

**$Y_i$ -DARBOUX SLANT HELICES ON SURFACES**CEYDA TOSUN<sup>1</sup>, HUSEYIN KOCAYIGIT<sup>1</sup>, TUBA AGIRMAN AYDIN<sup>2</sup>

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**Abstract.** In this study, a new class of surface curves, namely  $Y_i$  – Darboux slant helices, is investigated. The conditions under which a surface curve is a  $Y_i$  – Darboux slant helix with respect to the Osculating Darboux (OD), Normal Darboux (ND), and Rectifying Darboux (RD) frames are established, and several characterizations are obtained. In particular, necessary and sufficient conditions are derived in terms of the curvature functions associated with these frames. Furthermore, relationships between  $Y_i$  – Darboux slant helices and classical helices and slant helices are established for special surface curves, including geodesics, asymptotic curves, and lines of curvature. Concrete examples on conical and cylindrical surfaces are also presented to illustrate the theoretical results.

**Keywords:** Darboux frame; osculating Darboux frame; rectifying Darboux frame; normal Darboux frame;  $Y_i$  – Darboux slant helix.

**Mathematics Subject Classification:** 53A04.

**1. INTRODUCTION**

The theory of curves, due to its extensive applications in various fields, is one of the most attractive topics in differential geometry, and numerous mathematicians continue to conduct research in this area. Among the main focuses of curve theory are certain special curves, the most well-known of which are helices and slant helices, owing to their frequent appearances in everyday contexts. A helix is defined as a curve whose tangent vectors make a constant angle with a fixed direction [1]. Similarly, Izumiya and Takeuchi introduced slant helices as curves whose principal normal vectors make a constant angle with a fixed direction, and they provided their characterizations [2].

More recently, E. Ziplar et al. defined a special type of curve, called the Darboux helix, whose Darboux instantaneous rotation vector of the Frenet frame makes a constant angle with a fixed direction, and they derived its characterizations [3]. Following these developments, many studies have been devoted to these three types of helices, not only in Euclidean spaces but also in Minkowski spaces [1,2,4-11].

In addition to the characterizations of certain special space curves, various studies have examined their counterparts lying on surfaces. Moreover, beyond these analogs, researchers have introduced and analyzed other special surface curves using the Darboux frame, which serves as the frame of the curve-surface pair. The concept of helices lying on surfaces was investigated by J. Puig-Pey et al., who also provided their position vectors [12].

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Subsequently, the geometric properties and characterizations of these curves were studied in detail both in Euclidean and Lorentzian spaces by F. Doğan and Y. Yaylı [8,9].

In a similar vein, N. Macit and M. Döldül introduced another type of special curve on a surface, called the relatively normal slant helix, and investigated its geometric properties [13]. Furthermore, S. Hananoi et al. defined three new vector fields on surface curves [10]. Building upon these definitions, M. Önder introduced three new types of special curves on surfaces, each denoted by an index  $i \in \{o, n, r\}$  corresponding to the osculating, normal, and rectifying planes of the surface curve. He referred to these curves as  $D_i$  – Darboux slant helices and established their relationships with certain newly defined helices, termed associated helices [14].

Later, A. Alkan et al. defined three new frames-namely, the Osculating Darboux (OD) Frame, the Normal Darboux (ND) Frame, and the Rectifying Darboux (RD) Frame [6]. Using these frames, they obtained the equations characterizing  $D_i$  – Darboux slant helices by establishing the relationships between the curvature functions of these newly defined frames and those of the classical Darboux frame [15,16].

Motivated by these developments, the present study introduces a complementary class of surface curves, called  $Y_i$  – Darboux slant helices, with respect to the OD, ND, and RD frames. For each of these frames,  $Y_i$  – is the third unit vector of the corresponding orthonormal frame and is given by the cross product of its first two frame vectors. Unlike  $D_i$  – Darboux slant helices, whose defining condition is imposed on the  $D_i$  – direction, the present approach focuses on the corresponding  $Y_i$  – direction, requiring it to make a constant angle with a fixed direction. For each  $i \in \{o, n, r\}$ , we derive necessary and sufficient conditions characterizing  $Y_i$  – Darboux slant helices in terms of the associated curvature functions. We also investigate their axes and establish connections with classical helices and slant helices when the underlying surface curve is a geodesic, an asymptotic curve, or a line of curvature. Finally, examples on conical and cylindrical surfaces are given to illustrate the obtained results.

## 2. MATERIALS AND METHODS

### 2.1. PRELIMINARIES

Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ , be a unit speed curve in the three-dimensional Euclidean space. Along the curve  $\alpha$ , the vectors forming the Frenet frame  $\{T, N, B\}$

$$T = \frac{d\alpha}{ds}, N = \frac{1}{\kappa} T', B = T \times N.$$

where  $T$  denotes the unit tangent vector field,  $N$  the principal normal vector field, and  $B$  the binormal vector field of the curve. Here,  $\kappa$  represents the curvature, and  $s$  is the arc-length parameter. The function  $\tau = -\langle B', N \rangle$  is called the torsion (or the second curvature) of the curve [17]. The Frenet–Serret formulas describing the derivatives of the frame vectors are expressed as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

as given in [17-19].

A curve  $\alpha$  is said to be a helix if its unit tangent vector  $T(s)$  makes a constant angle with a fixed direction. The necessary and sufficient condition for  $\alpha$  to be a helix is that the ratio of torsion to curvature remains constant, i.e.,

$$f(s) = \frac{\tau(s)}{\kappa(s)} = \text{constant}, \quad (1)$$

(see [1,11,19]).

Similarly, if the principal normal vector  $N(s)$  of the curve  $\alpha$  makes a constant angle with a fixed direction, then the curve is called a slant helix. The necessary and sufficient condition for the curve  $\alpha$  to be a slant helix is that

$$\sigma(s) = \left( \frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left( \frac{\tau}{\kappa} \right)' \right) (s) = \text{constant}, \quad (2)$$

(see [2]).

Let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve lying on the surface  $M$ . Along the curve  $\alpha$ , let  $\{T, V, U\}$  denote the Darboux frame, where the corresponding derivative formulas of the Darboux frame are given by

$$T' = k_g V + k_n U, \quad V' = -k_g T + \tau_g U, \quad U' = -k_n T - \tau_g V \quad (3)$$

(see [9,13,18]). Here,  $T$  denotes the unit tangent vector field,  $U$  is the unit normal vector field of the surface  $M$  along the curve  $\alpha$ , and  $V = U \times T$ .  $k_g$ ,  $k_n$ , and  $\tau_g$  represent the geodesic curvature, normal curvature, and geodesic torsion of the curve  $\alpha$ , respectively [17].

Let  $\phi$  be the angle between the surface of the normal vector field  $U$  and the binormal vector field  $B$ . The relationships between the curvature and torsion of the Frenet frame and the corresponding curvatures and torsion of the Darboux frame can be expressed as follows:

$$\begin{cases} \kappa^2 = k_g^2 + k_n^2 \\ k_g = \kappa \cos \phi \\ k_n = \kappa \sin \phi \\ \tau_g = \tau - \phi'. \end{cases} \quad (4)$$

The relationship between the Darboux frame and the Frenet frame of the curve  $\alpha$  can be expressed as

$$\begin{cases} T = T \\ N = \cos \phi V + \sin \phi U \quad \text{or conversely,} \\ B = -\sin \phi V + \cos \phi U \\ V = \cos \phi N - \sin \phi B \\ U = \sin \phi N + \cos \phi B \end{cases} \quad (5)$$

(see [9]).

Let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve with the Darboux frame  $\{T, V, U\}$  and the corresponding curvatures  $k_g, k_n$ , and  $\tau_g$ . Along the curve  $\alpha$ , the following vector fields are defined as

$$\begin{aligned} D_o &= \tau_g T - k_n V, \\ D_r &= \tau_g T + k_g U, \\ D_n &= -k_n V + k_g U \end{aligned}$$

which are called, respectively, the osculating Darboux vector field, the rectifying Darboux vector field, and the normal Darboux vector field [10].

Alkan et al. [6] introduced three new frames on a surface by employing the aforementioned Darboux vector fields. These frames are referred to as the Osculating Darboux frame (OD-frame), the Normal Darboux frame (ND-frame), and the Rectifying Darboux frame (RD-frame), respectively. Let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve on the surface  $M$ . Along the curve  $\alpha$ , the Osculating Darboux frame (OD-frame) is denoted by  $\{\tilde{D}_o, U, Y_o\}$ . The derivative formulas of the OD-frame along the curve  $\alpha$  are given by

$$\begin{cases} \tilde{D}_o' = -\delta_o Y_o, \\ U' = \mu_o Y_o, \\ Y_o' = \delta_o \tilde{D}_o - \mu_o U, \end{cases} \quad (6)$$

where  $\tilde{D}_o = \frac{D_o}{\|D_o\|}$ ,  $U$  is the unit normal vector field of the surface  $M$ , and  $Y_o = \tilde{D}_o \times U$ . The

quantities  $\mu_o = \sqrt{k_n^2 + \tau_g^2}$  and  $\delta_o = \left( \frac{k_n^2}{k_n^2 + \tau_g^2} \right) \left( \frac{\tau_g}{k_n} \right)' + k_g$  represent the curvature functions of the curve  $\alpha$  with respect to the Osculating Darboux frame (OD-frame).

Along the curve  $\alpha$ , the Normal Darboux frame (ND-frame) is denoted by  $\{\tilde{D}_n, T, Y_n\}$ . The derivative formulas of the ND-frame along the curve  $\alpha$  are given by

$$\begin{cases} \tilde{D}_n' = -\delta_n Y_n, \\ T' = \mu_n Y_n, \\ Y_n' = \delta_n \tilde{D}_n - \mu_n T, \end{cases} \quad (7)$$

where  $\tilde{D}_n = \frac{D_n}{\|D_n\|}$ ,  $T$  denotes the unit tangent vector of the curve  $\alpha$ , and  $Y_n = \tilde{D}_n \times T$ . The quantities  $\mu_n = \sqrt{k_n^2 + k_g^2}$  and  $\delta_n = \left( \frac{k_g^2}{k_n^2 + k_g^2} \right) \left( \frac{k_g}{k_n} \right)' + \tau_g$  represent the curvature functions of the curve with respect to the Normal Darboux frame (ND-frame).

Along the curve  $\alpha$ , the Rectifying Darboux frame (RD-frame) is denoted by  $\{\tilde{D}_r, V, Y_r\}$ . The derivative formulas of the RD-frame along the curve  $\alpha$  are given by

$$\begin{cases} \tilde{D}_r' = -\delta_r Y_r, \\ V' = \mu_r Y_r, \\ Y_r' = \delta_r \tilde{D}_r - \mu_r V, \end{cases} \quad (8)$$

where  $\tilde{D}_r = \frac{D_r}{\|D_r\|}$ ,  $V = U \times T$  and  $Y_r = \tilde{D}_r \times V$ . The quantities  $\mu_r = \sqrt{k_g^2 + \tau_g^2}$  and  $\delta_r = \left( \frac{k_g^2}{k_g^2 + \tau_g^2} \right) \left( \frac{\tau_g}{k_g} \right)' - k_n$  represent the curvature functions of the curve with respect to the Rectifying Darboux frame (RD-frame) [6].

Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on this surface. If the vector field  $D_i$ ,  $i \in \{o, n, r\}$  of the curve  $\alpha$  (or equivalently the unit vector field  $\tilde{D}_i = \frac{D_i}{\|D_i\|}$ ) makes a constant angle with a fixed direction, then the curve  $\alpha$  is called a  $D_i$ -Darboux slant helix [14].

In the literature, it is stated that for a unit-speed curve  $\alpha$  lying on a surface  $M$ , the condition for  $\alpha$  to be classified as a  $D_i$ -Darboux slant helix is that the ratio between the associated Darboux frame functions  $\mu_i$  and  $\delta_i$  remains constant along the curve. This requirement is considered both necessary and sufficient for the curve to exhibit the  $D_i$ -Darboux slant helix property (see [6]).

In previous studies, it has been indicated that for a unit-speed curve  $\alpha$  lying on a surface  $M$ , several intrinsic characterizations can be given in terms of the geodesic torsion  $\tau_g$ , normal curvature  $k_n$ , and geodesic curvature  $k_g$ . More precisely, a curve is known to be a line of curvature when the geodesic torsion vanishes; it is asymptotic provided that the normal curvature becomes zero; and it is geodesic whenever the geodesic curvature is equal to zero. These characterizations are well-documented in the literature (see [17, 18]).

## 2.2. METHODS

In the present study,  $Y_i$ -Darboux slant helices are defined, and their differential equations are derived. These equations are obtained through the relationships between the curvatures of the new frames and those of the standard Darboux frame.

### 3. RESULTS AND DISCUSSION

#### 3.1. $Y_0$ – DARBOUX SLANT HELICES

In this section,  $Y_0$  – Darboux slant helices, a new type of surface curve lying on a surface  $M$  in  $E^3$ , are introduced, and several fundamental relations concerning these curves are established.

**Definition 3.1.1.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve along which the OD-frame  $\{\tilde{D}_o, U, Y_o\}$  is defined. If the vector field  $Y_o$  of the curve  $\alpha$  makes a constant angle with a fixed direction, then the curve  $\alpha$  is called a  $Y_0$  – Darboux slant helix.

**Theorem 3.1.2.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve possessing the OD-frame  $\{\tilde{D}_o, U, Y_o\}$  along the curve. A necessary and sufficient condition for the curve  $\alpha$  to be a  $Y_0$  – Darboux slant helix is that the function

$$\sigma_{Y_o}(s) = \frac{\delta_o^2}{(\delta_o^2 + \mu_o^2)^{3/2}} \left( \frac{\mu_o}{\delta_o} \right)'(s)$$

remains constant along the curve, where  $\mu_o \neq 0, \delta_o \neq 0$ .

*Proof:* Let  $\alpha$  be a  $Y_0$  – Darboux slant helix with respect to the Darboux frame  $\{\tilde{D}_o, U, Y_o\}$ . From Definition 3.1.1, we have  $\langle Y_o, d_{Y_o} \rangle = \cos \theta$ . Taking the derivative of both sides gives  $\langle Y_o', d_{Y_o} \rangle = 0$ , and by using equations (6), we obtain  $\langle \tilde{D}_o, d_{Y_o} \rangle = \frac{\mu_o}{\delta_o} \langle U, d_{Y_o} \rangle$ . Since  $d_{Y_o} \in \text{span}\{\tilde{D}_o, U, Y_o\}$ , we can write

$$d_{Y_o} = \frac{\mu_o}{\sqrt{\mu_o^2 + \delta_o^2}} \sin \theta \tilde{D}_o + \frac{\delta_o}{\sqrt{\mu_o^2 + \delta_o^2}} \sin \theta U + \cos \theta Y_o. \quad (9)$$

Differentiating both sides of equation (9), applying equations (5), and considering that the vector  $d_{Y_o}$  is constant, we have

$$\left( \left( \frac{\mu_o}{\delta_o} \right)' \frac{\delta_o^3}{(\mu_o^2 + \delta_o^2)^{3/2}} \sin \theta + \delta_o \cos \theta \right) \tilde{D}_o + \left( - \left( \frac{\mu_o}{\delta_o} \right)' \frac{\delta_o^2 \mu_o}{(\mu_o^2 + \delta_o^2)^{3/2}} \sin \theta - \mu_o \cos \theta \right) U = 0.$$

Since  $\tilde{D}_o \perp U$ , it follows that  $\left(\frac{\mu_o}{\delta_o}\right)' \frac{\delta_o^2}{(\mu_o^2 + \delta_o^2)^{3/2}} = -\cot \theta = \text{constant}$ . Conversely,

let  $\left(\frac{\mu_o}{\delta_o}\right)' \frac{\delta_o^2}{(\mu_o^2 + \delta_o^2)^{3/2}} = -\cot \theta = \text{constant}$ . From this, we can write

$$\left(\frac{\mu_o}{\delta_o}\right)' \frac{\delta_o^3}{(\mu_o^2 + \delta_o^2)^{3/2}} \sin \theta = -\delta_o \cos \theta.$$

Taking the derivative of the vector  $d_{Y_o}$  and using equations (6), we obtain  $d'_{Y_o} = 0$ , which means that the vector  $d_{Y_o}$  is constant and  $\langle Y_o, d_{Y_o} \rangle = \cos \theta = \text{constant}$ . Hence, by Definition 3.1.1, the curve  $\alpha$  is a  $Y_o$ -Darboux slant helix. Thus, the proof is completed.

**Corollary 3.1.3.** If the curve  $\alpha$  is a  $Y_o$ -Darboux slant helix such that  $\sigma_{Y_o}(s) = 0$ , then  $\alpha$  is also a  $D_o$ -Darboux slant helix.

**Corollary 3.1.4.** With respect to the OD-frame, the axis of the  $Y_o$ -Darboux slant helix is given by

$$d_{Y_o} = \frac{\mu_o}{\sqrt{\mu_o^2 + \delta_o^2}} \sin \theta \tilde{D}_o + \frac{\delta_o}{\sqrt{\mu_o^2 + \delta_o^2}} \sin \theta U + \cos \theta Y_o.$$

**Corollary 3.1.5.** With respect to the Darboux frame, the axis of the  $Y_o$ -Darboux slant helix is given by

$$\begin{aligned} d_{Y_o} = & \left( \frac{\tau_g}{\sqrt{\varphi_1^2 + k_n^2 + \tau_g^2}} \sin \theta - \frac{k_n}{\sqrt{\tau_g^2 + k_n^2}} \cos \theta \right) T + \left( \frac{\varphi_1}{\sqrt{\varphi_1^2 + k_n^2 + \tau_g^2}} \sin \theta \right) U \\ & - \left( \frac{k_n}{\sqrt{\varphi_1^2 + k_n^2 + \tau_g^2}} \sin \theta + \frac{\tau_g}{\sqrt{\tau_g^2 + k_n^2}} \cos \theta \right) V, \end{aligned} \quad (10)$$

where  $\varphi_1 = \frac{k_n^2}{k_n^2 + \tau_g^2} \left( \frac{\tau_g}{k_n} \right)' + k_g$ .

**Corollary 3.1.6.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$ . Denote by  $k_n, k_g$ , and  $\tau_g$  the geodesic curvature, normal curvature, and geodesic torsion of the Darboux frame along  $\alpha$ , respectively. Then,  $\alpha$  is a  $Y_o$ -Darboux slant helix if and only if, for  $\forall s \in I$ , the function

$$\sigma_{Y_o}(s) = \frac{\varphi_1^2}{(\varphi_1^2 + k_n^2 + \tau_g^2)^{3/2}} \left( \frac{\sqrt{k_n^2 + \tau_g^2}}{\varphi_1} \right)'(s) \quad (11)$$

is constant, where  $\varphi_1 = \frac{k_n^2}{k_n^2 + \tau_g^2} \left( \frac{\tau_g}{k_n} \right)' + k_g$ .

**Corollary 3.1.7.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$  with  $OD$ – frame  $\{\tilde{D}_o, U, Y_o\}$  and curvatures  $\delta_o, \mu_o$  such that  $\alpha$  is a  $Y_o$ –Darboux slant helix. Then:

i) If  $\alpha$  is an asymptotic curve on  $M$  with  $k_g \neq 0$ , then  $\alpha$  is a slant helix with axis

$$d_{Y_o} = \frac{\tau \sin \theta}{\sqrt{\kappa^2 + \tau^2}} T - \cos \theta N + \frac{\kappa \sin \theta}{\sqrt{\kappa^2 + \tau^2}} B.$$

ii) If  $\alpha$  is a line of curvature on  $M$ , then  $\alpha$  is a helix with axis  $d_{Y_o} = -\cos \theta T + \sin \theta B$ .

*Proof:*

i) Let  $\alpha$  be a  $Y_o$ –Darboux slant helix lying on the surface  $M$ . Suppose that  $\alpha$  is an asymptotic curve. Then, the normal curvature satisfies  $k_n = 0$ , and from equation (1) we obtain  $k_g = \mp \kappa$  ve  $\tau_g = \tau$ . Substituting these relations into equation (11), we have

$$\sigma_{Y_o} = \left( \frac{\tau}{\kappa} \right)' \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} = \text{constant.} \text{ Hence, according to equation (2), } \alpha \text{ is a slant helix.}$$

Furthermore, using equations (4) and (5) in equation (10), the axis of the  $Y_o$ –Darboux slant helix, expressed in terms of the Frenet frame vectors of  $\alpha$

$$d_{Y_o} = \frac{\tau \sin \theta}{\sqrt{\kappa^2 + \tau^2}} T - \cos \theta N + \frac{\kappa \sin \theta}{\sqrt{\kappa^2 + \tau^2}} B.$$

ii) Let  $\alpha$  be a line of curvature on the surface  $M$ . Thus, geodesic torsion satisfies  $\tau_g = 0$ . Substituting this into equation (11) yields

$$\sigma_{Y_o}(s) = \left( \left( \frac{k_n}{k_g} \right)' \frac{k_g^2}{(k_n^2 + k_g^2)^{3/2}} \right) (s) = \text{constant.}$$

Furthermore, from equations (4), we have  $\tau = \phi'$ ,  $\frac{k_n}{k_g} = \tan \phi$  and  $k_n^2 + k_g^2 = \kappa^2$ . Using

these relations, we obtain  $\sigma_{Y_o}(s) = \frac{\tau(s)}{\kappa(s)}(s) = \text{constant}$ . Hence, according to equation (1),  $\alpha$

is a helix. Finally, using equations (4), (5), and  $\tau_g = 0$  in equation (10), the axis of the  $Y_o$ –Darboux slant helix, expressed in terms of the Frenet frame vectors, is given by  $d_{Y_o} = -\cos \theta T + \sin \theta B$ .

**Example 3.1.8.** Let  $M$  be the conical surface given by the parametrization  $X(u, v) = (u, u \sin v, u \cos v)$ ,  $u > 0$ . Choose the parameter functions  $u = \frac{s}{\sqrt{3}}$  ( $s > 0$ ) and  $v = \ln \frac{s}{3}$  and consider the curve on  $M$ ,  $\alpha(s) = \left( \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}} \sin \left( \ln \frac{s}{3} \right), \frac{s}{\sqrt{3}} \cos \left( \ln \frac{s}{3} \right) \right)$ . The Darboux frame fields and the associated curvature functions of  $\alpha$  are computed as

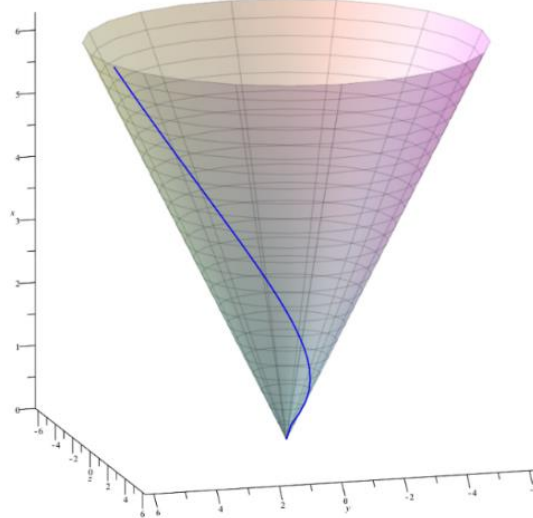
$$T(s) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \left( \sin \left( \ln \frac{s}{3} \right) + \cos \left( \ln \frac{s}{3} \right) \right), \frac{1}{\sqrt{3}} \left( \cos \left( \ln \frac{s}{3} \right) - \sin \left( \ln \frac{s}{3} \right) \right) \right),$$

$$U_\alpha(s) = \frac{1}{\sqrt{2}} \left( -1, \sin \left( \ln \frac{s}{3} \right), \cos \left( \ln \frac{s}{3} \right) \right),$$

$$V(s) = \frac{1}{\sqrt{6}} \left( -1, 2 \cos \left( \ln \frac{s}{3} \right) - \sin \left( \ln \frac{s}{3} \right), -\cos \left( \ln \frac{s}{3} \right) - 2 \sin \left( \ln \frac{s}{3} \right) \right), \quad k_n = -\frac{1}{\sqrt{6}s}, \quad k_g = \frac{1}{\sqrt{2}s},$$

$$\tau_g = -\frac{1}{\sqrt{3}s}.$$

By Corollary 3.1.6 we have  $\sigma_{Y_o}(s) = 0$  (constant), hence  $\alpha$  is a  $Y_o$ -Darboux slant helix on  $M$ . Then, by Corollary 3.1.3,  $\alpha$  is also a  $D_o$ -Darboux slant helix. The graph of  $\alpha$  is shown in Figure 1.



**Figure 1.** On the surface  $X(u, v) = (u, u \sin v, u \cos v)$ , consider the curve.

$$\alpha(s) = \left( \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}} \sin \left( \ln \frac{s}{3} \right), \frac{s}{\sqrt{3}} \cos \left( \ln \frac{s}{3} \right) \right).$$

### 3.2. $Y_n$ – DARBOUX SLANT HELICES

In this section, the concept of the  $Y_n$  – Darboux slant helix, which represents a new type of surface curve lying on a surface  $M$  in the three-dimensional Euclidean space  $E^3$ , is introduced. The necessary and sufficient condition for a surface curve to be a  $Y_n$  – Darboux slant helix, together with several important related relations, is presented. Here, the curvatures of the ND- frame are of the form  $\delta_n \neq 0, \mu_n \neq 0$ . Since the theorems in this section are constructed analogously to those in Section 3, their proofs are omitted.

**Definition 3.2.1.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  a unit-speed curve lying on  $M$  with the ND-frame  $\{\tilde{D}_n, T, Y_n\}$  and curvatures  $\delta_n, \mu_n$ . If the vector field  $Y_n$  forms a constant angle with a fixed direction, then the curve  $\alpha$  is called a  $Y_n$  – Darboux slant helix on the surface  $M$ .

**Theorem 3.2.2.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve possessing the ND-frame  $\{\tilde{D}_n, T, Y_n\}$  along the curve. A necessary and sufficient condition for the curve  $\alpha$  to be a  $Y_n$  – Darboux slant helix is that the function

$$\sigma_{Y_n}(s) = \frac{\delta_n^2}{(\delta_n^2 + \mu_n^2)^{3/2}} \left( \frac{\mu_n}{\delta_n} \right)'(s)$$

remains constant along the curve, where  $\mu_n \neq 0, \delta_n \neq 0$ .

**Corollary 3.2.3.** With respect to the ND-frame, the axis of the  $Y_n$  – Darboux slant helix is given by

$$d_{Y_n} = \frac{\mu_n}{\sqrt{\mu_n^2 + \delta_n^2}} \sin \beta \tilde{D}_n + \frac{\delta_n}{\sqrt{\mu_n^2 + \delta_n^2}} \sin \beta T + \cos \beta Y_n.$$

**Corollary 3.2.4.** If the curve  $\alpha$  is a  $Y_n$  – Darboux slant helix such that  $\sigma_{Y_n}(s) = 0$ , then  $\alpha$  is also a  $D_n$  – Darboux slant helix.

**Corollary 3.2.5.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$ . Denote by  $k_n, k_g$ , and  $\tau_g$  the geodesic curvature, normal curvature, and geodesic torsion of the Darboux frame along  $\alpha$ , respectively. Then,  $\alpha$  is a  $Y_n$  – Darboux slant helix if and only if, for  $\forall s \in I$ , the function

$$\sigma_{Y_n}(s) = \frac{\varphi_2^2}{(\varphi_2^2 + k_n^2 + k_g^2)^{3/2}} \left( \frac{\sqrt{k_n^2 + k_g^2}}{\varphi_2} \right)'(s)$$

is constant, where  $\varphi_2 = \frac{k_g^2}{k_n^2 + k_g^2} \left( \frac{k_n}{k_g} \right)' + \tau_g$ .

**Corollary 3.2.6.** With respect to the Darboux frame, the axis of the  $Y_n$  – Darboux slant helix is given by

$$d_{Y_n} = \left( \frac{\varphi_2}{\sqrt{\varphi_2^2 + k_n^2 + k_g^2}} \sin \beta \right) T - \left( \frac{k_n \sin \beta}{\sqrt{\varphi_2^2 + k_n^2 + k_g^2}} + \frac{k_g \cos \beta}{\sqrt{k_g^2 + k_n^2}} \right) V \\ + \left( \frac{k_g \sin \beta}{\sqrt{\varphi_2^2 + k_n^2 + k_g^2}} + \frac{k_n \cos \beta}{\sqrt{k_g^2 + k_n^2}} \right) U,$$

where  $\varphi_2 = \frac{k_g^2}{k_n^2 + k_g^2} \left( \frac{k_n}{k_g} \right)' + \tau_g$ .

**Corollary 3.2.7.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$  with ND-frame  $\{\tilde{D}_n, T, Y_n\}$  and curvatures  $\delta_n, \mu_n$  such that  $\alpha$  is a  $Y_n$  – Darboux slant helix. Then:

i) If  $\alpha$  is a geodesic curve on the surface with  $k_n \neq 0$ , then  $\alpha$  is a slant helix with axis

$$d_{Y_n} = \frac{\tau \sin \beta}{\sqrt{\kappa^2 + \tau^2}} T + \cos \beta N + \frac{\kappa \sin \beta}{\sqrt{\kappa^2 + \tau^2}} B.$$

ii) If  $\alpha$  is an asymptotic curve on  $M$  with  $k_g \neq 0$ , then  $\alpha$  is a slant helix with axis

$$d_{Y_n} = \frac{\tau \sin \beta}{\sqrt{\kappa^2 + \tau^2}} T - \cos \beta N + \frac{\kappa \sin \beta}{\sqrt{\kappa^2 + \tau^2}} B.$$

iii) If  $\alpha$  is a line of curvature on  $M$ , then  $\alpha$  is a helix with axis

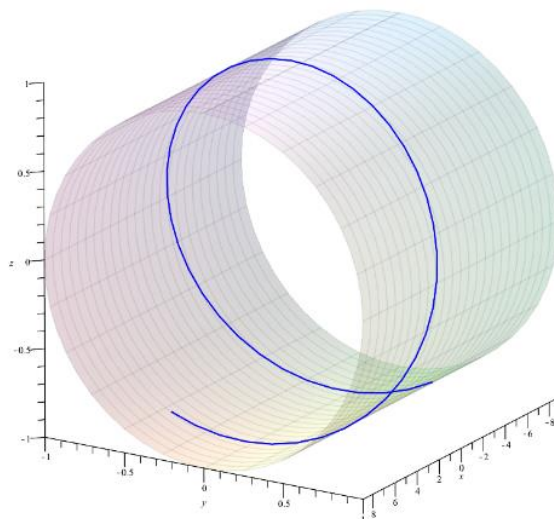
$$d_{Y_n} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} \sin \beta T + \cos \beta N + \frac{\kappa \sin \beta}{\sqrt{\tau^2 + \kappa^2}} B.$$

**Example 3.2.8.** Let  $M$  be the cylindrical surface given by the parametrization  $X(u, v) = (\sqrt{2}v, \sin u, \cos u)$ . Choose the parameter functions  $u = \frac{s}{\sqrt{3}}$  and  $v = \frac{s}{\sqrt{3}}$  and consider the curve on  $M$ ,  $\alpha(s) = \left( \frac{\sqrt{2}s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}}, \cos \frac{s}{\sqrt{3}} \right)$ . The Darboux frame fields and the associated curvature functions of  $\alpha$  are computed as

$$T(s) = \left( \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cos \frac{s}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \sin \frac{s}{\sqrt{3}} \right), \quad U_\alpha(s) = \left( 0, -\sin \frac{s}{\sqrt{3}}, -\cos \frac{s}{\sqrt{3}} \right),$$

$$V(s) = \left( \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}} \cos \frac{s}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \sin \frac{s}{\sqrt{3}} \right), \quad k_n = \frac{1}{3}, \quad k_g = 0, \quad \tau_g = -\frac{\sqrt{2}}{3}.$$

By Theorem 3.2.2, we have  $\sigma_{Y_n}(s) = 0$  (constant), hence  $\alpha$  is a  $Y_n$ –Darboux slant helix on  $M$ . The graph of  $\alpha$  is shown in Figure 2.



**Figure 2.** On the surface  $X(u, v) = (\sqrt{2}v, \sin u, \cos u)$ , consider the curve

$$\alpha(s) = \left( \frac{\sqrt{2}s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}}, \cos \frac{s}{\sqrt{3}} \right).$$

### 3.3. $Y_r$ – DARBOUX SLANT HELICES

In this section, the concept of the  $Y_r$  – Darboux slant helix, which represents a new type of surface curve lying on a surface  $M$  in the three-dimensional Euclidean space  $E^3$  is introduced. The necessary and sufficient condition for a surface curve to be a  $Y_r$  – Darboux slant helix, together with several important related relations, is presented. Here, the curvatures of the RD- frame are of the form  $\delta_r \neq 0$ ,  $\mu_r \neq 0$ . Since the theorems in this section are constructed analogously to those in Section 3, their proofs are omitted.

**Definition 3.3.1.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  a unit-speed curve lying on  $M$  with the RD-frame  $\{\tilde{D}_r, V, Y_r\}$  and curvatures  $\delta_r, \mu_r$ . If the vector field  $Y_r$  forms a constant angle with a fixed direction, then the curve  $\alpha$  is called a  $Y_r$  – Darboux slant helix on the surface  $M$ .

**Theorem 3.3.2.** Let  $M$  be a surface in  $E^3$ , and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit speed curve possessing the RD-frame  $\{\tilde{D}_r, V, Y_r\}$  along the curve. A necessary and sufficient condition for the curve  $\alpha$  to be a  $Y_r$ -Darboux slant helix is that the function

$$\sigma_{Y_r}(s) = \frac{\delta_r^2}{(\delta_r^2 + \mu_r^2)^{3/2}} \left( \frac{\mu_r}{\delta_r} \right)'(s)$$

remains constant along the curve, where  $\delta_r \neq 0$ ,  $\mu_r \neq 0$ .

**Corollary 3.3.3.** With respect to the RD-frame, the axis of the  $Y_r$ -Darboux slant helix is given by

$$d_{Y_r} = \frac{\mu_r \sin \omega}{\sqrt{\mu_r^2 + \delta_r^2}} \tilde{D}_r + \frac{\delta_r \sin \omega}{\sqrt{\mu_r^2 + \delta_r^2}} V + \cos \omega Y_r.$$

**Corollary 3.3.4.** If the curve  $\alpha$  is a  $Y_r$ -Darboux slant helix such that  $\sigma_{Y_r}(s) = 0$ , then  $\alpha$  is also a  $D_r$ -Darboux slant helix.

**Corollary 3.3.5.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$ . Denote by  $k_n, k_g$ , and  $\tau_g$  the geodesic curvature, normal curvature, and geodesic torsion of the Darboux frame along  $\alpha$ , respectively. Then,  $\alpha$  is a  $Y_r$ -Darboux slant helix if and only if, for  $\forall s \in I$ , the function

$$\sigma_{Y_r}(s) = \frac{\varphi_3^2}{(\varphi_3^2 + k_g^2 + \tau_g^2)^{3/2}} \left( \frac{\sqrt{k_g^2 + \tau_g^2}}{\varphi_3} \right)'(s)$$

is constant, where  $\varphi_3 = \left( \frac{k_g^2}{k_g^2 + \tau_g^2} \left( \frac{\tau_g}{k_g} \right)' - k_n \right)$ .

**Corollary 3.3.6.** With respect to the Darboux frame, the axis of the  $Y_r$ -Darboux slant helix is given by

$$d_{Y_r} = \left( \frac{\tau_g \sin \omega}{\sqrt{\varphi_3^2 + k_g^2 + \tau_g^2}} - \frac{k_g \cos \omega}{\sqrt{\tau_g^2 + k_g^2}} \right) T + \frac{\varphi_3}{\sqrt{\varphi_3^2 + k_g^2 + \tau_g^2}} \sin \omega V \\ + \left( \frac{k_g \sin \omega}{\sqrt{\varphi_3^2 + k_g^2 + \tau_g^2}} + \frac{\tau_g \cos \omega}{\sqrt{\tau_g^2 + k_g^2}} \right) U,$$

where  $\varphi_3 = \left( \frac{k_g^2}{k_g^2 + \tau_g^2} \left( \frac{\tau_g}{k_g} \right)' - k_n \right)$ .

**Corollary 3.3.7.** Let  $M \subset E^3$  be a surface and let  $\alpha: I \subset \mathbb{R} \rightarrow M$  be a unit-speed curve lying on  $M$  with RD-frame  $\{\tilde{D}_r, V, Y_r\}$  and curvatures  $\delta_r, \mu_r$  such that  $\alpha$  is a  $Y_r$ -Darboux slant helix. Then:

i) If  $\alpha$  is a geodesic curve on the surface with  $k_n \neq 0$ , then  $\alpha$  is a slant helix with axis

$$d_{Y_r} = \frac{\tau \sin \omega}{\sqrt{\kappa^2 + \tau^2}} T + \cos \omega N + \frac{\kappa \sin \omega}{\sqrt{\kappa^2 + \tau^2}} B.$$

ii) If  $\alpha$  is a line of curvature on  $M$ , then  $\alpha$  is a helix with axis

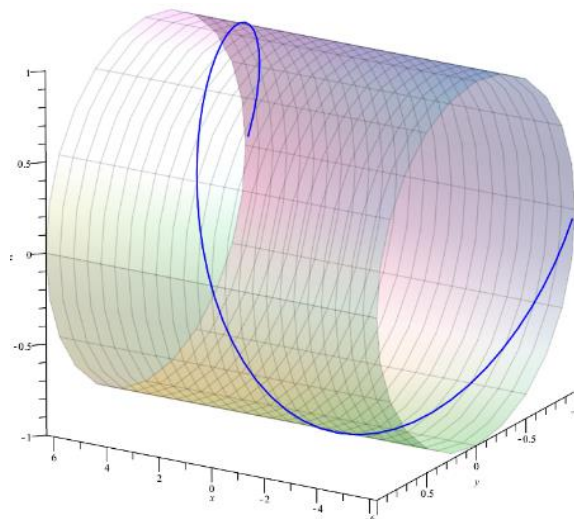
$$d_{Y_r} = (-\cos \omega) T + (\sin \omega \sin 2\phi) N + (\sin \omega \cos 2\phi) B.$$

**Example 3.3.8.** Let  $M$  be the cylindrical surface given by the parametrization  $X(u, v) = (u, \cos v, \sin v)$ . Choose the parameter functions  $u = \frac{2}{\sqrt{5}}s$ ,  $v = \frac{s}{\sqrt{5}}$  and consider the curve on  $M$ ,  $\gamma(s) = \left(\frac{2s}{\sqrt{5}}, \cos \frac{s}{\sqrt{5}}, \sin \frac{s}{\sqrt{5}}\right)$ . The Darboux frame fields and the associated curvature functions of  $\alpha$  are computed as

$$T = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \sin \frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}} \cos \frac{s}{\sqrt{5}}\right), \quad V = \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \sin \frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}} \cos \frac{s}{\sqrt{5}}\right),$$

$$U_\gamma = \left(0, -\cos \frac{s}{\sqrt{5}}, -\sin \frac{s}{\sqrt{5}}\right), \quad k_n = \frac{1}{5}, \quad k_g = 0, \quad \tau_g = \frac{2}{5}$$

By Theorem 3.3.2, we have  $\sigma_{Y_r}(s) = 0$  (constant), hence  $\gamma$  is a  $Y_r$ -Darboux slant helix on  $M$ . Then, by Corollary 3.3.4,  $\gamma$  is also a  $D_r$ -Darboux slant helix. The graph of  $\gamma$  is shown in Fig. 3.



**Figure 3.** On the surface  $X(u, v) = (u, \cos v, \sin v)$ , consider the curve

$$\gamma(s) = \left(\frac{2s}{\sqrt{5}}, \cos \frac{s}{\sqrt{5}}, \sin \frac{s}{\sqrt{5}}\right).$$

## 4. CONCLUSIONS

In this study, we have systematically introduced and analyzed a novel class of surface curves, namely the  $Y_i$ -Darboux slant helices  $i \in \{o, n, r\}$  lying on surfaces in the three-dimensional Euclidean space  $E^3$ . Building upon the classical Darboux frame and the newly defined osculating, normal, and rectifying Darboux frames, we derived the necessary and sufficient conditions for a curve to be classified as a  $Y_i$ -Darboux slant helix. These conditions were expressed in terms of the curvature functions associated with the respective frames.

Several fundamental properties and geometric characteristics of these curves were established, including their relationships with lines of curvature, asymptotic curves, geodesics, helices, and slant helices. The axes of the  $Y_i$ -Darboux slant helices were explicitly formulated with respect to both the corresponding Darboux frames and the classical Frenet frames. Unlike classical helices and slant helices, which are characterized by the constant-angle conditions of the tangent and principal normal vectors, respectively, the  $Y_i$ -Darboux slant helices are characterized by the corresponding  $Y_i$ -directions of the associated Darboux frames. In this respect, the  $Y_i$ -Darboux slant helices provide a complementary class to the previously studied  $D_i$ -Darboux slant helices, which are defined by the  $D_i$ -directions. Concrete examples on conical and cylindrical surfaces illustrated the applicability of the theoretical results, and graphical representations were provided to visualize the behavior of the curves.

Future studies may extend the present framework to Lorentzian and other non-Euclidean spaces and investigate possible applications of  $Y_i$ -Darboux slant helices in geometric modeling.

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