ORIGINAL PAPER

DOMINATING ENERGY IN CUBIC BIPOLAR FUZZY GRAPHS

LINCY EBANEZAR[1], MAHESWARI VANNIARAJAN[1]

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Abstract. *Fuzzy sets and interval-valued fuzzy sets are combined to create cubic sets, which were proposed by Y.B. Jun et al. The cubic graph is defined using this cubic set. The classical fuzzy graph is extended by the bipolar fuzzy graph, which has both positive and negative membership functions. Jan et al. introduced the cubic bipolar fuzzy graph (CBFG) by combining these two graph classes. In this article, the dominating set in CBFGs is established, and a formal definition of dominating energy, based on the eigenvalues of the corresponding dominating matrix, is presented. The computation of dominating energy is illustrated with an example. Additionally, the characteristics of dominating energy are investigated, and its bounds are determined.*

Keywords: *cubic bipolar fuzzy graph; dominating matrix; dominating energy.*
Mathematics Subject Classification: *05C72; 03E72; 94D05*

1. INTRODUCTION

The phenomenon of fuzzy sets (FSs) $\mathcal{X}$ is characterized by the grade of membership for each $x \in \mathcal{X}$. Interval-valued fuzzy sets (IVFS) [2] and intuitionistic fuzzy sets (IFSs) [3] are among the numerous FS generalizations that have been investigated. The cubic fuzzy set, which combines FS and IVFS, was initiated by Jun et al. [4] as a more versatile method for simulating ambiguity and uncertainty. In [5], bipolar fuzzy sets (BPFSs) were first introduced, unifying polarity and fuzziness in a single model. The membership values in BPFSs were limited to the range [-1, 1]. Fuzzy graphs [6] moved the traditional theory of graphs to fuzzy set theory by introducing uncertainty. In [7], interval-valued fuzzy graphs (IVFGs), an expansion of FGs, were presented, and some properties were investigated. In [8,9], cubic graphs, which are a generalized structure of FGs, were defined, and in [10], cubic fuzzy graphs were proposed due to their unique ability to represent membership degrees using both interval and fuzzy values. Akram provided the first introduction to bipolar fuzzy graphs in [11] and discussed properties of these graphs. Jan et al. introduced CBFGs in [12] using cubic sets and presented some graph-theoretic terms in CBFG.

The energy of a graph, which was initially presented in [13], is an important concept in spectral and chemical graph theories. Haynes et al. [14] studied fundamentals of domination in graphs in 1998. Somasundaram et al. [15] demonstrated domination with effective edges in fuzzy graphs. Further, the concept of domination in fuzzy graphs was proposed using strong arcs in [16], and the domination number was also obtained. The idea of minimum dominating energy in FG, which combines domination and energy, was proposed in

[1] A.P.C. Mahalaxmi College for Women, PG & Research Department of Mathematics, 628002 Thoothukudi, Tamilnadu, India; Affiliated to Manonmaniam Sundaranar University, Abishekapatti, 627012 Tirunelveli, Tamilnadu, India. E-mail: lincy.je@gmail.com; mahiraj2005@gmail.com.

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[17]. Akram et al. [18] presented the energy of double-dominating bipolar FGs in 2019 by analysing their positive and negative membership values.

In this paper, domination in CBFGs is defined based on bipolar strong edges, following the concepts in [19], and the idea of a dominating matrix in cubic bipolar fuzzy graphs is presented. Using eigenvalues calculated from the dominating matrix in CBFG, the dominating energy in CBFG is defined. Theorems on domination in CBFG are studied in detail. Additionally, we explore the properties of eigenvalues and the boundaries of dominant energy in CBFGs.

2. PRELIMINARIES

Definition 2.1. *An FS on $\mathcal{X}$ is $\mathcal{K} = \{\langle \mathfrak{u}, \varsigma_{\mathcal{K}}(\mathfrak{u})\rangle : \mathfrak{u} \in \mathcal{X}\}$ where $\mathcal{X}$ is the universal set and $\varsigma_{\mathcal{K}}(\mathfrak{u}) : \mathcal{X} \to [0,1]$ is the membership grade of FS on $\mathcal{K}$. A FG $\mathcal{G}^* = (\mathcal{K}, \mathcal{L})$ is non-$\emptyset$ set $\mathcal{X}$ with fuzzy set $\mathcal{K} : \mathcal{X} \to [0,1]$ and fuzzy relation $\mathcal{L} : \mathcal{X} \times \mathcal{X} \to [0,1]$ given by $\varsigma_{\mathcal{L}}(\mathfrak{u}t) \le \varsigma_{\mathcal{K}}(\mathfrak{u}) \wedge \varsigma_{\mathcal{K}}(t) \forall \mathfrak{u}, t \in \mathcal{X}$*

Definition 2.2. *A cubic set (CS) on $\mathcal{X}$ is $\mathcal{C} = \{\langle [\delta(\mathfrak{c}), \beta(\mathfrak{c})], \varepsilon(\mathfrak{c})\rangle, : \mathfrak{c} \in \mathcal{X}\}$ where $\mathcal{X}$ is non-ϕ set and $[\delta(\mathfrak{c}), \beta(\mathfrak{c})]$ is an interval-valued membership grade, $\varepsilon(\mathfrak{c})$ is fuzzy membership value and $(\mathfrak{c}), \beta(\mathfrak{c}), \varepsilon(\mathfrak{c}) : \mathcal{X} \to [0,1]$. A cubic graph on $\mathcal{X}$ is $\mathcal{G} = (\mathcal{R}, \mathcal{S})$ where $\mathcal{R}$ and $\mathcal{S}$ are CFS in $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ respectively, such that $\delta_{\mathcal{S}}(\mathscr{b}\mathfrak{c}) \le \delta_{\mathcal{R}}(\mathscr{b}) \wedge \delta_{\mathcal{R}}(c)$, $\beta_{\mathcal{S}}(\mathscr{b}\mathfrak{c}) \le \beta_{\mathcal{R}}(\mathscr{b}) \wedge \beta_{\mathcal{R}}(c)$ and $\varepsilon_{\mathcal{S}}(\mathscr{b}\mathfrak{c}) \le \varepsilon_{\mathcal{R}}(\mathscr{b}) \wedge \varepsilon_{\mathcal{R}}(c) \forall \mathscr{b}, \mathfrak{c} \in \mathcal{X}$.*

Definition 2.3. *A BFS $\mathfrak{B}$ on non-$\emptyset$ set $\mathcal{X}$ is $\mathfrak{B} = \{\langle z, \psi_{\mathfrak{B}}^{+}(z), \psi_{\mathfrak{B}}^{-}(z)\rangle : z \in \mathcal{X}\}$ where $\psi_{\mathfrak{B}}^{+}(z) : \mathcal{X} \to [0,1]$ and $\psi_{\mathfrak{B}}^{-}(z) : \mathcal{X} \to [-1,0]$. A BFG $\mathbb{G} = (\mathcal{A}, \mathfrak{B})$ where $\mathcal{A} = (\psi_{\mathcal{A}}^{+}(z), \psi_{\mathcal{A}}^{-}(z))$ is bipolar FS on $\mathcal{X}$ and $\mathfrak{B} = (\psi_{\mathfrak{B}}^{+}(z), \psi_{\mathfrak{B}}^{-}(z))$ is a bipolar relation on $\mathcal{X} \times \mathcal{X}$ such that $\psi_{\mathfrak{B}}^{+}(z\mathfrak{s}) \le \psi_{\mathcal{A}}^{+}(z) \wedge \psi_{\mathcal{A}}^{+}(\mathfrak{s})$ and $\psi_{\mathfrak{B}}^{-}(z\mathfrak{s}) \ge \psi_{\mathcal{A}}^{-}(z) \vee \psi_{\mathcal{A}}^{-}(\mathfrak{s}) \forall z, \mathfrak{s} \in \mathcal{X}$.*

Definition 2.4. *A CBFS $\mathcal{T}$ on non-$\emptyset$ set $\mathcal{X}$ is defined as $\mathcal{T} = \{\langle \mathfrak{x}, [\rho^{\ell}(\mathfrak{x}), \rho^{u}(\mathfrak{x})], [\eta^{\ell}(\mathfrak{x}), \eta^{u}(\mathfrak{x})], \rho(\mathfrak{x}), \eta(\mathfrak{x}) : \mathfrak{x} \in \mathcal{X}\rangle\}$ where $[\rho^{\ell}(\mathfrak{x}), \rho^{u}(\mathfrak{x})] \subseteq [0,1]$, $[\eta^{\ell}(\mathfrak{x}), \eta^{u}(\mathfrak{x})] \subseteq [-1,0]$, $\rho(\mathfrak{x}) : \mathcal{X} \to [0,1]$ and $\eta(\mathfrak{x}) : \mathcal{X} \to [-1,0]$. A graph $\tilde{\mathfrak{J}} = (\mathfrak{U}, \mathfrak{R})$ is known as a CBFG when $\mathfrak{U} = \{\langle [\rho_{\mathfrak{U}}^{\ell}, \rho_{\mathfrak{U}}^{u}], [\eta_{\mathfrak{U}}^{\ell}, \eta_{\mathfrak{U}}^{u}], \rho_{\mathfrak{U}}, \eta_{\mathfrak{U}}\rangle\}$ is CBFS and $\mathfrak{R}$ is a CBF relation given as $\rho_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \le min(\rho_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \rho_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$, $\rho_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \le min(\rho_{\mathfrak{U}}^{u}(\mathfrak{f}), \rho_{\mathfrak{U}}^{u}(\mathfrak{j}))$ along with $\rho_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \ge min(\rho_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \rho_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$, $\eta_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \ge max(\eta_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \eta_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$ along with $\eta_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \le max(\eta_{\mathfrak{U}}^{u}(\mathfrak{f}), \eta_{\mathfrak{U}}^{u}(\mathfrak{j}))$, $\eta_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \ge max(\eta_{\mathfrak{U}}^{u}(\mathfrak{f}), \eta_{\mathfrak{U}}^{u}(\mathfrak{j}))$, $\rho_{\mathfrak{R}}(\mathfrak{f}\mathfrak{j}) \le min(\rho_{\mathfrak{U}}(\mathfrak{f}), \rho_{\mathfrak{U}}(\mathfrak{j}))$ and $\eta_{\mathfrak{R}}(\mathfrak{f}\mathfrak{j}) \ge max(\eta_{\mathfrak{U}}(\mathfrak{f}), \eta_{\mathfrak{U}}(\mathfrak{j}))$ $\forall \mathfrak{f}, \mathfrak{j} \in \mathcal{X}$.*

Definition 2.5. *A sequence of $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_{\mathscr{k}}$ is known as a path in CBFG such that $(\rho_{\mathfrak{R}}^{\ell}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \rho_{\mathfrak{R}}^{u}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \rho_{\mathfrak{R}}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})) > 0$, $(\eta_{\mathfrak{R}}^{\ell}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \eta_{\mathfrak{R}}^{u}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \eta_{\mathfrak{R}}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})) < 0$ for $\mathrm{I} \in \{2, \ldots, \mathscr{k}\}$. The strength of a path, shown by $(\mathcal{S}^{\rho}, \mathcal{S}^{\eta})$, where positive strength and negative strength are*

$$\mathcal{S}^{\rho} = \left(\left(\wedge_{\mathrm{I}=2}^{\mathscr{k}} \rho_{\mathfrak{R}}^{\ell}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \wedge_{\mathrm{I}=2}^{\mathscr{k}} \rho_{\mathfrak{R}}^{u}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})\right), \wedge_{\mathrm{I}=2}^{\mathscr{k}} \rho_{\mathfrak{R}}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})\right)$$
$$\mathcal{S}^{\eta} = \left(\left(\vee_{\mathrm{I}=2}^{\mathscr{k}} \eta_{\mathfrak{R}}^{\ell}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}}), \vee_{\mathrm{I}=2}^{\mathscr{k}} \eta_{\mathfrak{R}}^{u}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})\right), \vee_{\mathrm{I}=2}^{\mathscr{k}} \eta_{\mathfrak{R}}(\mathfrak{p}_{\mathrm{I}-1}, \mathfrak{p}_{\mathrm{I}})\right).$$

respectively. The positive strength and negative strength of connectivity between $\mathfrak{g}$ and $\acute{\mathfrak{g}}$, respectively, is given by

$$\mathcal{F}^{\rho,\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \left(\left[\mathcal{F}^{\rho^{\ell},\infty}(\mathfrak{g},\acute{\mathfrak{g}}), \mathcal{F}^{\rho^{u},\infty}(\mathfrak{g},\acute{\mathfrak{g}})\right], \mathcal{F}^{\rho,\infty}(\mathfrak{g},\acute{\mathfrak{g}})\right)$$
$$\mathcal{F}^{\eta,\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \left(\left[\mathcal{F}^{\eta^{\ell},\infty}(\mathfrak{g},\acute{\mathfrak{g}}), \mathcal{F}^{\eta^{u},\infty}(\mathfrak{g},\acute{\mathfrak{g}})\right], \mathcal{F}^{\eta,\infty}(\mathfrak{g},\acute{\mathfrak{g}})\right)$$

where

$$\mathcal{F}^{\rho^{\ell},\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \max_{k}\left\{\bigwedge_{\mathrm{I}=2}^{k} \rho_{\Re}^{\ell}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\}, \ \mathcal{F}^{\rho^{u},\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \max_{k}\left\{\bigwedge_{\mathrm{I}=2}^{k} \rho_{\Re}^{u}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\}$$
$$\mathcal{F}^{\rho,\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \max_{k}\left\{\bigwedge_{\mathrm{I}=2}^{k} \rho_{\Re}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\}, \ \mathcal{F}^{\eta^{\ell},\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \min_{k}\left\{\bigvee_{\mathrm{I}=2}^{k} \eta_{\Re}^{\ell}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\},$$
$$\mathcal{F}^{\eta^{u},\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \min_{k}\left\{\bigvee_{\mathrm{I}=2}^{k} \eta_{\Re}^{u}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\}, \ \mathcal{F}^{\eta,\infty}(\mathfrak{g},\acute{\mathfrak{g}}) = \min_{k}\left\{\bigvee_{\mathrm{I}=2}^{k} \eta_{\Re}(\mathfrak{g}_{\mathrm{I}-1},\mathfrak{g}_{\mathrm{I}})\right\}$$

Definition 2.6. *An edge $\mathfrak{q}\mathfrak{x}$ in CBFG is*

i. positive strong if $\rho_{\Re}^{\ell}(\mathfrak{q},\mathfrak{x}) \geq \mathcal{F}^{\rho^{\ell},\infty}(\mathfrak{q},\mathfrak{x}), \rho_{\Re}^{u}(\mathfrak{q},\mathfrak{x}) \geq \mathcal{F}^{\rho^{u},\infty}(\mathfrak{q},\mathfrak{x})$ and $\rho_{\Re}(\mathfrak{q},\mathfrak{x}) \geq \mathcal{F}^{\rho,\infty}(\mathfrak{q},\mathfrak{x})$.

ii. negative strong if $\eta_{\Re}^{\ell}(\mathfrak{q},\mathfrak{x}) \leq \mathcal{F}^{\eta^{\ell},\infty}(\mathfrak{q},\mathfrak{x})$, $\eta_{\Re}^{u}(\mathfrak{q},\mathfrak{x}) \leq \mathcal{F}^{\eta^{u},\infty}(\mathfrak{q},\mathfrak{x})$ and $\eta_{\Re}(\mathfrak{q},\mathfrak{x}) \leq \mathcal{F}^{\eta,\infty}(\mathfrak{q},\mathfrak{x})$

iii. bipolar strong if it is both positive strong and negative strong.

3. DOMINATION IN CBFG

Definition 3.1. *Let $\tilde{\mathcal{J}} = (\mathfrak{U},\Re)$ be a CBFG of a crisp graph and $h, m \in \mathfrak{U}$. Then h dominates m if there exists bipolar strong edge between them*. Here, domination is symmetric on $\mathfrak{U}$. A set $\Upsilon \subseteq \mathfrak{U}$ is known as a dominating set (DS) if $\forall \mathfrak{w} \in \mathfrak{U}\backslash\Upsilon, \exists \mathfrak{s} \in \Upsilon$ such that $\mathfrak{s}$ dominates $\mathfrak{w}$. A DS $\mathcal{Q}$ in CBFG is called minimal DS if $\mathcal{H} \subset \mathcal{Q}$ and $\mathcal{H}$ is not a DS.

Example 3.2. Consider a CBFG as shown below with vertex set=$\{\mathcal{Z},\mathcal{E},\mathcal{F},\mathcal{K},\mathcal{P}\}$. The bipolar strong edges are $(\mathcal{Z},\mathcal{P}), (\mathcal{E},\mathcal{F}), (\mathcal{E},\mathcal{K})$ and $(\mathcal{F},\mathcal{P})$. Then $\mathcal{Z}$ dominates $\mathcal{P}$; $\mathcal{E}$ dominates $\mathcal{K}$&$\mathcal{F}$; $\mathcal{F}$ dominates $\mathcal{E}$&$\mathcal{P}$; $\mathcal{K}$ dominates $\mathcal{E}$; $\mathcal{P}$ dominates $\mathcal{Z}$&$\mathcal{F}$.

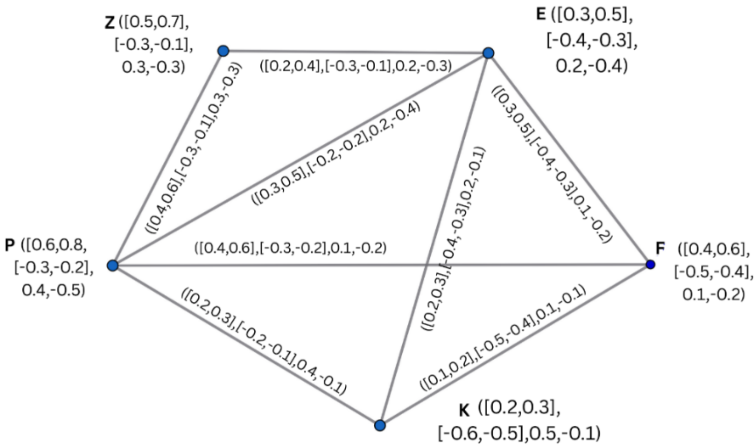

Figure 1. CBFG.

Definition 3.3. *The vertex cardinality of CBFG is given by*

$$|\mathfrak{U}| = \sum_{\mathfrak{s}\in\mathfrak{U}} \frac{1-\rho^{\ell}_{\mathfrak{U}}(\mathfrak{s})+\rho^{u}_{\mathfrak{U}}(\mathfrak{s})+\rho_{\mathfrak{U}}}{3} + \sum_{\mathfrak{s}\in\mathfrak{U}} \frac{1-\eta^{\ell}_{\mathfrak{U}}(\mathfrak{s})+\eta^{u}_{\mathfrak{U}}(\mathfrak{s})+\eta_{\mathfrak{U}}}{3}$$

Domination number $|\Upsilon|$ is the lowest cardinality of all the minimal DS in CBFG, and minimum dominating set Υ is the DS with the lowest cardinality. The notation $\mathfrak{n}(\Upsilon)$ indicates how many vertices there are in Υ.

Example 3.4. Some minimal DS of $\tilde{\mathfrak{J}}$ given in Fig.1 are $\Upsilon_1 = \{\mathcal{K},\mathcal{P}\}, \Upsilon_2 = \{\mathcal{E},\mathcal{P}\}, \Upsilon_3 = \{\mathcal{Z},\mathcal{E}\}$ and $\Upsilon_4 = \{\mathcal{Z},\mathcal{F},\mathcal{K}\}$ and $|\Upsilon_1| = 1.59, |\Upsilon_2| = 1.427, |\Upsilon_3| = 1.497$ and $|\Upsilon_4| = 2.39$. Hence, the minimum DS is Υ_2.

Definition 3.5. *The bipolar strong neighbourhood of vertex $\mathfrak{b}$ in CBFG is given by $\mathcal{N}(\mathfrak{b}) = \{\mathcal{r}: \mathcal{r} \in \mathfrak{U}, (\mathfrak{b},\mathcal{r})$ is a bipolar strong edge$\}$. A vertex $g \in \mathfrak{U}$ in CBFG is known as an isolated vertex if $\mathcal{N}(g) = \emptyset$. i.e. no vertex in $\tilde{\mathfrak{J}}$ will be dominated by an isolated vertex.*

Definition 3.6. *Two vertices are said to be independent in CBFG if there is no bipolar strong edge between them. An independent set (IS) $\mathcal{W}$ is known as a maximal IS, if for each $\mathfrak{j} \in \mathfrak{U}\backslash\mathcal{W}$, the set $\mathcal{W} \cup \{\mathfrak{j}\}$ is not an IS.*

Theorem 3.7. *A DS Υ of CBFG is a minimal DS iff for each $\mathfrak{r} \in \Upsilon$ one of the following conditions holds*:

(a) $\mathfrak{r}$ is not a bipolar strong neighbour of any vertex in Υ.
(b) $\exists$ a vertex $\mathfrak{t} \in \mathfrak{U}\backslash\Upsilon \ni \mathcal{N}(\mathfrak{t}) \cap \Upsilon = \mathfrak{r}$.

Proof: Let Υ be a minimal DS of $\tilde{\mathfrak{J}}$. Then for every vertex $\mathfrak{r} \in \Upsilon$, $\Upsilon - \{\mathfrak{r}\}$ is not DS. Hence $\exists\, \mathfrak{t} \in \mathfrak{U}\backslash(\Upsilon - \{\mathfrak{r}\})$ which is not dominated by any vertex in $\Upsilon - \{\mathfrak{r}\}$. If $\mathfrak{t} = \mathfrak{r}$, then $\mathfrak{r}$ is not a bipolar strong neighbour of any vertex in Υ. If $\mathfrak{t} \neq \mathfrak{r}$, $\mathfrak{t}$ is not dominated by $\Upsilon - \{\mathfrak{r}\}$, but $\mathfrak{t}$ is dominated by Υ. Therefore, $\mathfrak{t}$ is a bipolar strong neighbour only to $\mathfrak{r}$ in Υ. Hence, $\mathcal{N}(\mathfrak{t}) \cap \Upsilon = \mathfrak{r}$.

Conversely, let Υ be DS and for each $\mathfrak{r} \in \Upsilon$ one of the two condition holds. Assume Υ is not minimal DS. Hence $\exists$ vertex $\mathfrak{r} \in \Upsilon \ni \Upsilon - \{\mathfrak{r}\}$ is DS. Then, $\mathfrak{r}$ is bipolar strong neighbor to at least one vertex in $\Upsilon - \{\mathfrak{r}\}$, hence condition (a) fails. If $\Upsilon - \{\mathfrak{r}\}$ is DS, then every vertex in $\mathfrak{U}\backslash\Upsilon$ is bipolar strong neighbour to at a minimum one vertex in $\Upsilon-\{\mathfrak{r}\}$, hence condition (b) fails. This contradicts that at a minimum one of the conditions holds as in our assumption. Hence Υ is minimal DS.

Theorem 3.8. *Let $\tilde{\mathfrak{J}}$ be CBFG without isolated vertices and Υ be its minimal DS. Then $\mathfrak{U}\backslash\Upsilon$ is DS of $\tilde{\mathfrak{J}}$.*

Proof: Let Υ be the minimal DS of $\tilde{\mathfrak{J}}$ and $\mathfrak{i}\in\Upsilon$. Since $\tilde{\mathfrak{J}}$ is without isolated vertices, $\exists$ $\mathfrak{f}$ such that $\mathfrak{f}\in\mathcal{N}(\mathfrak{i})$. Then $\mathfrak{i}$ is dominated by at a minimum one vertex in $\Upsilon-\{\mathfrak{i}\}$. Hence $\Upsilon-\{\mathfrak{i}\}$ is a DS. By theorem:3.7, $\mathfrak{f}\in\mathfrak{U}\backslash\Upsilon$. Therefore, every vertex in Υ is dominated by at least one vertex in $\mathfrak{U}\backslash\Upsilon$. Hence $\mathfrak{U}\backslash\Upsilon$ is a DS.

Theorem 3.9. *Every maximal independent set (IS) in CBFG $\tilde{\mathfrak{J}}$ is a minimal DS.*

Proof: Let $\mathcal{W}$ be a maximal IS of CBFG. Then $\mathcal{W}$ is a DS, and assume that $\mathcal{W}$ is not a minimal DS. Hence, $\exists$ at a minimum one vertex $\mathfrak{x}\in\mathcal{W}$ such that $\mathcal{W}-\{\mathfrak{x}\}$ is DS. But if $\mathcal{W}-\{\mathfrak{x}\}$ dominates $\mathfrak{U}\backslash\mathcal{W}-\{\mathfrak{x}\}$, then at a minimum one vertex in $\mathcal{W}-\{\mathfrak{x}\}$ is a bipolar strong neighbour of $\mathfrak{x}$. This is a contradiction to $\mathcal{W}$ is IS of $\tilde{\mathfrak{J}}$. Hence $\mathcal{W}$ is minimal DS.

4. DOMINATING ENERGY IN CBFG

Definition 4.1. *Let $\tilde{\mathfrak{J}}$ be a CBFG and Υ be its minimum DS.* The dominating matrix of CBFG is given by $\mathcal{A}_{\mathfrak{D}}(\tilde{\mathfrak{J}})=[k_{\mathfrak{r}\mathfrak{w}}]$, where

$$k_{\mathfrak{r}\mathfrak{w}}=\begin{cases}\left(\left[\rho_{\mathfrak{R}}^{\ell}(\mathfrak{r}\mathfrak{w}),\rho_{\mathfrak{R}}^{u}(\mathfrak{r}\mathfrak{w})\right],\left[\eta_{\mathfrak{R}}^{\ell}(\mathfrak{r}\mathfrak{w}),\eta_{\mathfrak{R}}^{u}(\mathfrak{r}\mathfrak{w})\right],\rho_{\mathfrak{R}}(\mathfrak{r}\mathfrak{w}),\eta_{\mathfrak{R}}(\mathfrak{r}\mathfrak{w})\right) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ (1,1,-1,-1,1,-1) & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ (0,0,0,0,0,0) & otherwise\end{cases}$$

The dominating matrix of CBFG is shown in six distinct matrices as $\mathcal{A}_{\mathfrak{D}}(\tilde{\mathfrak{J}})=\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}}),\rho_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}}),\rho_{\mathfrak{D}}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}(\tilde{\mathfrak{J}})\right)$, where

$$\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}})=\begin{cases}\rho_{\mathfrak{R}}^{\ell}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ 1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases},\ \rho_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}})=\begin{cases}\rho_{\mathfrak{R}}^{u}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ 1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases},$$

$$\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}})=\begin{cases}\eta_{\mathfrak{R}}^{\ell}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ -1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases},\ \eta_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}})=\begin{cases}\eta_{\mathfrak{R}}^{u}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ -1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases},$$

$$\rho_{\mathfrak{D}}(\tilde{\mathfrak{J}})=\begin{cases}\rho_{\mathfrak{R}}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ 1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases}\text{ and }\eta_{\mathfrak{D}}(\tilde{\mathfrak{J}})=\begin{cases}\eta_{\mathfrak{R}}(\mathfrak{r}\mathfrak{w}) & if \quad \mathfrak{r}\mathfrak{w}\in\mathfrak{R}\\ -1 & if \quad \mathfrak{r}=\mathfrak{w}\ and\ \mathfrak{r}\in\Upsilon\\ 0 & otherwise\end{cases}$$

Definition 4.2. *The spectrum of the dominating matrix of CBFG is defined by $\left(\underline{\sigma},\overline{\sigma},\underline{\phi},\overline{\phi},\tau,\omega\right)$ where $\underline{\sigma},\overline{\sigma},\underline{\phi},\overline{\phi},\tau,\omega$ are sets of eigenvalues of $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}}),\rho_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}^{u}(\tilde{\mathfrak{J}}),\rho_{\mathfrak{D}}(\tilde{\mathfrak{J}}),\eta_{\mathfrak{D}}(\tilde{\mathfrak{J}})$ respectively.*

Definition 4.3. *The dominating energy of CBFG is defined as*

$$d_e(\tilde{J}) = \left(\left\{d_e\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{J})\right), d_e(\rho_{\mathfrak{D}}^{u}(\tilde{J}))\right\}, \left\{d_e\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{J})\right), d_e(\eta_{\mathfrak{D}}^{u}(\tilde{J}))\right\}, d_e(\rho_{\mathfrak{D}}(\tilde{J})), d_e(\eta_{\mathfrak{D}}(\tilde{J}))\right)$$
$$= \left(\left\{\sum_{I=1}^{\mathfrak{s}}\left|\underline{\sigma_I}\right|, \sum_{I=1}^{\mathfrak{s}}\left|\overline{\sigma}_I\right|\right\}, \left\{\sum_{I=1}^{\mathfrak{s}}\left|\underline{\phi_I}\right|, \sum_{I=1}^{\mathfrak{s}}\left|\overline{\phi}_I\right|\right\}, \sum_{I=1}^{\mathfrak{s}}|\tau_I|, \sum_{I=1}^{\mathfrak{s}}|\omega_I|\right)$$

Example 4.4. Consider a CBFG $\tilde{J}$ with vertex set=$\{\mathcal{Z}, \mathcal{E}, \mathcal{F}, \mathcal{K}, \mathcal{P}\}$ and minimum DS $\Upsilon = \{\mathcal{E}, \mathcal{P}\}$ as in Fig.1. The dominating energy of $\tilde{J}$ is given by

$$\rho_{\mathfrak{D}}^{\ell}(\tilde{J}) = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0.4 \\ 0.2 & 1 & 0.3 & 0.2 & 0.3 \\ 0 & 0.3 & 0 & 0.1 & 0.4 \\ 0 & 0.2 & 0.1 & 0 & 0.2 \\ 0.4 & 0.3 & 0.4 & 0.2 & 1 \end{bmatrix}$$

Spectrum of $\rho_{\mathfrak{D}}^{\ell}(\tilde{J}) = \{1.62586, 0.720913, -0.295497, -0.0898383, 0.0385658\}$. The dominating energy of $\rho_{\mathfrak{D}}^{\ell}(\tilde{J})$ matrix is $d_e\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{J})\right) = \sum_{I=1}^{\mathfrak{s}}\left|\underline{\sigma_I}\right| = 1.62586 + 0.720913 + 0.295497 + 0.0898383 + 0.0385658 = 2.7706741$.

$$\rho_{\mathfrak{D}}^{u}(\tilde{J}) = \begin{bmatrix} 0 & 0.4 & 0 & 0 & 0.6 \\ 0.4 & 1 & 0.5 & 0.3 & 0.5 \\ 0 & 0.5 & 0 & 0.2 & 0.6 \\ 0 & 0.3 & 0.2 & 0 & 0.3 \\ 0.6 & 0.5 & 0.6 & 0.3 & 1 \end{bmatrix}$$

Spectrum of $\rho_{\mathfrak{D}}^{u}(\tilde{J}) = \{2.13707, -0.566188, 0.522294, -0.179307, 0.0861307\}$. The dominating energy of $\rho_{\mathfrak{D}}^{u}(\tilde{J})$ matrix is $d_e\left(\rho_{\mathfrak{D}}^{u}(\tilde{J})\right) = \sum_{I=1}^{\mathfrak{s}}|\overline{\sigma}_I| = 2.13707 + 0.566188 + 0.522294 + 0.179307 + 0.0861307 = 3.4909897$.

$$\eta_{\mathfrak{D}}^{\ell}(\tilde{J}) = \begin{bmatrix} 0 & -0.3 & 0 & 0 & -0.3 \\ -0.3 & -1 & -0.4 & -0.4 & -0.2 \\ 0 & -0.4 & 0 & -0.5 & -0.3 \\ 0 & -0.4 & -0.5 & 0 & -0.2 \\ -0.3 & -0.2 & -0.3 & -0.2 & -1 \end{bmatrix}$$

Spectrum of $\eta_{\mathfrak{D}}^{\ell}(\tilde{J}) = \{-1.67573, -0.824262, 0.504273, -0.223737, 0.219458\}$. The dominating energy of $\eta_{\mathfrak{D}}^{\ell}(\tilde{J})$ matrix is $d_e\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{J})\right) = \sum_{I=1}^{\mathfrak{s}}\left|\underline{\phi_I}\right| = 1.67573 + 0.824262 + 0.504273 + 0.223737 + 0.219458 = 3.44746$.

$$\eta_{\mathfrak{D}}^{u}(\tilde{J}) = \begin{bmatrix} 0 & -0.1 & 0 & 0 & -0.1 \\ -0.1 & -1 & -0.3 & -0.3 & -0.2 \\ 0 & -0.3 & 0 & -0.4 & -0.2 \\ 0 & -0.3 & -0.4 & 0 & -0.1 \\ -0.1 & -0.2 & -0.2 & -0.1 & -1 \end{bmatrix}$$

Spectrum of $\eta_{\mathfrak{D}}^{u}(\tilde{J}) = \{-1.42083, -0.826863, 0.403789, -0.183642, 0.0275492\}$. The dominating energy of $\eta_{\mathfrak{D}}^{u}(\tilde{J})$ matrix is $d_e\left(\eta_{\mathfrak{D}}^{u}(\tilde{J})\right) = \sum_{I=1}^{\mathfrak{s}}\left|\overline{\phi}_I\right| = 1.42083 + 0.826863 + 0.403789 + 0.183642 + 0.0275492 = 2.8626732$.

$$\rho_{\mathfrak{D}}(\tilde{\mathcal{J}}) = \begin{bmatrix} 0 & 0.2 & 0 & 0 & 0.3 \\ 0.2 & 1 & 0.1 & 0.2 & 0.2 \\ 0 & 0.1 & 0 & 0.1 & 0.1 \\ 0 & 0.2 & 0.1 & 0 & 0.4 \\ 0.3 & 0.2 & 0.1 & 0.4 & 1 \end{bmatrix}$$

Spectrum of $\rho_{\mathfrak{D}}(\tilde{\mathcal{J}}) = \{1.43831, 0.816407, -0.237847, -0.0717591, 0.0548855\}$. The dominating energy of $\rho_{\mathfrak{D}}(\tilde{\mathcal{J}})$ matrix is $d_e\left(\rho_{\mathfrak{D}}(\tilde{\mathcal{J}})\right) = \sum_{I=1}^{s} |\tau_I| = 1.43831 + 0.816407 + 0.237847 + 0.0717591 + 0.0548855 = 2.6192086$

$$\eta_{\mathfrak{D}}(\tilde{\mathcal{J}}) = \begin{bmatrix} 0 & -0.3 & 0 & 0 & -0.3 \\ -0.3 & -1 & -0.2 & -0.1 & -0.4 \\ 0 & -0.2 & 0 & -0.1 & -0.2 \\ 0 & -0.1 & -0.1 & 0 & -0.1 \\ -0.3 & -0.4 & -0.2 & -0.1 & -1 \end{bmatrix}$$

Spectrum of $\eta_{\mathfrak{D}}(\tilde{\mathcal{J}}) = \{-1.58062, -0.6, 0.166742, 0.0898689, -0.075996\}$. The dominating energy of $\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})$ matrix is $d_e(\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})) = \sum_{I=1}^{s} |\omega_I| = 1.62586 + 0.720913 + 0.295497 + 0.0898383 + 0.0385658 = 2.5132269$.

Hence, the dominating energy of CBFG is

$$d_e(\tilde{\mathcal{J}}) = \left(\left\{d_e\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right), d_e(\rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}))\right\}, \left\{d_e\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right), d_e(\eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}))\right\}, d_e(\rho_{\mathfrak{D}}(\tilde{\mathcal{J}})), d_e(\eta_{\mathfrak{D}}(\tilde{\mathcal{J}}))\right)$$
$$d_e(\tilde{\mathcal{J}}) = \left(\left\{\sum_{I=1}^{s} |\underline{\sigma_I}|, \sum_{I=1}^{s} |\overline{\sigma}_I|\right\}, \left\{\sum_{I=1}^{s} |\underline{\phi_I}|, \sum_{I=1}^{s} |\overline{\phi}_I|\right\}, \sum_{I=1}^{s} |\tau_I|, \sum_{I=1}^{s} |\omega_I|\right)$$
$$d_e(\tilde{\mathcal{J}}) = (\{2.7706741, 3.4909897\}, \{3.44746, 2.8626732\}, 2.6192086, 2.5132269)$$

Theorem 4.5. *Let $\tilde{\mathcal{J}}$ be CBFG and Υ be its minimum DS. If $\underline{\sigma_I}, \overline{\sigma}_I, \underline{\phi_I}, \overline{\phi}_I, \tau_I$ and ω_I are eigenvalues of dominating matrices $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}(\tilde{\mathcal{J}})$ and $\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})$ respectively, then*

(1) $\sum_{I=1}^{s} \underline{\sigma_I} = \sum_{I=1}^{s} \overline{\sigma}_I = \sum_{I=1}^{s} \tau_I = \mathfrak{n}(\Upsilon)$
(2) $\sum_{I=1}^{s} \underline{\phi_I} = \sum_{I=1}^{s} \overline{\phi}_I = \sum_{I=1}^{s} \omega_I = -\mathfrak{n}(\Upsilon)$
(3) $\sum_{I=1}^{s} \underline{\sigma_I}^2 = \sum_{I=1}^{s} \left(\rho_{II}^{\ell}\right)^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \rho_{I\mathfrak{w}}^{\ell} \rho_{\mathfrak{w}I}^{\ell}$
(4) $\sum_{I=1}^{s} \overline{\sigma}_I^2 = \sum_{I=1}^{s} (\rho_{II}^{u})^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \rho_{I\mathfrak{w}}^{u} \rho_{\mathfrak{w}I}^{u}$
(5) $\sum_{I=1}^{s} \underline{\phi_I}^2 = \sum_{I=1}^{s} \left(\eta_{II}^{\ell}\right)^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \eta_{I\mathfrak{w}}^{\ell} \eta_{\mathfrak{w}I}^{\ell}$
(6) $\sum_{I=1}^{s} \overline{\phi}_I^2 = \sum_{I=1}^{s} (\eta_{II}^{u})^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \eta_{I\mathfrak{w}}^{u} \eta_{\mathfrak{w}I}^{u}$
(7) $\sum_{I=1}^{s} \tau_I^2 = \sum_{I=1}^{s} (\rho_{II})^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \rho_{I\mathfrak{w}} \rho_{\mathfrak{w}I}$
(8) $\sum_{I=1}^{s} \omega_I^2 = \sum_{I=1}^{s} (\eta_{II})^2 + 2\sum_{1 \le I < \mathfrak{w} \le s} \eta_{I\mathfrak{w}} \eta_{\mathfrak{w}I}$

Proof:
(1) By using the property that the trace of a matrix $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})$ equals the sum of eigenvalues, we obtain

$$\sum_{I=1}^{s} \underline{\sigma_I} = \sum \rho_{II}^{\ell} = \mathfrak{n}(\Upsilon)$$

In a similar manner, the proof of $\sum_{\mathrm{I}=1}^{\mathfrak{s}} \overline{\sigma}_{\mathrm{I}}=\sum_{\mathrm{I}=1}^{\mathfrak{s}} \tau_{\mathrm{I}} = \mathfrak{n}(\Upsilon)$ is done.

(2) By using the property that the trace of a matrix $\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})$ equals the sum of eigenvalues, we obtain

$$\sum_{\mathrm{I}=1}^{\mathfrak{s}} \underline{\sigma_{\mathrm{I}}} = \sum \eta_{\mathrm{II}}^{\ell} = -\mathfrak{n}(\Upsilon)$$

In a similar manner, the proof of $\sum_{\mathrm{I}=1}^{\mathfrak{s}} \overline{\phi}_{\mathrm{I}} = \sum_{\mathrm{I}=1}^{\mathfrak{s}} \omega_{\mathrm{I}} = -\,\mathfrak{n}(\Upsilon)$ is done.

(3) Apparently, the total squares of eigenvalues of $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})$ equals the trace of $\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)^2$. Hence

$$tr\left(\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)^2\right) = \sum_{\mathrm{I}=1}^{\mathfrak{s}} \underline{\sigma_{\mathrm{I}}}^2$$
$$= \rho_{11}^{\ell}\rho_{11}^{\ell} + \rho_{12}^{\ell}\rho_{21}^{\ell} + \cdots + \rho_{1\mathfrak{s}}^{\ell}\rho_{\mathfrak{s}1}^{\ell} + \rho_{21}^{\ell}\rho_{21}^{\ell} + \rho_{22}^{\ell}\rho_{22}^{\ell} + \cdots + \rho_{2\mathfrak{s}}^{\ell}\rho_{2\mathfrak{s}}^{\ell} + \cdots$$
$$+ \rho_{\mathfrak{s}1}^{\ell}\rho_{1\mathfrak{s}}^{\ell} + \rho_{\mathfrak{s}2}^{\ell}\rho_{2\mathfrak{s}}^{\ell} + \cdots + \rho_{\mathfrak{s}\mathfrak{s}}^{\ell}\rho_{\mathfrak{s}\mathfrak{s}}^{\ell}$$
$$\sum_{\mathrm{I}=1}^{\mathfrak{s}} \underline{\sigma_{\mathrm{I}}}^2 = \sum_{\mathrm{I}=1}^{\mathfrak{s}} \left(\rho_{\mathrm{II}}^{\ell}\right)^2 + 2\sum_{1\le \mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}$$

In the same way, (4), (5), (6), (7), and (8) can be stated.

Theorem 4.6. Let $\tilde{\mathcal{J}}$ be CBFG and Υ be its minimum DS. Then

a. $d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}\right)}$

b. $d_{\mathfrak{e}}(\rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}})) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{u}\right)^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{u}\rho_{\mathfrak{w}\mathrm{I}}^{u}\right)}$

c. $d_{\mathfrak{e}}\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\eta_{\mathrm{II}}^{\ell}\right)^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \eta_{\mathrm{I}\mathfrak{w}}^{\ell}\eta_{\mathfrak{w}\mathrm{I}}^{\ell}\right)}$

d. $d_{\mathfrak{e}}(\eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}})) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\eta_{\mathrm{II}}^{u}\right)^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \eta_{\mathrm{I}\mathfrak{w}}^{u}\eta_{\mathfrak{w}\mathrm{I}}^{u}\right)}$

e. $d_{\mathfrak{e}}(\rho_{\mathfrak{D}}(\tilde{\mathcal{J}})) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\rho_{\mathrm{II}})^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}\rho_{\mathfrak{w}\mathrm{I}}\right)}$

f. $d_{\mathfrak{e}}(\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})) \le \sqrt{\mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\eta_{\mathrm{II}})^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \eta_{\mathrm{I}\mathfrak{w}}\eta_{\mathfrak{w}\mathrm{I}}\right)}$

Proof: a. Applying $a_{\mathrm{I}} = 1$ and $b_{\mathrm{I}} = \left|\underline{\sigma_{\mathrm{I}}}\right|$ to the Cauchy-Schwarz inequality $(\sum_{\mathrm{I}=1}^{\mathfrak{s}} a_{\mathrm{I}}b_{\mathrm{I}})^2 \le (\sum_{\mathrm{I}=1}^{\mathfrak{s}} a_{\mathrm{I}}^2)(\sum_{\mathrm{I}=1}^{\mathfrak{s}} b_{\mathrm{I}}^2)$ and by Theorem 4.5,

$$\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left|\underline{\sigma_{\mathrm{I}}}\right|\right)^2 \le \left(\sum_{\mathrm{I}=1}^{\mathfrak{s}} 1\right)\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left|\underline{\sigma_{\mathrm{I}}}\right|^2\right)$$
$$\left(d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right)^2 \le \mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^2 + 2\sum_{1\le\mathrm{I}<\mathfrak{w}\le\mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}\right)$$

$$d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \leq \sqrt{\mathfrak{s}\left(\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right)}$$

In the same way, we can show b, c, d, e and f.

Theorem 4.7. Let $\tilde{\mathcal{J}}$ be CBFG and Υ be its minimum DS. Let $|\underline{\mathcal{J}}|, |\overline{\mathcal{J}}|, |\underline{\mathcal{K}}|, |\overline{\mathcal{K}}|, |\mathcal{F}|$ and $|\mathcal{L}|$ denote determinants of dominating matrices $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}(\tilde{\mathcal{J}})$ and $\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})$ respectively. Then

a. $d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathfrak{l}}^{\ell} + \mathfrak{s}(\mathfrak{s}-1)|\underline{\mathcal{J}}|^{\frac{2}{\mathfrak{s}}}}$

b. $d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{u}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{u}\rho_{\mathfrak{w}\mathfrak{l}}^{u} + \mathfrak{s}(\mathfrak{s}-1)|\overline{\mathcal{J}}|^{\frac{2}{\mathfrak{s}}}}$

c. $d_{\mathfrak{e}}\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\eta_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\eta_{\mathfrak{l}\mathfrak{w}}^{\ell}\eta_{\mathfrak{w}\mathfrak{l}}^{\ell} + \mathfrak{s}(\mathfrak{s}-1)|\underline{\mathcal{K}}|^{\frac{2}{\mathfrak{s}}}}$

d. $d_{\mathfrak{e}}\left(\eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\eta_{\mathfrak{l}\mathfrak{l}}^{u}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\eta_{\mathfrak{l}\mathfrak{w}}^{u}\eta_{\mathfrak{w}\mathfrak{l}}^{u} + \mathfrak{s}(\mathfrak{s}-1)|\overline{\mathcal{K}}|^{\frac{2}{\mathfrak{s}}}}$

e. $d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}(\rho_{\mathfrak{l}\mathfrak{l}})^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}\rho_{\mathfrak{w}\mathfrak{l}} + \mathfrak{s}(\mathfrak{s}-1)|\mathcal{F}|^{\frac{2}{\mathfrak{s}}}}$

f. $d_{\mathfrak{e}}\left(\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})\right) \geq \sqrt{\sum_{\mathfrak{l}=1}^{\mathfrak{s}}(\eta_{\mathfrak{l}\mathfrak{l}})^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\eta_{\mathfrak{l}\mathfrak{w}}\eta_{\mathfrak{w}\mathfrak{l}} + \mathfrak{s}(\mathfrak{s}-1)|\mathcal{L}|^{\frac{2}{\mathfrak{s}}}}$

Proof: a.

$$\left(d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right)^2 = \left(\sum_{\mathfrak{l}=1}^{\mathfrak{s}}|\underline{\sigma}_{\mathfrak{l}}|\right)^2 = \sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left|\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right|^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\left|\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\right|\left|\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right|$$
$$= \left(\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right) + \frac{2\mathfrak{s}(\mathfrak{s}-1)}{2}\underset{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}{AM}\left\{\left|\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\right|\left|\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right|\right\}$$

Since

$$\underset{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}{AM}\left\{\left|\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\right|\left|\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right|\right\} \geq \underset{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}{GM}\left\{\left|\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\right|\left|\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right|\right\}$$
$$= \left(\prod_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\left|\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\right|\left|\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right|\right)^{\frac{2}{\mathfrak{s}(\mathfrak{s}-1)}}$$
$$= \left(\prod_{1\leq\mathfrak{l}\leq\mathfrak{s}}\left|\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right|^{\mathfrak{s}-1}\right)^{\frac{2}{\mathfrak{s}(\mathfrak{s}-1)}} = \left(\prod_{1\leq\mathfrak{l}\leq\mathfrak{s}}\left|\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right|\right)^{\frac{2}{\mathfrak{s}}} = |\underline{\mathcal{J}}|^{\frac{2}{\mathfrak{s}}}$$
$$\left(d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right)^2 \geq \left(\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right) + \mathfrak{s}(\mathfrak{s}-1)|\underline{\mathcal{J}}|^{\frac{2}{\mathfrak{s}}}$$
$$d_{\mathfrak{e}}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right) \geq \sqrt{\left(\sum_{\mathfrak{l}=1}^{\mathfrak{s}}\left(\rho_{\mathfrak{l}\mathfrak{l}}^{\ell}\right)^2 + 2\sum_{1\leq\mathfrak{l}<\mathfrak{w}\leq\mathfrak{s}}\rho_{\mathfrak{l}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathfrak{l}}^{\ell}\right) + \mathfrak{s}(\mathfrak{s}-1)|\underline{\mathcal{J}}|^{\frac{2}{\mathfrak{s}}}}$$

Similarly, we can show b, c, d, e, and f.

Theorem 4.8. Let $\tilde{\mathcal{J}}$ be CBFG with dominating matrices $\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}}), \eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}), \rho_{\mathfrak{D}}(\tilde{\mathcal{J}})$ and $\eta_{\mathfrak{D}}(\tilde{\mathcal{J}})$. Let $\rho^{\ell}(\tilde{\mathcal{J}}), \rho^{u}(\tilde{\mathcal{J}}), \eta^{\ell}(\tilde{\mathcal{J}}), \eta^{u}(\tilde{\mathcal{J}}), \rho(\tilde{\mathcal{J}})$ and $\eta(\tilde{\mathcal{J}})$ be the adjacency matrices of $\tilde{\mathcal{J}}$. Then

a. $\left[d_{e}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^{2}+\left(\varepsilon\left(\rho^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2}\right]$

b. $[d_{e}(\rho_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}))]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\rho_{\mathrm{II}}^{u})^{2}+\left(\varepsilon(\rho^{u}(\tilde{\mathcal{J}}))\right)^{2}\right]$

c. $\left[d_{e}\left(\eta_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\eta_{\mathrm{II}}^{\ell}\right)^{2}+\left(\varepsilon\left(\eta^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2}\right]$

d. $[d_{e}(\eta_{\mathfrak{D}}^{u}(\tilde{\mathcal{J}}))]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\eta_{\mathrm{II}}^{u})^{2}+\left(\varepsilon(\eta^{u}(\tilde{\mathcal{J}}))\right)^{2}\right]$

e. $[d_{e}(\rho_{\mathfrak{D}}(\tilde{\mathcal{J}}))]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\rho_{\mathrm{II}})^{2}+\left(\varepsilon\left(\rho(\tilde{\mathcal{J}})\right)\right)^{2}\right]$

f. $[d_{e}(\eta_{\mathfrak{D}}(\tilde{\mathcal{J}}))]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}(\eta_{\mathrm{II}})^{2}+\left(\varepsilon(\eta(\tilde{\mathcal{J}}))\right)^{2}\right]$

Proof:

a.

$$\left(\varepsilon\left(\rho^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2} \geq 2\sum_{1\leq \mathrm{I}<\mathfrak{w}\leq \mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}+\mathfrak{s}(\mathfrak{s}-1)\left|\underline{\mathcal{J}}\right|^{\frac{2}{\mathfrak{s}}} \geq 2\sum_{1\leq \mathrm{I}<\mathfrak{w}\leq \mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}$$

$$\text{i.e. } 2\sum_{1\leq \mathrm{I}<\mathfrak{w}\leq \mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell} \leq \left(\varepsilon\left(\rho^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2} \tag{1}$$

Now

$$\left[d_{e}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right]^{2} \leq \mathfrak{s}\left(\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^{2}+2\sum_{1\leq \mathrm{I}<\mathfrak{w}\leq \mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}\right)$$

$$=\mathfrak{s}\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^{2}+2\mathfrak{s}\sum_{1\leq \mathrm{I}<\mathfrak{w}\leq \mathfrak{s}} \rho_{\mathrm{I}\mathfrak{w}}^{\ell}\rho_{\mathfrak{w}\mathrm{I}}^{\ell}$$

$$\left[d_{e}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right]^{2} \leq \mathfrak{s}\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^{2}+\mathfrak{s}\left(\varepsilon\left(\rho^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2} \text{ [by (1)]}$$

$$\left[d_{e}\left(\rho_{\mathfrak{D}}^{\ell}(\tilde{\mathcal{J}})\right)\right]^{2} \leq \mathfrak{s}\left[\sum_{\mathrm{I}=1}^{\mathfrak{s}}\left(\rho_{\mathrm{II}}^{\ell}\right)^{2}+\left(\varepsilon\left(\rho^{\ell}(\tilde{\mathcal{J}})\right)\right)^{2}\right]$$

Similarly, we can show b, c, d, e, and f.

5. APPLICATION OF DOMINATION IN CBFG

Social Network Exploration (SNE) is the process of applying graph theory and networks to the study of social structures. The components of a social network include Nodes (vertices), which can be people, organizations, or groups and edges (links), which can be connections or exchanges between them (communication, influence, friendship, trust, etc.).

Relationships in social networks in the actual world are not binary. Hence, cubic bipolar fuzzy graphs are useful in this situation. Every individual (node) and relationship (edge) may have:
1) A positive degree of membership: friendliness, cooperation, and trust.
2) A negative degree of membership: rivalry, conflict, and mistrust.
3) Interval-valued: To show the degree of uncertainty.

The concept of domination using bipolar strong edge of CBFG is applied to SNE. Some benefits of domination in SNE are determining influencers, i.e. finding a small subset of users who have the power to affect the network, which is beneficial for political campaigns; identifying key players within communities or subgroups; controlling misinformation in social media and so on. Hence, domination provides a strong tool for social network exploration in cubic bipolar fuzzy graphs.

5.1. Proposed Algorithm

Input: CBFG with vertices and edges representing the social network along with membership values.

Output: Identification of minimum dominating set.

Step 1. Construct the CBFG $\tilde{\mathcal{J}} = (\mathfrak{U}, \mathfrak{R})$ with vertices $\mathfrak{U} = \{\langle[\rho_{\mathfrak{U}}^{\ell}, \rho_{\mathfrak{U}}^{u}], [\eta_{\mathfrak{U}}^{\ell}, \eta_{\mathfrak{U}}^{u}], \rho_{\mathfrak{U}}, \eta_{\mathfrak{U}}\rangle\}$ is CBFS and edges $\mathfrak{R}$ satisfying the CBF relation given as $\rho_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \leq min(\rho_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \rho_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$, $\rho_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \leq min(\rho_{\mathfrak{U}}^{u}(\mathfrak{f}), \rho_{\mathfrak{U}}^{u}(\mathfrak{j}))$ along with $\rho_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \geq min(\rho_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \rho_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$, $\eta_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \geq max(\eta_{\mathfrak{U}}^{\ell}(\mathfrak{f}), \eta_{\mathfrak{U}}^{\ell}(\mathfrak{j}))$ along with $\eta_{\mathfrak{R}}^{\ell}(\mathfrak{f}\mathfrak{j}) \leq max(\eta_{\mathfrak{U}}^{u}(\mathfrak{f}), \eta_{\mathfrak{U}}^{u}(\mathfrak{j}))$, $\eta_{\mathfrak{R}}^{u}(\mathfrak{f}\mathfrak{j}) \geq max(\eta_{\mathfrak{U}}^{u}(\mathfrak{f}), \eta_{\mathfrak{U}}^{u}(\mathfrak{j}))$, $\rho_{\mathfrak{R}}(\mathfrak{f}\mathfrak{j}) \leq min(\rho_{\mathfrak{U}}(\mathfrak{f}), \rho_{\mathfrak{U}}(\mathfrak{j}))$ and $\eta_{\mathfrak{R}}(\mathfrak{f}\mathfrak{j}) \geq max(\eta_{\mathfrak{U}}(\mathfrak{f}), \eta_{\mathfrak{U}}(\mathfrak{j}))$ $\forall \mathfrak{f}, \mathfrak{j} \in \mathcal{X}$.

Step 2. Compute the bipolar strong edges, which are both positive strong and negative strong, by using the conditions: positive strong if $\rho_{\mathfrak{R}}^{\ell}(\mathfrak{q}, \mathfrak{x}) \geq \mathcal{F}^{\rho^{\ell},\infty}(\mathfrak{q}, \mathfrak{x}), \rho_{\mathfrak{R}}^{u}(\mathfrak{q}, \mathfrak{x}) \geq \mathcal{F}^{\rho^{u},\infty}(\mathfrak{q}, \mathfrak{x})$ and $\rho_{\mathfrak{R}}(\mathfrak{q}, \mathfrak{x}) \geq \mathcal{F}^{\rho,\infty}(\mathfrak{q}, \mathfrak{x})$ and negative strong if $\eta_{\mathfrak{R}}^{\ell}(\mathfrak{q}, \mathfrak{x}) \leq \mathcal{F}^{\eta^{\ell},\infty}(\mathfrak{q}, \mathfrak{x})$, $\eta_{\mathfrak{R}}^{u}(\mathfrak{q}, \mathfrak{x}) \leq \mathcal{F}^{\eta^{u},\infty}(\mathfrak{q}, \mathfrak{x})$ and $\eta_{\mathfrak{R}}(\mathfrak{q}, \mathfrak{x}) \leq \mathcal{F}^{\eta,\infty}(\mathfrak{q}, \mathfrak{x})$.

Step 3. Find all dominating sets Υ where $\Upsilon \subseteq \mathfrak{U}$ is known as a dominating set (DS) if $\forall \mathfrak{w} \in \mathfrak{U} \backslash \Upsilon, \exists \mathfrak{s} \in \Upsilon$ such that $\mathfrak{s}$ dominates $\mathfrak{w}$ and calculate its domination number using

$$|\boldsymbol{\Upsilon}| = \sum_{\mathfrak{s} \in \boldsymbol{\Upsilon}} \frac{\mathbf{1} - \boldsymbol{\rho}_{\mathfrak{U}}^{\boldsymbol{\ell}}(\mathfrak{s}) + \boldsymbol{\rho}_{\mathfrak{U}}^{\boldsymbol{u}}(\mathfrak{s}) + \boldsymbol{\rho}_{\mathfrak{U}}}{\mathbf{3}} + \sum_{\mathfrak{s} \in \boldsymbol{\Upsilon}} \frac{\mathbf{1} - \boldsymbol{\eta}_{\mathfrak{U}}^{\boldsymbol{\ell}}(\mathfrak{s}) + \boldsymbol{\eta}_{\mathfrak{U}}^{\boldsymbol{u}}(\mathfrak{s}) + \boldsymbol{\eta}_{\mathfrak{U}}}{\mathbf{3}}$$

Step 4. Obtain minimum domination number $|\Upsilon|$ which is the lowest cardinality of all the minimal DS in CBFG, and the minimum dominating set Υ is the DS with the lowest cardinality.

5.2. Example on Viral Marketing in Social Networks

A viral marketing campaign uses word-of-mouth communication to disseminate information. In recent days, many people utilize it to share information on social networks. Let us consider an example of a company $\mathcal{Y}$ needs to find active individuals to target for marketing initiatives in order to promote a new product or innovation, then a social network is represented as a simple undirected CBFG $\tilde{\mathfrak{I}} = (\mathfrak{U}, \mathfrak{R})$ where each vertex in $\mathfrak{U}$ represents a customer in marketing and the edges in $\mathfrak{R}$ represent the connection between two persons if they correspond to mutual influence between them. Let $\mathfrak{U} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N\}$ denote vertices of CBFG with $\langle \mathfrak{x}, [\rho^{\ell}(\mathfrak{x}), \rho^{u}(\mathfrak{x})], [\eta^{\ell}(\mathfrak{x}), \eta^{u}(\mathfrak{x})], \rho(\mathfrak{x}), \eta(\mathfrak{x}) : \mathfrak{x} \in \mathcal{X} \rangle$ values as given below:

$$A = ([0.3,0.5], [-0.4,0.2], 0.3, -0.5), B = ([0.4,0.6], [-0.6, -0.3], 0.2, -0.7),$$
$$C = ([0.2,0.4], [-0.3, -0.1], 0.2, -0.2), D([0.1,0.3], [-0.4, -0.2], 0.2, -0.3),$$
$$E = (0.2,0.3], [-0.5, -0.3], 0.4, -0.5), F = ([0.4,0.5], [-0.3, -0.1], 0.2, -0.3),$$
$$G = ([0.1,0.2], [-0.3, -0.1], 0.2, -0.2), H = ([0.3,0.5], [-0.5, -0.3], 0.3,0.6),$$
$$I = ([0.1,0.4], [-0.3, -0.1], 0.2, -0.2), J = ([0.2,0.6], [-0.6, -0.2], 0.5, -0.3),$$
$$K = ([0.3,0.5], [-0.6, -0.4], 0.3, -0.4), L = ([0.2,0.5], [-0.6, -0.2], 0.3, -0.3),$$
$$M = ([0.1,0.3], [-0.3, -0.1], 0.3, -0.4), N = ([0.3,0.7], [-0.4, -0.1], 0.3, -0.2)$$

In this context, a dominating set is a group of highly influential individuals who exert a powerfully favourable influence on others and maintain direct or indirect relationships with everyone. Here, a positive degree of membership of vertices indicates customers' support for the product, and a negative degree of membership indicates customers' resistance; a positive edge membership degree denotes a strong spread of influence, and a negative degree denotes the possibility of rejection. In order to find persons with high influence, we consider each edge in CBFG as bipolar strong edges. Hence using the definition of domination in CBFG, a minimum dominating set is determined. Here $\Upsilon = \{B, D, G, K, M\}$ with $|\Upsilon| = 3.74$. The people in the dominating set are key influencers who can reach most customers in the network, enabling the product to spread easily at minimal cost.

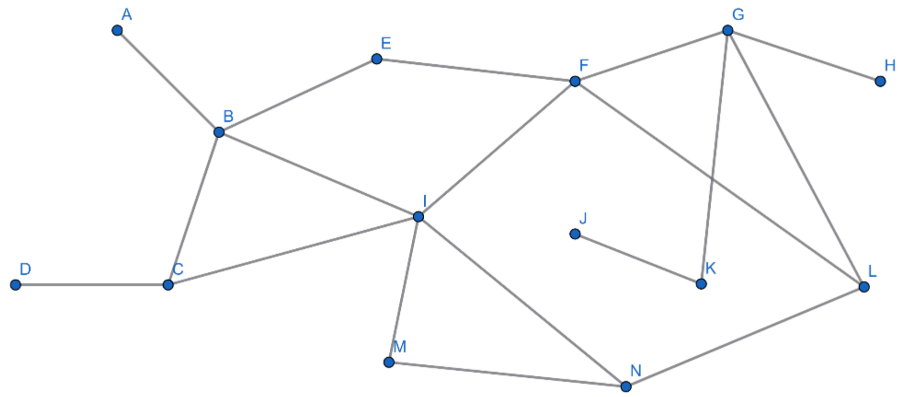

Figure 2. Social network as CBFG.

5.3. Comparison

In the proposed algorithm, CBFG is used for social network exploration because, in complex, real-world systems, membership values can denote both positive and negative aspects of each individual within a social network. Our proposed approach using CBFG provides a more comprehensive representation because it uses interval-valued bipolar membership for uncertain data and single-valued bipolar membership for precise data when compared with the cubic fuzzy graph and bipolar fuzzy graph. Moreover, it proves valuable for assessing the most influential individual in a communication network by using the concept of a dominating set in CBFG, since each node depends on its membership value and the domination number is not constant within the network.

6. CONCLUSIONS

The idea of domination in cubic bipolar fuzzy graphs based on bipolar strong edges was examined in this work. The dominating matrix in CBFG is defined, and the eigenvalues are computed. Next, using the previously determined eigenvalues, the dominant energy in CBFG is computed. Bounds of dominating energy in CBFG are also established, and theorems on domination in CBFG are examined. Finally, we demonstrated that dominating sets in cubic bipolar fuzzy graphs are important for determining the influence of different nodes, and the corresponding dominating energy values provide a novel way to compare different configurations of CBFGs. Dominating energy in additional classes of cubic bipolar fuzzy graphs may be investigated in future research.

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