ORIGINAL PAPER

SYMMETRIC TRAPEZOID-TYPE INEQUALITIES FOR (α, m)-CONVEX FUNCTIONS WITH APPLICATIONS

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Manuscript received: 28.08.2025; Accepted paper: 10.03.2026;
Published online: 30.06.2026.

Abstract. *In the current study, we develop a modified version of the right symmetric quantum trapezoid-type inequalities using (α, m)-convex differentiable functions with m ∈ [0, 1]. Recent results have been obtained using fundamental inequalities such as the Power mean and Hölder's. The obtained results generalize both the classical trapezoid-type inequalities and the q- trapezoid-type inequalities for convex functions. Additionally, we provide examples and numerical results that support our main outcomes. It is assumed that these concepts and methodologies will stimulate new research.*

Keywords: *Quantum calculus; symmetric quantum calculus; convex functions; (α, m)-convex functions; trapezoid- type inequalities.*

Mathematics Subject Classification: *05A30; 26A51; 26D10; 26D15.*

1. INTRODUCTION

Quantum Calculus (q-calculus) is the study of calculus without limitations. The q-analogues of mathematical objects can be retrieved using $q \to 1^-$. Euler (1707-1783) established the q-parameter in Newton's infinite series and developed q-calculus. In 1910, Jackson [1] expanded on Euler's concept to develop the q-integral and q-derivative of continuous functions on the interval (0, ∞), often known as calculus without bounds. Al-Salam [2] investigated q-fractional and q-Riemann-Liouville fractional integral inequalities in 1966. Kac and Cheung's book [3] summarizes the core ideas of q-calculus, with further references [4-6]. Tariboon and Ntouyas introduced the q-integral and q-derivative for continuous functions on finite intervals in [7]. Recently, some authors have investigated the existence theory for q-boundary value problems [8-10]. Some new results concerning integral inequalities have been obtained [11,12].

In convexity theory, Hermite-Hadamard is a well-known inequality developed by Hermite and Hadamard (see [13,14]). Convexity is a basic and straightforward notion used to address various mathematical issues. The study of convex functions has expanded to include other types such as quasi-convex functions [15-17], log convex functions [18], coordinated convex functions [19,20], harmonically convex functions [21], GA-convex functions [22, 23], and (α, m)-convex functions [24]. Inequalities are a natural consequence of convexity. A function $\phi: T \to R$ is said to be convex if

$$\phi\,(\sigma t_1 + (1-\sigma)t_2) \leq \sigma\phi\,(t_1) + (1-\sigma)\phi\,(t_2),$$

where $t_1, t_2 \epsilon$ T and $\sigma \in [0,1]$.

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https://doi.org/10.46939/J.Sci.Arts-26.2-a15

Definition 1 ([25]). *A mapping ϕ: $[0, x_2) \rightarrow R$, is m-convex, if the inequality*

$$\phi(\sigma t_1 + m(1-\sigma)t_2) \leq \sigma\phi(t_1) + m(1-\sigma)\phi(t_2),$$

holds $\forall\, t_1, t_2 \in [0, x_2), \sigma \in [0,1]$ and $m \in [0,1]$.

Definition 2 ([24]). *A mapping $\phi : [0, x_2) \rightarrow R$, is (α,m)-convex, if the inequality*

$$\phi(\sigma t_1 + m(1-\sigma)t_2) \leq \sigma^\alpha\phi(t_1) + m(1-\sigma^\alpha)\phi(t_2),$$

holds $\forall\, t_1, t_2 \in [0, x_2), \sigma \in [0,1]$ and $m \in [0,1]$.

Furthermore, the function ϕ is regarded as convex if and only if it fulfills the Hermite-Hadamard inequality, given as:

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \phi(t)\, dt \leq \frac{\phi(x_1) + \phi(x_2)}{2},$$

where $\phi: T \rightarrow R$ is a convex function and $x_1, x_2 \in T$ with $x_1 < x_2$. Convex functions are used in various mathematical domains, such as optimization [26], time scale [27,28], quantum and post-quantum calculus [29]. Alp et al. employed the left-quantum integrals in [29] to establish the following quantum Hermite-Hadamard type for convex functions:

$$\phi\left(\frac{qx_1 + x_2}{[2]_q}\right) \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \phi(t)\ {}_{x_1}dt \leq \frac{q\phi(x_1) + \phi(x_2)}{[2]_q}. \tag{1}$$

In a recent work, Bermudo et al. [30] proved the Hermite–Hadamard type inequality for convex functions using right-quantum integrals:

$$\phi\left(\frac{x_1 + q\,x_2}{[2]_q}\right) \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \phi(t)\ {}^{x_2}dt \leq \frac{\phi(x_1) + q\phi(x_2)}{[2]_q}. \tag{2}$$

The literature [31-38] includes estimates of inequalities (1) and (2) on the left and right. Noor et al. [39] presented a generalized version of (1). The authors of [40-43] employed q-calculus to demonstrate Simpson's and Newton's type inequalities using convexity and coordinated convexity. For an examination of Ostrowski's disparities, see [44-46]. For information on calculus theory and applications, see [47-49].

In light of ongoing research, we employ symmetric quantum calculus to create novel trapezoid-type inequalities for $(\alpha, m)-$convex functions. These results can create both quantum trapezoid-type inequalities for convex functions [29-38] and classical trapezoid-type inequalities for convex functions [50], without requiring separate proofs.

1. Preliminaries and Definitions

We provide definitions and properties for q-integrals and q̃-integrals. Assume $0 < q < 1$ a constant throughout this paper.

The q-analogue of any number n is given as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots\dots + q^{n-1}.$$

Definition 3 ([51]). *Let $\phi: [x_1, x_2] \to R$ be a continuous function. Then ${}_{x_1}D_q\phi$ at $t \in [x_1, x_2]$ is defined by*

$$ {}_{x_1}D_q\phi\,(t) = \begin{cases} \dfrac{\phi(t) - \phi(qt + (1-q)x_1)}{(1-q)(t - x_1)}, \text{if } t \neq x_1; \\ \lim\limits_{t \to x_1} {}_{x_1}D_q\phi\,(t)\,, \text{if } t = x_1. \end{cases} \tag{3} $$

The function ϕ is said to be q-differentiable function if ${}_{x_1}D_q\phi\,(t)$ exists for all $t \in [x_1, x_2]$. Given that $x_1 = 0$ and ${}_0D_q\phi\,(t) = D_q\phi(t)$, then equation (3) simplifies to

$$ D_q\phi(t) = \begin{cases} \dfrac{\phi(t) - \phi(qt)}{(1-q)t} \text{ if } t \neq 0, \\ \lim\limits_{t \to x_1} D_q\phi(t) \text{ if } t = 0, \end{cases} $$

which is the q-Jackson derivative [51,1].

Definition 4 ([51]). *Let $\phi: [x_1, x_2] \to R$ be a continuous function. Then ${}_{x_1}q$-integrals of ϕ at $t \in [x_1, x_2]$ is defined by*

$$ \int_{x_1}^{b} \phi(t)\ {}_{x_1}d_qt = (1-q)(b - x_1) \sum_{n=0}^{\infty} q^n\, \phi(q^nb + (1-q^n)x_1). \tag{4} $$

The function ϕ is said to be a q-integrable function on $[x_1, x_2]$. If $\int_{x_1}^{b} \phi(t)\ {}_{x_1}d_qt$ exists for all $b \in [x_1, x_2]$. When $x_1 = 0$, then equation (4) simplifies to

$$ \int_0^b \phi(t)\ {}_0d_qt = \int_0^b \phi(t)\, d_qt = (1-q)b \sum_{n=0}^{\infty} q^n\, \phi(q^nb), $$

which is the q-Jackson integral [51,1].

Bermudo et al. [30] defined the right q-derivative and the right q-integral, respectively.

Definition 5. *Let $\phi: [x_1, x_2] \to R$ be a continuous function. Then ${}^{x_2}D_q\phi$ at $t \in [x_1, x_2]$ is defined by*

$$ {}^{x_2}D_q\phi(t) = \begin{cases} \frac{\phi(qt + (1-q)x_2) - \phi(t)}{(1-q)(x_2 - t)}, \text{if } t \neq x_2; \\ \lim_{t \to x_2} {}^{x_2}D_q\phi(t), \text{if } t = x_2. \end{cases} $$

Definition 6. *Let $\phi: [x_1, x_2] \to R$ be a continuous function. Then ${}^{x_2}q$-integrals of ϕ at $t \in [x_1, x_2]$ is defined by*

$$ \int_b^{x_2} \phi(t)\ {}^{x_2}d_qt = (1-q)(x_2 - b) \sum_{n=0}^{\infty} q^n\, \phi(q^nb + (1-q^n)x_2). $$

In this paper, we will derive trapezoid-type inequalities within the framework of symmetric quantum calculus. The concept of symmetric quantum calculus was first proposed by Da Cruz et al. [52]. Quantum mechanics heavily relies on the q-symmetric calculus. It plays a crucial role in developing fundamental hypergeometric functions, the generalized linear Schrödinger equation, and the quantum dynamical equation in quantum mechanics [51].

The structure of this paper is arranged as follows: Section 2 provides a brief explanation of the principles of q-calculus and $\tilde{q}-$calculus. In Section 3, we develop a key finding that is core to the development of the paper's main findings with the help of $\tilde{q}$-integrals. In Section 4, the trapezoid-type inequalities for $\tilde{q}-$differentiable functions are derived using symmetric q-integrals. Section 5 includes illustrative examples. In Section 6, a summary of the results is addressed.

For a real parameter $0 < q < 1$, in symmetric quantum, we have $[n]_q$ as:

$$[n]_q = \frac{1-q^{2n}}{1-q^2}, n \in \mathbb{R}.$$

Definition 7 ([53]). *Let $\phi: [x_1, x_2] \to R$ be a continuous function. Then ${}_{x_1}q$-symmetric derivative $t \in [x_1, x_2]$ is defined by*

$${}_{x_1}\tilde{D}_q \phi(t) = \frac{\tilde{d}_q \phi(t)}{\tilde{d}_q t} = \frac{\phi(qt+(1-q)x_1) - \phi(q^{-1}t + (1-q^{-1})x_1)}{(q-q^{-1})(t-x_1)}, t \neq x_1,$$

It follows that

$$\tilde{D}_q \phi(t) = \frac{\tilde{d}_q \phi(t)}{\tilde{d}_q t} = \frac{\phi(qt) - \phi(q^{-1}t)}{(q-q^{-1})t}, t \neq 0.$$

Definition 8 ([53]). *Let $\phi : [x_1, x_2] \to \mathbb{R}$ be a continuous function. Then, ${}_{x_1}q$-symmetric integral is defined by*

$$\int_{x_1}^{y} \phi(t)\ {}_{x_1}\tilde{d}_q t = (q^{-1} - q)(y - x_1) \sum_{n=0}^{\infty} q^{2n+1} \phi(q^{2n+1}y + (1 - q^{2n+1})x_1).$$

Here, $y \in [x_1, x_2]$, or

$$\int_{x_1}^{y} \phi(t)\ {}_{x_1}\tilde{d}_q t = (1 - q^2)(y - x_1) \sum_{n=0}^{\infty} q^{2n} \phi(q^{2n+1}y + (1 - q^{2n+1})x_1). \quad (5)$$

Let $\phi(t) = 1$, Equation (5) reduces to

$$\int_{x_1}^{y} \phi(t)\ {}_{x_1}\tilde{d}_q t = y - x_1$$

and if $x_1 = 0$ in (5), then

$$\int_0^y \phi(t)\ {}_0\tilde{d}_q t = \int_0^y \phi(t)\ \tilde{d}_q t$$

or

$$\int_0^y \phi(t)\ {}_0\tilde{d}_q t = \int_0^y \phi(t)\ \tilde{d}_q t = (1 - q^2)y \sum_{n=0}^{\infty} q^{2n} \phi(q^{2n+1}y).$$

If $u \in (x_1, y)$, then the ${}_{x_1}\tilde{q}$-definite integral on $[u, y]$ is given as

$$\int_{u}^{y}\phi(t)\ {}_{x_1}\tilde{d}_q t = \int_{x_1}^{y}\phi(t)\ {}_{x_1}\tilde{d}_q t - \int_{x_1}^{u}\phi(t)\ {}_{x_1}\tilde{d}_q t.$$

Definition 9. *Let $\phi : [x_1, x_2] \to \mathbb{R}$ be a continuous function. Then, ${}^{x_2}q$-symmetric derivative at t ∈ $[x_1, x_2]$ is defined by*

$$^{x_2}\tilde{D}_q\phi(t) = \frac{\tilde{d}_q\phi(t)}{\tilde{d}_q t} = \frac{\phi(qt + (1-q)x_2) - \phi(q^{-1}t + (1-q^{-1})x_2)}{(q^{-1} - q)(x_2 - t)}, t \neq x_2. \tag{6}$$

Definition 10. *Let $\phi : [x_1, x_2] \to \mathbb{R}$ be a continuous function. Then, ${}^{x_2}q$-definite integral on $[x_1, x_2]$ is defined by*

$$\int_{y}^{x_2}\phi(t)\ {}^{x_2}\tilde{d}_q t = (q^{-1} - q)(x_2 - y)\sum_{n=0}^{\infty} q^{2n+1}\,\phi(q^{2n+1}y + (1 - q^{2n+1})x_2)$$

for $y \in [x_1, x_2]$, or

$$\int_{y}^{x_2}\phi(t)\ {}^{x_2}\tilde{d}_q t = (1 - q^2)(x_2 - y)\sum_{n=0}^{\infty} q^{2n}\,\phi(q^{2n+1}y + (1 - q^{2n+1})x_2). \tag{7}$$

Formula for integration by parts: Let ϕ, ψ : [p, u] → $\mathbb{R}$ be q-differentiable and continuous mappings with p, u ∈ $\mathbb{R}$; then,

$$\int_{p}^{u}\phi(qs)\big(\tilde{D}_q\,\psi\big)(s)\,\tilde{d}_q s = [\phi(s)\psi(s)]\big|_{p}^{u} - \int_{p}^{u}\psi(q^{-1}s)\big(\tilde{D}_q\,\phi\big)(s)\,\tilde{d}_q s. \text{ [5]}$$

3. FUNDAMENTAL IDENTITY

In this section, we demonstrate a symmetric quantum integral identity using the integration by parts method to develop the principal findings.

Lemma 1. For $m \in (0, 1]$ with $0 < q < 1$, let us assume an arbitrary function $\phi : [0, \infty) \to \mathbb{R}$ such that ${}^{x_2}\tilde{D}_q\phi$ is continuous and integrable on $[x_1, x_2]$. Then,

$$\begin{aligned}&\frac{q\phi(x_1) + (1 - q + q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\ {}^{mx_2}\tilde{d}_q t \\ &= \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\int_0^1 \big(1 - [2]_q t\big)\,{}^{x_2}\tilde{D}_q\phi(qtx_1 + m(1 - qt)x_2)\tilde{d}_q t\right].\end{aligned} \tag{8}$$

Proof: Consider the R. H. S of Equation (8),

$$\frac{q^2(mx_2 - x_1)}{[2]_q}\left[\int_0^1 \big(1 - [2]_q t\big)\,{}^{x_2}\tilde{D}_q\phi(qtx_1 + m(1 - qt)x_2)\tilde{d}_q t\right].$$

Using the q-symmetric integration by parts we get,

$$\left[\int_0^1 \left(1-[2]_q t\right) {}^{x_2}\widetilde{D}_q \phi(qtx_1+m(1-qt)x_2)\tilde{d}_q t\right]$$
$$= -\left.\frac{(1-(1+q^2)t)\phi(qtx_1+m(1-qt)x_2)}{(mx_2-x_1)}\right|_0^1 - \frac{1+q^2}{(mx_2-x_1)}\int_0^1 \phi(q^2tx_1+m(1-q^2t)x_2)\tilde{d}_q t$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{1+q^2}{(mx_2-x_1)}\int_0^1 \phi(q^2tx_1+m(1-q^2t)x_2)\tilde{d}_q t$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{(1+q^2)(1-q^2)}{(mx_2-x_1)}\sum_{n=0}^{\infty} q^{2n}\,\phi(q^{2n+3}x_1+(1-q^{2n+3})x_2)$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{(1+q^2)(1-q^2)}{q^2(mx_2-x_1)}\sum_{n=0}^{\infty} q^{2n+2}\;\phi(q^{2n+2+1}x_1+ (1-q^{2n+2+1})x_2)$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{(1+q^2)(1-q^2)}{q^2(mx_2-x_1)}\left(\sum_{n=0}^{\infty} q^{2k}\,\phi\left(q^{2k+1}x_1+\left(1-q^{2k+1}\right)x_2\right)\right) \tag{9}$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{(1+q^2)(1-q^2)}{q^2(mx_2-x_1)}\left(\sum_{n=0}^{\infty} q^{2n}\,\phi(q^{2n+1}x_1+(1-q^{2n+1})x_2) - \phi(qx_1+m(1-q)x_2)\right)$$
$$= \frac{q^2\,\phi(qx_1+m(1-q)x_2)+\phi(mx_2)}{(mx_2-x_1)} - \frac{(1+q^2)}{q^2(mx_2-x_1)^2}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t + \frac{(1+q^2)(1-q^2)}{q^2(mx_2-x_1)}\phi(qx_1+m(1-q)x_2)$$
$$= \frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{q^2(mx_2-x_1)} - \frac{(1+q^2)}{q^2(mx_2-x_1)^2}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t.$$

Multiplying $\frac{q^2(mx_2-x_1)}{[2]_q}$ by both sides of (9),

$$\frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t$$
$$= \frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t$$

Hence, the proof is achieved.

Remark 1. In Lemma 1, if we put α = m = 1 and $q \to 1^-$, we get Lemma 2.1 in [50].

4. TRAPEZOID- TYPE INEQUALITIES FOR (α, m)-CONVEX FUNCTIONS

In this section, we prove Trapezoid-type inequalities for differentiable (α, m)-convex functions.

Theorem 1. *If $\phi:[x_1,x_2]\to\mathbb{R}$ is a ${}^{x_2}q$- symmetric differentiable function on (x_1,x_2) such that ${}^{x_2}\widetilde{D}_q\phi$ is continuous and integrable on $[x_1,x_2]$. If $|{}^{x_2}\widetilde{D}_q\phi|$ is (α, m)-convex on $[x_1,x_2]$, then we have the inequality*

$$\left|\frac{Q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right|$$
$$\leq \frac{q^2(mx_2-x_1)}{[2]_q}\left[\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|\{R_1(q)-R_2(q)\} + m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|\{S_1(q)-S_2(q)\}\right], \tag{10}$$

where

$$R_1(q) = \int_0^{\frac{1}{[2]_q}}\left(qt^{\alpha} - [2]_q qt^{\alpha+1}\right)\tilde{d}_q t = \frac{q^{\alpha+1}[\alpha+2]_q - q^{\alpha+2}[\alpha+1]_q}{[2]_q^{\alpha+1}[\alpha+1]_q\,[\alpha+2]_q},$$

$$R_2(q) = \int_{\frac{1}{[2]_q}}^1\left(qt^{\alpha} - [2]_q qt^{\alpha+1}\right)\tilde{d}_q t = \frac{q^{\alpha+1}}{[\alpha+1]_q} - \frac{q^{\alpha+2}[2]_q}{[\alpha+2]_q} - \frac{q^{\alpha+1}[\alpha+2]_q - q^{\alpha+2}[\alpha+1]_q}{[2]_q^{\alpha+1}[\alpha+1]_q\,[\alpha+2]_q},$$

$$S_1(q) = \int_0^{\frac{1}{[2]_q}}(1 - [2]_q t)(1 - qt^{\alpha})\,\tilde{d}_q t = \frac{1}{[2]_q} - \frac{q}{[2]_q^2} - \frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q} + \frac{q^{\alpha+2}}{[2]_q^{\alpha+1}[\alpha+2]_q},$$

$$S_2(q) = \int_{\frac{1}{[2]_q}}^1\left(1 - [2]_q t\right)(1 - qt^{\alpha})\tilde{d}_q t = 1 - q - \frac{q^{\alpha+1}}{[\alpha+1]_q} + \frac{q^{\alpha+2}[2]_q}{[\alpha+2]_q} - \frac{1}{[2]_q} + \frac{q}{[2]_q^2} + \frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q} - \frac{q^{\alpha+2}}{[2]_q^{\alpha+1}[\alpha+2]_q}.$$

Proof: *Taking the modulus of (8) and applying the definition of (α, m)-convexity, we have*

$$\begin{aligned}
&\left|\frac{q\phi(x_1) + (1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right| \\
&= \frac{q^2(mx_2-x_1)}{[2]_q}\left|\int_0^1 (1-(1+q^2)t)\,{}^{x_2}\tilde{D}_q\phi(qtx_1 + m(1-qt)x_2)\tilde{d}_q t\right| \\
&\le \frac{q^2(mx_2-x_1)}{[2]_q}\int_0^1\left|\left(1 - [2]_q t\right)\right|\,\left|{}^{x_2}\tilde{D}_q\phi(qtx_1 + m(1-qt)x_2)\right|\tilde{d}_q t \\
&\le \frac{q^2(mx_2-x_1)}{[2]_q}\int_0^1\left|\left(1 - [2]_q t\right)\right|\,\left[qt^{\alpha}\left|{}^{x_2}\tilde{D}_q\phi(x_1)\right| + m(1-qt^{\alpha})\left|{}^{x_2}\tilde{D}_q\phi(x_2)\right|\,\right]\tilde{d}_q t \\
&\le \frac{q^2(mx_2-x_1)}{[2]_q}\left[\int_0^1\left|\left(1 - [2]_q t\right)\right|\,qt^{\alpha}\left|{}^{x_2}\tilde{D}_q\phi(x_1)\right|\tilde{d}_q t\right. \\
&\qquad\left. + m\int_0^1\left|\left(1 - [2]_q t\right)\right|\,(1-qt^{\alpha})\left|{}^{x_2}\tilde{D}_q\phi(x_2)\right|\tilde{d}_q t\right] \qquad (11)\\
&\le \frac{q^2(mx_2-x_1)}{[2]_q}\left[\left|{}^{x_2}\tilde{D}_q\phi(x_1)\right|\int_0^1\left|\left(1 - [2]_q t\right)\right|\,qt^{\alpha}\tilde{d}_q t\right. \\
&\qquad\left. + m\left|{}^{x_2}\tilde{D}_q\phi(x_2)\right|\int_0^1\left|\left(1 - [2]_q t\right)\right|\,(1-qt^{\alpha})\tilde{d}_q t\right] \\
&\le \frac{q^2(mx_2-x_1)}{[2]_q}\left[\{R_1(q) - R_2(q)\}\left|{}^{x_2}\tilde{D}_q\phi(x_1)\right|\right. \\
&\qquad\left. + m\{S_1(q) - S_2(q)\}\left|{}^{x_2}\tilde{D}_q\phi(x_2)\right|\,\right].
\end{aligned}$$

Now consider,

$$\int_0^1\left|\left(1 - [2]_q t\right)\right|\,qt^{\alpha}\tilde{d}_q t = \left[\int_0^{\frac{1}{[2]_q}}\left(1 - [2]_q t\right)qt^{\alpha}\tilde{d}_q t + \int_{\frac{1}{[2]_q}}^1\left(1 - [2]_q t\right)qt^{\alpha}\,\tilde{d}_q t\right] \qquad (12)$$

$$\begin{aligned}
R_1(q) &= \int_0^{\frac{1}{[2]_q}}\left(qt^{\alpha} - [2]_q qt^{\alpha+1}\right)\tilde{d}_q t \\
&= \frac{q(1-q^2)}{[2]_q}\sum_{n=0}^{\infty} q^{2n}\left(\frac{q^{2n+1}}{[2]_q}\right)^{\alpha} - \frac{q[2]_q(1-q^2)}{[2]_q}\sum_{n=0}^{\infty} q^{2n}\left(\frac{q^{2n+1}}{[2]_q}\right)^{\alpha+1} \\
&= \frac{q(1-q^2)}{[2]_q^{\alpha+1}}\sum_{n=0}^{\infty} q^{2n}\left(q^{2n+1}\right)^{\alpha} - \frac{q(1-q^2)}{[2]_q^{\alpha+1}}\sum_{n=0}^{\infty} q^{2n}\left(q^{2n+1}\right)^{\alpha+1} \\
&= \frac{q(1-q^2)}{[2]_q^{\alpha+1}}\left(\frac{q^{\alpha}}{1-q^{2(\alpha+1)}}\right) - \frac{q(1-q^2)}{[2]_q^{\alpha+1}}\left(\frac{q^{\alpha+1}}{1-q^{2(\alpha+2)}}\right) \\
&= \frac{q^{\alpha+1}[\alpha+2]_q - q^{\alpha+2}[\alpha+1]_q}{[2]_q^{\alpha+1}[\alpha+1]_q\,[\alpha+2]_q},
\end{aligned}$$

$$R_2(q) = \int_{\frac{1}{[2]_q}}^{1}\left(qt^{\alpha} - [2]_q qt^{\alpha+1}\right)\tilde{d}_q t = \int_0^1\left(qt^{\alpha} - [2]_q qt^{\alpha+1}\right)\tilde{d}_q t -$$

$$\int_0^{\frac{1}{[2]_q}}(qt^{\alpha} - [2]_q qt^{\alpha+1})\tilde{d}_q t$$

$$= q\,(1 - q^2)\sum_{n=0}^{\infty} q^{2n}\,(q^{2n+1})^{\alpha} - \frac{q^{\alpha+1}\,[\alpha+2]_q - q^{\alpha+2}\,[\alpha+1]_q}{[2]_q^{\alpha+1}[\alpha+1]_q\,[\alpha+2]_q}$$

$$= \frac{q^{\alpha+1}}{[\alpha+1]_q} - \frac{q^{\alpha+2}[2]_q}{[\alpha+2]_q} - \frac{q^{\alpha+1}\,[\alpha+2]_q - q^{\alpha+2}\,[\alpha+1]_q}{[2]_q^{\alpha+1}[\alpha+1]_q\,[\alpha+2]_q},$$

$$\int_0^1\left|\left(1 - [2]_q t\right)\right|(1 - qt^{\alpha})\tilde{d}_q t = \left[\int_0^{\frac{1}{[2]_q}}\left(1 - [2]_q t\right)(1 - qt^{\alpha})\,\tilde{d}_q t + \int_{\frac{1}{[2]_q}}^{1}\left(1 - [2]_q t\right)(1 - qt^{\alpha})\tilde{d}_q t\right] \quad (13)$$

$$S_1(q) = \int_0^{\frac{1}{[2]_q}}(1 - [2]_q t)(1 - qt^{\alpha})\tilde{d}_q t = \int_0^{\frac{1}{[2]_q}} 1\,\tilde{d}_q t - [2]_q\int_0^{\frac{1}{[2]_q}} t\,\tilde{d}_q t - q\int_0^{\frac{1}{[2]_q}} t^{\alpha}\,\tilde{d}_q t + q[2]_q\int_0^{\frac{1}{[2]_q}} t^{\alpha+1}\,\tilde{d}_q t$$

$$= \frac{1}{[2]_q} - \frac{q}{[2]_q^2} - \frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q} + \frac{q^{\alpha+2}}{[2]_q^{\alpha+1}[\alpha+2]_q},$$

$$S_2(q) = \int_{\frac{1}{[2]_q}}^{1}\left(1 - [2]_q t\right)(1 - qt^{\alpha})\tilde{d}_q t = \int_0^1\left(1 - [2]_q t\right)(1 - qt^{\alpha})\tilde{d}_q t$$

$$-\int_0^{\frac{1}{[2]_q}}\left(1 - [2]_q t\right)(1 - qt^{\alpha})\,\tilde{d}_q t$$

$$= \int_0^1 1\,\tilde{d}_q t - [2]_q\int_0^1 t\,\tilde{d}_q t - q\int_0^1 t^{\alpha}\,\tilde{d}_q t + q[2]_q\int_0^1 t^{\alpha+1}\,\tilde{d}_q t - \int_0^{\frac{1}{[2]_q}} 1\,\tilde{d}_q t + [2]_q\int_0^{\frac{1}{[2]_q}} t\,\tilde{d}_q t + q\int_0^{\frac{1}{[2]_q}} t^{\alpha}\,\tilde{d}_q t - q[2]_q\int_0^{\frac{1}{[2]_q}} t^{\alpha+1}\,\tilde{d}_q t$$

$$= 1 - q - \frac{q^{\alpha+1}}{[\alpha+1]_q} + \frac{q^{\alpha+2}[2]_q}{[\alpha+2]_q} - \frac{1}{[2]_q} + \frac{q}{[2]_q^2} + \frac{q^{\alpha+1}}{[2]_q^{\alpha+1}[\alpha+1]_q} - \frac{q^{\alpha+2}}{[2]_q^{\alpha+1}[\alpha+2]_q},$$

which completes the proof.

Remark 2. In Theorem 1 if we put α = m = 1 and $q \to 1^-$, we get Theorem 2.2 in [50].

Theorem 2. *If $\phi : [x_1, x_2] \to \mathbb{R}$ is a ${}^{x_2}q$- symmetric differentiable function on (x_1, x_2) such that ${}^{x_2}\widetilde{D}_q\phi$ is continuous and integrable on $[x_1, x_2]$. If $\left|{}^{x_2}\widetilde{D}_q\phi\right|^r$ is (α, m)-convex on $[x_1, x_2]$, $r \geq 1$ then we have the inequality*

$$\left|\frac{q\phi(x_1) + (1 - q + q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right|$$
$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q^{3-\frac{2}{r}}}\left[\zeta(q)\left(\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\{R_1(q) - R_2(q)\} + m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\{S_1(q) - S_2\right.\right. \quad (14)$$

where

$$\zeta(q) = (1 - q + 2q^3 - q^4 + q^5)^{1-\frac{1}{r}}$$

Proof: Taking the modulus on both sides of Lemma 1, we obtain

$$\left|\frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q}-\frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right|$$
$$=\frac{q^2(mx_2-x_1)}{[2]_q}\left|\int_0^1\left(1-[2]_q t\right)^{x_2}\widetilde{D}_q\phi(qtx_1+m(1-qt)x_2)\tilde{d}_q t\right|$$
$$\leq\frac{q^2(mx_2-x_1)}{[2]_q}\left[\int_0^1\left|\left(1-[2]_q t\right)\right|\left|{}^{x_2}\widetilde{D}_q\phi(qtx_1+m(1-qt)x_2)\right|\tilde{d}_q t\right].$$

Using the power mean inequality,

$$\left|\frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q}-\frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right|$$
$$\leq\frac{q^2(mx_2-x_1)}{[2]_q}\left[\left(\int_0^1\left|\left(1-[2]_q t\right)\right|\tilde{d}_q t\right)^{1-\frac{1}{r}}\right.$$
$$\left.\times\left(\int_0^1\left|\left(1-[2]_q t\right)\right|\left|{}^{x_2}\widetilde{D}_q\phi(qtx_1+m(1-qt)x_2)\right|^r\tilde{d}_q t\right)^{\frac{1}{r}}\right].$$

By using the definition of $(\alpha, m)-$ convexity, we have

$$\begin{aligned}&\left|\frac{q\phi(x_1)+(1-q+q^2)\phi(mx_2)}{[2]_q}-\frac{1}{mx_2-x_1}\int_{x_1}^{mx_2}\phi(t)\,{}^{mx_2}\tilde{d}_q t\right|\\&\leq\frac{q^2(mx_2-x_1)}{[2]_q}\left[\left(\int_0^1\left|\left(1-[2]_q t\right)\right|\tilde{d}_q t\right)^{1-\frac{1}{r}}\right.\\&\times\left(\int_0^1\left|\left(1-[2]_q t\right)\right|qt^\alpha\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\tilde{d}_q t\right.\\&\left.\left.+\int_0^1\left|\left(1-[2]_q t\right)\right|m(1-qt^\alpha)\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\tilde{d}_q t\right)^{\frac{1}{r}}\right]\end{aligned}\tag{15}$$

$$\leq\frac{q^2(mx_2-x_1)}{[2]_q}\left[\left(\frac{1-q+2q^3-q^4+q^5}{[2]_q^2}\right)^{1-\frac{1}{r}}\left\{\int_0^1\left|\left(1-[2]_q t\right)\right|\left(qt^\alpha\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\right.\right.\right.$$
$$\left.\left.\left.+m(1-qt^\alpha)\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\right)\tilde{d}_q t\right\}^{\frac{1}{r}}\right]$$

$$\leq\frac{q^2(mx_2-x_1)}{{[2]_q}^{3-\frac{2}{r}}}\left[\zeta(q)\left\{\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\left(\int_0^{\frac{1}{[2]_q}}(qt^\alpha-[2]_q qt^{\alpha+1})\tilde{d}_q t-\int_{\frac{1}{[2]_q}}^1(qt^\alpha-[2]_q qt^{\alpha+1})\tilde{d}_q t\right)\right.\right.$$
$$+m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\left(\int_0^{\frac{1}{[2]_q}}\left(1-[2]_q t\right)(1-qt^\alpha)\tilde{d}_q t\right.$$
$$\left.\left.\left.-\int_{\frac{1}{[2]_q}}^1\left(1-[2]_q t\right)(1-qt^\alpha)\tilde{d}_q t\right)\right\}^{\frac{1}{r}}\right]$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q{}^{3-\frac{2}{r}}}\left[\zeta(q)\{\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\left(R_1(q) - R_2(q)\right) + m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\left(S_1(q)\ S_2(q)\right)\}^{\frac{1}{r}}\right].$$

Using (12-13) from Theorem 1 in (15), we get the required result.

Theorem 3. *If $\phi : [x_1, x_2] \to \mathbb{R}$ is a ${}^{x_2}q$- symmetric differentiable function on (x_1, x_2) such that ${}^{x_2}\widetilde{D}_q\phi$ is continuous and integrable on $[x_1, x_2]$. If $\left|{}^{x_2}\widetilde{D}_q\phi\right|^s$ is (α , m)-convex on $[x_1, x_2]$, $s > 1$ then we have the inequality*

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\upsilon(q)\{\ \xi_1(q)\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^s + m\ \xi_2(q)\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^s\}^{\frac{1}{s}}\right], \tag{16}$$

where

$$\upsilon(q) = \int_0^1 \left(\left|\left(1 - [2]_q t\right)\right|\right)^r \tilde{d}_q t,$$

$$\xi_1(q) = \int_0^1 qt^\alpha\, \tilde{d}_q t = \frac{q^{\alpha+1}}{[\alpha+1]_q},$$

$$\xi_2(q) = \int_0^1 (1 - qt^\alpha)\tilde{d}_q t = 1 - \frac{q^{\alpha+1}}{[\alpha+1]_q}.$$

Proof: Taking the absolute value of Lemma 1 and applying Hölder's inequality, we get

$$\left|\frac{q\phi(x_1) + (1 - q + q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\ {}^{mx_2}\tilde{d}_q t\right|$$

$$= \frac{q^2(mx_2 - x_1)}{[2]_q}\left|\int_0^1 \left(1 - [2]_q t\right)\left({}^{x_2}\widetilde{D}_q\phi(qtx_1 + m(1 - qt)x_2)\right)\tilde{d}_q t\right|$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\int_0^1 \left|\left(1 - [2]_q t\right)\right|\ \left|{}^{x_2}\widetilde{D}_q\phi(qtx_1 + m(1 - qt)x_2)\right|\tilde{d}_q t\right]$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\left(\int_0^1 \left|\left(1 - [2]_q t\right)\right|^r \tilde{d}_q t\right)^{\frac{1}{r}}\left\{\int_0^1 \left|{}^{x_2}\widetilde{D}_q\phi(qtx_1 + m(1 - qt)x_2)\right|^s \tilde{d}_q t\right\}^{\frac{1}{s}}\right].$$

By using the definition of $(\alpha, m) -$ convexity, we have

$$\left|\frac{q\phi(x_1) + (1 - q + q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\ {}^{mx_2}\tilde{d}_q t\right|$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\left(\int_0^1 \left|\left(1 - [2]_q t\right)\right|^r \tilde{d}_q t\right)^{\frac{1}{r}}\left\{\int_0^1 \left(qt^\alpha\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^s \right.\right.\right.$$

$$\left.\left.\left. + m(1 - qt^\alpha)\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^s\right)\tilde{d}_q t\right\}^{\frac{1}{s}}\right] \tag{17}$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\left(\frac{2q^2}{[2]_q[r+1]_q}\right)^{\frac{1}{r}}\left\{\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^s\int_0^1 qt^\alpha\tilde{d}_q t + m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^s\int_0^1 (1 - qt^\alpha)\tilde{d}_q t\right\}^{\frac{1}{s}}\right]$$

$$\leq \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\ \left(\upsilon(q)\right)^{\frac{1}{r}}\{\ \xi_1(q)\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^s + m\ \xi_2(q)\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^s\}^{\frac{1}{s}}\right].$$

Now consider

$$\xi_1(q) = \int_0^1 qt^{\alpha}\,\tilde{d}_q t = q(1-q^2)\sum_{n=0}^{\infty} q^{2n}\,(q^{2n+1})^{\alpha} = \frac{q^{\alpha+1}(1-q^2)}{(1-q^{2(1+\alpha)})} = \frac{q^{\alpha+1}}{[\alpha+1]_q}, \tag{18}$$

$$\xi_2(q) = \int_0^1 (1-qt^{\alpha})\tilde{d}_q t = (1-q)(1-q^2)\sum_{n=0}^{\infty} q^{2n}\,(q^{2n+1})^{\alpha} = \frac{[\alpha+1]_q - q^{\alpha+1}}{[\alpha+1]_q}. \tag{19}$$

Putting values (18-19) in (17) we get the desired result.

Remark 3. In Theorem 3 if we put $\alpha = m = 1$ and $q \to 1^-$, we get Theorem 2.3 in [50].

5. EXAMPLES

Here we present examples to show our main outcomes.

Example 1. We consider a convex function $\phi : [0,1] \to \mathbb{R}$, $\phi(t) = t^2$. From Theorem 1 with $m = \frac{1}{2}$, $q = 1$ and $\alpha = 1$, we have

$$\int_{x_1}^{mx_2} \phi(t)\ {}^{mx_2}\tilde{d}_q t = \int_0^{\frac{1}{2}} t^2\ {}^{\frac{1}{2}}\tilde{d}_q t$$
$$= \frac{1-q^2}{8}\sum_{n=0}^{\infty} q^{2n}(1-q^{2n+1})^2$$
$$= \frac{1}{8}\left[1 - \frac{2q}{[2]_q} + \frac{q^2}{[3]_q}\right].$$

Then L.H.S of the inequality becomes

$$\left|\frac{q\phi(x_1) + (1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2} \phi(t)\ {}^{mx_2}\tilde{d}_q t\right|$$
$$= \left|\frac{1}{8}\left\{\frac{1-q+q^2}{[2]_q} - \left[1 - \frac{2q}{[2]_q} + \frac{q^2}{[2]_q}\right]\right\}\right|$$
$$= |0.125\,\{0.5 - [1 - 1 + 0.33]\}|$$
$$= 0.021.$$

By definition (9), we get

$${}^{x_2}\widetilde{D}_q\phi(t) = {}^{1}\widetilde{D}_q t^2 = \frac{\left(qt+(1-q)\right)^2 - \left(q^{-1}t+(1-q^{-1}\right)^2}{(q^{-1}-q)(1-t)}.$$

Hence,

$$\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right| = \left|{}^{1}\widetilde{D}_q\phi(0)\right| = \left|2 - [2]_q\right|,$$
$$\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right| = \left|{}^{1}\widetilde{D}_q\phi(1)\right| = 2.$$

Therefore, the R.H.S of the inequality becomes

$$\le \frac{q^2(mx_2 - x_1)}{[2]_q}\left[\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|\{R_1(q) - R_2(q)\} + m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|\{S_1(q) - S_2(q)\}\right]$$
$$= \frac{q^2}{4}\left[\left\{2\left(\frac{q^2[3]_q - q^3[2]_q}{[2]_q^3\,[3]_q}\right) - \frac{q^2}{[2]_q} + \frac{q^3[2]_q}{[3]_q}\right\}\left|{}^{1}\widetilde{D}_q\phi(0)\right| + \frac{1}{2}\left\{\frac{2+q^2}{[2]_q} - \frac{2q}{[2]_q^2} - \frac{2q^2}{[2]_q^3} + \frac{2q^3}{[2]_q^2[3]_q} - \right.\right. \tag{5.1}$$

$$\left.\frac{q^3[2]_q}{[3]_q}\right\} \left| {}^{1}\widetilde{D}_q\phi(1)\right| \Big]$$
$$= 0.25\,[0.25]$$
$$= 0.063.$$

By (10), we have inequality as

$$\left|\frac{1}{8}\left\{\frac{1-q+q^2}{[2]_q} - \left[1 - \frac{2q}{[2]_q} + \frac{q^2}{[2]_q}\right]\right\}\right|$$
$$\leq \frac{q^2}{4}\left[\left\{2\left(\frac{q^2[3]_q - q^3[2]_q}{[2]_q^3\,[3]_q}\right) - \frac{q^2}{[2]_q} + \frac{q^3[2]_q}{[3]_q}\right\}\left|2 - [2]_q\right| + \left\{\frac{2+q^2}{[2]_q} - \frac{2q}{[2]_q^2} - \frac{2q^2}{[2]_q^3} + \frac{2q^3}{[2]_q^2[3]_q} - \frac{q^3[2]_q}{[3]_q}\right\}\right]. \tag{5.2}$$

It is clear that $0.021 < 0.063$.

Table 1. Numerical results of R_1, R_2, S_1, S_2 for different values of q.

	q = 0.7	**q = 0.8**	**q = 0.9**	**q = 1**
R_1	0.046	0.045	0.044	0.042
R_2	-0.023	-0.067	-0.129	-0.208
S_1	0.300	0.264	0.233	0.208
S_2	-0.022	-0.042	-0.048	-0.042

Example 2. We consider a convex function $\phi : [0,1] \to \mathbb{R}$, $\phi(t) = t^2$. From Theorem 2 with $m = \frac{1}{2}$, q = 1, r = 2 and $\alpha = 1$, we have

$$\int_{x_1}^{mx_2} \phi(t)\ {}^{mx_2}\tilde{d}_q t = \int_0^{\frac{1}{2}} t^2\ {}^{\frac{1}{2}}\tilde{d}_q t$$
$$= \frac{1-q^2}{8}\sum_{n=0}^{\infty} q^{2n}(1-q^{2n+1})^2$$
$$= \frac{1}{8}\left[1 - \frac{2q}{[2]_q} + \frac{q^2}{[3]_q}\right].$$

Then L.H.S of the inequality becomes

$$\left|\frac{q\phi(x_1) + (1-q+q^2)\phi(mx_2)}{[2]_q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2}\phi(t)\ {}^{mx_2}\tilde{d}_q t\right|$$
$$= \left|\frac{1}{8}\left\{\frac{1-q+q^2}{[2]_q} - \left[1 - \frac{2q}{[2]_q} + \frac{q^2}{[2]_q}\right]\right\}\right|$$
$$= |0.125\,\{0.5 - [1 - 1 + 0.33]\}|$$
$$= 0.021.$$

By definition (9), we get

$$ {}^{x_2}\widetilde{D}_q\phi(t) = {}^{1}\widetilde{D}_q t^2 = \frac{\left(qt + (1-q)\right)^2 - \left(q^{-1}t + (1-q^{-1})\right)^2}{(q^{-1}-q)(1-t)}.$$

Hence,

$$\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right| = \left|{}^{1}\widetilde{D}_q\phi(0)\right| = \left|2 - [2]_q\right|,$$
$$\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right| = \left|{}^{1}\widetilde{D}_q\phi(1)\right| = 2.$$

Therefore, the R.H.S of the inequality becomes

$$\frac{q^2(mx_2-x_1)}{[2]_q{}^{3-\frac{2}{r}}}\left[\zeta(q)\left(\left|{}^{x_2}\widetilde{D}_q\phi(x_1)\right|^r\{R_1(q)-R_2(q)\}+m\left|{}^{x_2}\widetilde{D}_q\phi(x_2)\right|^r\{S_1(q)-S_2(q)\}\right)^{\frac{1}{r}}\right]$$

$$\begin{aligned}&=\frac{q^2}{8}\Bigg[(1-q+2q^3-q^4+q^5)^{1-\frac{1}{2}}\Bigg(\left|{}^{1}\widetilde{D}_q\phi(0)\right|^2\left\{2\left(\frac{q^2[3]_q-q^3[2]_q}{[2]_q^3[3]_q}\right)-\frac{q^2}{[2]_q}+\frac{q^3[2]_q}{[3]_q}\right\}+\\&\qquad\frac{1}{2}\left|{}^{1}\widetilde{D}_q\phi(1)\right|^2\left\{\frac{2+q^2}{[2]_q}-\frac{2q}{[2]_q^2}-\frac{2q^2}{[2]_q^3}+\frac{2q^3}{[2]_q^2[3]_q}-\frac{q^3[2]_q}{[3]_q}\right\}\Bigg)^{\frac{1}{2}}\Bigg]\\&=0.125\left[1.4142\,(0\times 0.25+2\times 0.25)^{\frac{1}{2}}\right]\\&=0.125.\end{aligned}\tag{5.3}$$

By (14), we have inequality as

$$\begin{aligned}&\left|\frac{1}{8}\left\{\frac{1-q+q^2}{[2]_q}-\left[1-\frac{2q}{[2]_q}+\frac{q^2}{[2]_q}\right]\right\}\right|\\&\leq\frac{q^2}{8}\Bigg[(1-q+2q^3-q^4+q^5)^{1-\frac{1}{2}}\Bigg(\left|{}^{1}\widetilde{D}_q\phi(0)\right|^2\left\{2\left(\frac{q^2[3]_q-q^3[2]_q}{[2]_q^3[3]_q}\right)-\frac{q^2}{[2]_q}+\frac{q^3[2]_q}{[3]_q}\right\}+\\&\qquad\frac{1}{2}\left|{}^{1}\widetilde{D}_q\phi(1)\right|^2\left\{\frac{2+q^2}{[2]_q}-\frac{2q}{[2]_q^2}-\frac{2q^2}{[2]_q^3}+\frac{2q^3}{[2]_q^2[3]_q}-\frac{q^3[2]_q}{[3]_q}\right\}\Bigg)^{\frac{1}{2}}\Bigg]\end{aligned}\tag{5.4}$$

It is clear that $0.021<0.125$.

6. CONCLUSION

In the current paper, we presented symmetric quantum trapezoid-type inequalities utilizing the idea of (α, m) −convex functions to get new outcomes. Moreover, we demonstrated that the newly developed inequalities strongly generalize similar findings in the literature as q gets closer to 1. Using Power mean and Holder's integral inequalities, we extended the research of trapezoid-type integral inequalities and employed a novel methodology. Examples are given to prove that the current results provide more accurate approximations than previous data in the literature. It is fascinating to apply these findings to different convexities. We expect the recently revealed findings to be the subject of extensive research in this intriguing field.

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