}{2}$. Then

$$2 \frac{\sin x}{x} + \frac{\tan x}{x} - 3 < \frac{3}{20} x^4 \left(\frac{\tan x}{x} \right)^{15/14}, \quad (2)$$

where $\frac{3}{20}$ is the best constant in (2).

Theorem 1.3. Let $0 < x < \frac{\pi}{2}$. Then

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$$\frac{\sin x}{x} + 2 \frac{\tan \frac{x}{2}}{\frac{x}{2}} - 3 < \frac{1}{40} x^3 \tan x, \quad (3)$$

where the constant $\frac{1}{40}$ in (3) is the best possible.

Theorem 1.4. Let $0 < |x| < \frac{\pi}{2}$. Then

$$\frac{\sin x}{x} + 2 \frac{\tan \frac{x}{2}}{\frac{x}{2}} - 3 < \frac{1}{40} x^4 \left(\frac{\tan x}{x} \right)^{5/28}, \quad (4)$$

where the constant $\frac{1}{40}$ in (4) is the best possible.

Theorem 1.5. Let $0 < x < \frac{\pi}{2}$. Then

$$2 \frac{x}{\sin x} + \frac{x}{\tan x} - 3 < \frac{1}{60} x^3 \tan x, \quad (5)$$

where the constant $\frac{1}{60}$ in (5) is the best possible.

Theorem 1.6. Let $0 < |x| < \frac{\pi}{2}$. Then

$$2 \frac{x}{\sin x} + \frac{x}{\tan x} - 3 < \frac{1}{60} x^4 \left(\frac{\tan x}{x} \right)^{5/14}, \quad (6)$$

where the constant $\frac{1}{60}$ in (6) is the best possible.

The aim of this paper is to investigate which of the previous theorems remain valid when the circular trigonometric functions are replaced by the hyperbolic trigonometric functions.

2. MAIN RESULTS

Theorems 1.3 and 1.6 hold if trigonometric functions are replaced with the corresponding hyperbolic functions, and in this way, we obtain Theorems 2.3 and 2.6. Theorems 1.1, 1.2, 1.4, and 1.5 do not hold with such replacements. Based on Theorems 1.1 and 1.5, we give and prove Theorems 2.1 and 2.5. Based on Theorems 1.2 and 1.4, we propose Conjectures 2.2 and 2.4.

Theorem 2.1. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{3}{20} x^3 \tanh x - \frac{1}{280} \frac{x^6}{\cosh x} < 2 \frac{\sinh x}{x} + \frac{\tanh x}{x} - 3,$$

where $\frac{3}{20}$ is the best constant.

Proof: Let us consider the function

$$H(x) = 2 \frac{\sinh x}{x} + \frac{\tanh x}{x} - 3 - \frac{3}{20} x^3 \tanh x + \frac{1}{280} \frac{x^6}{\cosh x},$$

for $x \in \mathbb{R} \setminus \{0\}$.

Notice that

$$H(x) = \frac{h(x)}{20 x \cosh x},$$

where

$$h(x) = -3 x^4 \sinh x + 20 \sinh 2x - 60 x \cosh x + 20 \sinh x + \frac{1}{14} x^7.$$

By developing the hyperbolicus functions in the power series, we obtain

$$\begin{aligned} h(x) &= -3 \sum_{n=0}^{+\infty} \frac{x^{2n+5}}{(2n+1)!} + 20 \sum_{n=0}^{+\infty} \frac{2^{2n+1} x^{2n+1}}{(2n+1)!} - 60 \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n)!} + 20 \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} + \frac{1}{14} x^7 \\ &= -3 \sum_{n=2}^{+\infty} \frac{x^{2n+5}}{(2n+1)!} + 20 \sum_{n=2}^{+\infty} \frac{2^{2n+5} x^{2n+5}}{(2n+5)!} - 60 \sum_{n=2}^{+\infty} \frac{x^{2n+5}}{(2n+4)!} + 20 \sum_{n=2}^{+\infty} \frac{x^{2n+5}}{(2n+5)!} \\ &= \frac{1}{560} x^9 + \sum_{n=3}^{+\infty} \frac{a_n}{(2n+5)!} x^{2n+5}, \end{aligned}$$

where

$$a_n = -(48 n^4 + 336 n^3 + 852 n^2 + 1044 n + 640) + 640 \cdot 2^{2n}.$$

It holds that

$$\begin{aligned} a_n &= -(48 n^4 + 336 n^3 + 852 n^2 + 1044 n + 640) + 640 \cdot (1+1)^{2n} \\ &> -(48 n^4 + 336 n^3 + 852 n^2 + 1044 n + 640) + \\ &\quad + 640 \left(1 + \binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \binom{2n}{4} \right) \\ &= \frac{1136}{3} n^4 - \frac{2288}{3} n^3 + \frac{964}{3} n^2 - \frac{892}{3} n \\ &= \frac{1}{3} n^3 (1136 n - 2288) + \frac{1}{3} n (964 n - 892). \end{aligned}$$

Since $1136 n - 2288 > 0$ for $n \geq 3$ and $964 n - 892 > 0$ for $n \geq 1$, it holds that $a_n > 0$ for $n \geq 3$. Therefore, $H(x) > 0$ for $x \in \mathbb{R} \setminus \{0\}$.

Considering the family $\{\varphi_p(x)\}_{p \in \mathbb{R}}$, defined by

$$\varphi_p(x) = \begin{cases} 2 \frac{\sinh x}{x} + \frac{\tanh x}{x} - 3 - p x^3 \tanh x + \frac{1}{280} \frac{x^6}{\cosh x}, & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0, \end{cases}$$

for $x \in \mathbb{R}$, based on the approach from [14, 15], we obtain that the constant $\frac{3}{20}$ is the best possible for the considered inequality. $\square$

Conjecture 2.2. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{3}{20} x^4 \left(\frac{\tanh x}{x} \right)^{15/14} - \frac{3}{15680} x^8 < 2 \frac{\sinh x}{x} + \frac{\tanh x}{x} - 3,$$

where $\frac{3}{20}$ is the best constant.

Theorem 2.3. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{1}{40} x^3 \tanh x < \frac{\sinh x}{x} + 2 \frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3,$$

where $\frac{1}{40}$ is the best constant.

Proof: The previous inequality is equivalent to the inequality

$$\frac{1}{40} x^3 \tanh x < \frac{\sinh x}{x} - \frac{4(1 - \cosh x)}{x \sinh x} - 3$$

for $x \in \mathbb{R} \setminus \{0\}$. Let us consider the function

$$H(x) = \frac{1}{40} x^3 \frac{\sinh x}{\cosh x} - \frac{\sinh x}{x} + \frac{4(1 - \cosh x)}{x \sinh x} + 3.$$

Notice that

$$H(x) = \frac{h(x)}{40 x \sinh x \cosh x},$$

where

$$\begin{aligned} h(x) &= x^4 \cosh^2 x - 40 \cosh^3 x + 120x \sinh x \cosh x - 160 \cosh^2 x + 200 \cosh x - x^4 \\ &= \frac{1}{2} x^4 \cosh 2x - \frac{1}{2} x^4 - 10 \cosh 3x + 170 \cosh x + 60x \sinh 2x - 80 \cosh 2x - 80. \end{aligned}$$

By developing the hyperbolic functions in the power series, we obtain

$$\begin{aligned} h(x) &= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{2^{2n} x^{2n+4}}{(2n)!} - 10 \sum_{n=0}^{+\infty} \frac{3^{2n} x^{2n}}{(2n)!} + 170 \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \\ &\quad + 60 \sum_{n=0}^{+\infty} \frac{2^{2n+1} x^{2n+2}}{(2n+1)!} - 80 \sum_{n=0}^{+\infty} \frac{2^{2n} x^{2n}}{(2n)!} - \frac{1}{2} x^4 - 80 \\ &= \frac{1}{2} \sum_{n=1}^{+\infty} \frac{2^{2n} x^{2n+4}}{(2n)!} - 10 \sum_{n=1}^{+\infty} \frac{3^{2n+4} x^{2n+4}}{(2n+4)!} + 170 \sum_{n=1}^{+\infty} \frac{x^{2n+4}}{(2n+4)!} + \\ &\quad + 60 \sum_{n=1}^{+\infty} \frac{2^{2n+3} x^{2n+4}}{(2n+3)!} - 80 \sum_{n=1}^{+\infty} \frac{2^{2n+4} x^{2n+4}}{(2n+4)!} \\ &= \sum_{n=1}^{+\infty} \frac{a_n}{(2n+4)!} x^{2n+4}, \end{aligned}$$

where

$$a_n = (8n^4 + 40n^3 + 70n^2 + 1010n + 652) 2^{2n} - 810 \cdot 3^{2n} + 170.$$

It holds that

$$\begin{aligned} a_n &= (8n^4 + 40n^3 + 70n^2 + 1010n + 652) 2^{2n} - 810 \cdot (1+2)^{2n} + 170 \\ &< (8n^4 + 40n^3 + 70n^2 + 1010n + 652) 2^{2n} \end{aligned}$$

$$\begin{aligned}
& -810 \left(1 + \left(\frac{1}{2^4} \binom{2n}{2n-4} + \frac{1}{2^3} \binom{2n}{2n-3} + \frac{1}{2^2} \binom{2n}{2n-2} + \frac{1}{2} \binom{2n}{2n-1} + 1 \right) 2^{2n} \right) + 170 \\
& = (8n^4 + 40n^3 + 70n^2 + 1010n + 652) 2^{2n} \\
& - 810 \left(1 + \left(\frac{1}{24} n^4 + \frac{1}{24} n^3 + \frac{35}{96} n^2 + \frac{77}{96} n + 1 \right) 2^{2n} \right) + 170,
\end{aligned}$$

i.e.

$$a_n < -\left(\frac{1}{4} n^3 (103n - 25) + \frac{3605}{16} n^2 - \frac{5765}{16} n + 158 \right) 2^{2n} - 640.$$

Since $104n - 25 > 0$ and $\frac{3605}{16} n^2 - \frac{5765}{16} n + 158 > 0$ for $n \geq 1$, it holds that $a_n < 0$ for $n \geq 1$. Therefore, $H(x) < 0$ for $x \in \mathbb{R} \setminus \{0\}$.

Considering the family $\{\varphi_p(x)\}_{p \in \mathbb{R}}$, defined by

$$\varphi_p(x) = \begin{cases} \frac{\sinh x}{x} + 2 \frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3 - px^3 \tanh x & , \quad x \in \mathbb{R} \setminus \{0\}, \\ 0 & , \quad x = 0, \end{cases}$$

for $x \in \mathbb{R}$, based on the approach from [14,15], we obtain that the constant $\frac{1}{40}$ is the best possible for the considered inequality. $\square$

Conjecture 2.4. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{1}{40} x^4 \left(\frac{\tanh x}{x} \right)^{5/28} - \frac{41}{188160} x^8 < \frac{\sinh x}{x} + 2 \frac{\tanh \frac{x}{2}}{\frac{x}{2}} - 3,$$

where $\frac{1}{40}$ is the best constant.

Theorem 2.5. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 < \frac{1}{60} x^3 \tanh x + \frac{1}{9979200} \frac{x^7 (235x^4 + 7326x^2 + 71280)}{\sinh(2x)},$$

where $\frac{1}{60}$ is the best constant.

Proof: Let us consider the function

$$H(x) = 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 - \frac{1}{60} x^3 \tanh x - \frac{1}{9979200} \frac{x^7 (235x^4 + 7326x^2 + 71280)}{\sinh(2x)}.$$

Notice that

$$H(x) = \frac{h(x)}{19958400 \sinh x \cosh x},$$

where

$$\begin{aligned}
h(x) = & -166320x^3 \cosh(2x) + 9979200 x \cosh(2x) + 39916800 x \cosh x \\
& - 29937600 \sinh(2x) - 235x^{11} - 7326x^9 - 71280x^7 + 166320x^3 + 9979200x.
\end{aligned}$$

By developing the hyperbolicus functions in the power series, we obtain

$$\begin{aligned}
h(x) &= -166320 \sum_{n=0}^{+\infty} \frac{2^{2n} x^{2n+3}}{(2n)!} + 9979200 \sum_{n=0}^{+\infty} \frac{2^{2n} x^{2n+1}}{(2n)!} + 39916800 \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n)!} x \\
&\quad - 29937600 \sum_{n=0}^{+\infty} \frac{2^{2n+1} x^{2n+1}}{(2n+1)!} - 235x^{11} - 7326x^9 - 71280x^7 \\
&\quad + 166320x^3 + 9979200x \\
&= -166320 \sum_{n=5}^{+\infty} \frac{2^{2n} x^{2n+3}}{(2n)!} + 9979200 \sum_{n=5}^{+\infty} \frac{2^{2n+2} x^{2n+3}}{(2n+2)!} \\
&\quad + 39916800 \sum_{n=6}^{+\infty} \frac{x^{2n+3}}{(2n+2)!} - 29937600 \sum_{n=6}^{+\infty} \frac{2^{2n+3} x^{2n+3}}{(2n+3)!} \\
&= \sum_{n=5}^{+\infty} \frac{a_n}{(2n+3)!} x^{2n+3},
\end{aligned}$$

where

$$\begin{aligned}
a_n &= (-1330560 n^3 - 3991680 n^2 + 76174560 n - 120748320) 2^{2n} \\
&\quad + 79833600 n + 119750400
\end{aligned}$$

for $n \geq 5$, it holds that

$$\begin{aligned}
a_n &= (-1330560 n^3 + 76174560 n - 120748320) 2^{2n} \\
&\quad - 3991680 n^2 2^{2n} + 79833600 n + 119750400 \\
&< (-1330560 n^3 + 76174560 n - 120748320) 2^{2n} \\
&\quad - 3991680 n^2 \cdot 64 + 79833600 n + 119750400.
\end{aligned}$$

Since $-3991680 n^2 \cdot 64 + 79833600 n + 119750400 < 0$ for $n \geq 5$, it holds that $a_n < 0$ for $n \geq 5$. Therefore, $H(x) < 0$ for $x \in \mathbb{R} \setminus \{0\}$.

Considering the family $\{\varphi_p(x)\}_{p \in \mathbb{R}}$, defined by

$$\varphi_p(x) = \begin{cases} 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 - px^3 \tanh x + \frac{1}{9979200} \frac{x^7 (235x^4 + 7326x^2 + 71280)}{\sinh(2x)} & , \quad x \in \mathbb{R} \setminus \{0\}, \\ 0 & , \quad x = 0, \end{cases}$$

for $x \in \mathbb{R}$, based on the approach from [14, 15], we obtain that the constant $\frac{1}{60}$ is the best possible for the considered inequality. $\square$

Theorem 2.6. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 < \frac{1}{60} x^4 \left(\frac{\tanh x}{x} \right)^{5/14},$$

where $\frac{1}{60}$ is the best constant.

Proof: Let us consider the function

$$H(x) = \frac{2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3}{\frac{1}{60} x^4 \left(\frac{\tanh x}{x} \right)^{5/14}}.$$

Therefore,

$$H'(x) = \frac{h(x)}{7x^5 \left(\frac{\tanh x}{x} \right)^{5/14} \sinh^2 x \cosh x},$$

where

$$\begin{aligned} h(x) &= -1110 x \sinh x \cosh^2 x - 840x^2 \cosh^2 x - 2220 x \sinh x \cosh x \\ &\quad + 4590 \cosh^3 x - 570x^2 \cosh x + 450 x \sinh x - 300x^2 - 4590 \cosh x \\ &= -\frac{555}{2} x \sinh(3x) + \frac{2295}{2} \cosh(3x) - \frac{2295}{2} \cosh x + \frac{345}{2} x \sinh x \\ &\quad - 1110 x \sinh(2x) - 420x^2 \cosh(2x) - 570x^2 \cosh x - 720x^2 \end{aligned}$$

By developing the hyperbolicus functions in the power series, we obtain

$$\begin{aligned} h(x) &= -\frac{555}{2} \sum_{n=0}^{+\infty} \frac{3^{2n+1} x^{2n+2}}{(2n+1)!} + \frac{2295}{2} \sum_{n=0}^{+\infty} \frac{3^{2n} x^{2n}}{(2n)!} - \frac{2295}{2} \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} + \frac{345}{2} \sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(2n+1)!} \\ &\quad - 1110 \sum_{n=0}^{+\infty} \frac{2^{2n+1} x^{2n+2}}{(2n+1)!} - 420 \sum_{n=0}^{+\infty} \frac{2^{2n} x^{2n+2}}{(2n)!} - 570 \sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(2n)!} - 720x^2 \\ &= -\frac{555}{2} \sum_{n=1}^{+\infty} \frac{3^{2n+1} x^{2n+2}}{(2n+1)!} + \frac{2295}{2} \sum_{n=1}^{+\infty} \frac{3^{2n+2} x^{2n+2}}{(2n+2)!} - \frac{2295}{2} \sum_{n=1}^{+\infty} \frac{x^{2n+2}}{(2n+2)!} - 720x^2 \\ &\quad + \frac{345}{2} \sum_{n=1}^{+\infty} \frac{x^{2n+2}}{(2n+1)!} - 1110 \sum_{n=1}^{+\infty} \frac{2^{2n+1} x^{2n+2}}{(2n+1)!} - 420 \sum_{n=1}^{+\infty} \frac{2^{2n} x^{2n+2}}{(2n)!} - 570 \sum_{n=1}^{+\infty} \frac{x^{2n+2}}{(2n)!} \\ &= \sum_{n=1}^{+\infty} \frac{a_n}{2(2n+2)!} x^{2n+2} = -\frac{263}{420} x^{10} - \frac{743}{5544} x^{12} + \sum_{n=6}^{+\infty} \frac{a_n}{2(2n+2)!} x^{2n+2}, \end{aligned}$$

where

$$\begin{aligned} a_n &= -((3330n - 17325) 3^{2n} + (3360n^2 + 13920n + 10560) 2^{2n} \\ &\quad + 4560n^2 + 6150n + 3885). \end{aligned}$$

Since $3360n^2 + 13920n + 10560 > 0$ and $4560n^2 + 6150n + 3885 > 0$ for $n \geq 1$ and $3330n - 17325 > 0$ for $n \geq 6$, it holds that $a_n < 0$ for $n \geq 6$. Therefore, the function $H(x)$ is decreasing for $x > 0$. Notice that $H(0+) = 1$, which concludes the proof.

Considering the family $\{\varphi_p(x)\}_{p \in \mathbb{R}}$, defined by

$$\varphi_p(x) = \begin{cases} 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 - p x^4 \left(\frac{\tanh x}{x} \right)^{5/14}, & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0, \end{cases}$$

for $x \in \mathbb{R}$, based on the approach from [14], we obtain that the constant $\frac{1}{60}$ is the best possible for the considered inequality. $\square$

Acknowledgments. The authors are greatly indebted to Tatjana Mirković for her invaluable contributions to this work. This work was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract numbers: 451-03-33/2026-03/200103 (for the first author) and 451-03-34/2026-03/200103 (for the second author). The research was partially conducted on the premises of the Palace of Science, Miodrag Kostić Endowment.

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