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QUASI-HEMI-SLANT SUBMANIFOLDS OF QUASI-SASAKIAN MANIFOLD WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. *In this paper, we define and study quasi-hemi-slant submanifolds as a generalization of slant submanifolds, semi-slant submanifolds, and hemi-slant submanifolds of contact metric manifolds. Further, we obtain necessary and sufficient conditions for integrability of distributions which are involved in the definition of quasi-hemi-slant submanifolds of quasi-Sasakian manifolds with semi-symmetric non-metric connection. We also investigate the necessary and sufficient conditions for quasi-hemi-slant submanifolds of quasi-Sasakian manifolds with semi-symmetric non-metric connection to be totally geodesic and study the geometry of foliations determined by the distribution.*

Keywords: *quasi-Sasakian manifold; quasi-hemi-slant submanifold; semi-symmetric non-metric connection; totally geodesic; integrability conditions.*

Mathematics Subject Classification: *53C15; 53C25; 53C40.*

1. INTRODUCTION AND PRELIMINARIES

The study of the geometry of submanifolds originates from the exploration of the extrinsic geometry of surfaces and has evolved over time to address the geometry of submanifolds in more general ambient spaces. This field plays a significant role in diverse applications such as economic modelling, computer-aided design, image processing and mathematical physics and mechanics. Due to its wide-ranging applications, the geometry of submanifolds remains a vibrant and compelling area of research. The concept of slant submanifolds, which generalizes both holomorphic and totally real immersions, was first introduced by B. Y. Chen [1]. Since then, many geometers have explored various aspects of this topic over the past two decades [2-14]. A. Lotta [15] studied the properties of slant immersions of Riemannian manifolds into an almost contact metric manifold.

Further contributions by M. Ahmad and J. B. Jun included the study of CR-submanifolds of nearly hyperbolic Kenmotsu manifolds under semi-symmetric non-metric connections [16].

The notion of semi-slant submanifolds in Kaehlerian manifolds was explored by N. Papaghuic [17]. Over time, the slant submanifolds have been further generalized into several categories, including semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds, and quasi-slant submanifolds, etc., in various differentiable manifolds [18-20].

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The theory of quasi-Sasakian structures was initially introduced and studied by D.E. Blair [21] in 1967. Subsequent studies, such as [22], have examined slant and pseudo-slant submanifolds within quasi-Sasakian manifolds. R. Prasad and collaborators [23] investigated quasi-hemi-slant submanifolds in Sasakian manifolds, focusing on the integrability of distributions and conditions for total geodesicity. More recently, T. Khan and S. Rizvi proposed the concept of quasi-hemi-slant submanifolds in Lorentzian concircular structures in [24].

Let $\bar{M}$ be a real $(2n+1)$dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1, 1), ξ a vector field, η is a $1-$form and g is a Riemannian metric on $\bar{M}$, such that

$$\phi^2 = -I + \eta \otimes \xi, \eta \text{ o } \phi = 0, \eta(\xi) = 1 \tag{1}$$

$$g(X, \xi) = \eta(X), \phi(\xi) = 0 \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for any vector field X, Y tangent $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$of $\bar{M}$. An almost contact metric structure $(\phi, \xi, \eta, g)$is said to be normal [3] if an almost complex structure J on the product manifold $\bar{M}$x $R$is given by

$$J\left(U, f\frac{d}{dt}\right) = \left(\phi U - f\xi, \eta(U)f\frac{d}{dt}\right),$$

where $J^2 = -I$ and f is a differentiable function on $\bar{M}$x R and J is integrable. The condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a quasi-Sasakian manifold [25] if

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y).\xi, \tag{4}$$

where A is a symmetric linear transformation field, ∇ which denotes the Riemannian connection of g on $\bar{M}$. We have, on a quasi-Sasakian manifold $\bar{M}$

$$\nabla_X \xi = \phi AX \tag{5}$$

Now, we define a semi-symmetric non-metric connection $\bar{\nabla}$as

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{6}$$

such that

$$\bar{\nabla}_X g(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y)$$

for any vector field X, Y, Z tangent to $\bar{M}$.

Using equations (4) and (6), we have

$$\begin{gathered}\bar{\nabla}_X \phi Y - \eta(\phi Y)X - \phi(\bar{\nabla}_X Y - \eta(Y)X) = \eta(Y)AX - g(AX, Y)\xi \\ \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) + \eta(Y)\phi X = \eta(Y)AX - g(AX, Y)\xi \\ (\bar{\nabla}_X \phi)Y + \eta(Y)\phi X = \eta(Y)AX - g(AX, Y)\xi \\ (\bar{\nabla}_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi - \eta(Y)\phi X\end{gathered} \tag{7}$$

Now, substituting $Y = \xi$ in (6), we have

$$\nabla_X \xi = \overline{\nabla}_X \xi - \eta(\xi) X \tag{8}$$

Now, from Equations (5) and (8), we have

$$\phi \mathrm{A} X = \overline{\nabla}_X \xi - \eta(\xi) X$$

Since, $\eta(\xi) = 1$, we have

$$\overline{\nabla}_X \xi = \phi \mathrm{A} X + X \tag{9}$$

for any vector field X, Y tangent $\overline{M}$. Now, let M be a Riemannian manifold isometrically immersed in $\overline{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{10}$$

$$\overline{\nabla}_X V = -\mathrm{A}_V X + \nabla_X^{\perp} V \tag{11}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp} M)$, where ∇ is an induced connection on the normal bundle $T^{\perp} M$ of M and A_V is the shape operator of M with the normal vector $V \in \Gamma(T^{\perp} M)$. Moreover, σ, the second fundamental form and A_V, the shape operator (also called the Weingarten map) is associated with V as

$$g(\mathrm{A}_V X, Y) = g(\sigma(X, Y), V) \tag{12}$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{n} trace(\sigma) = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i) \tag{13}$$

where n is the dimension of M and $\{e_1, e_2, \dots \dots \dots, e_n\}$ is a local orthogonal frame of M. A submanifold M of an almost contact metric manifold $\overline{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y) H \tag{14}$$

where H is the mean curvature vector. A submanifold M is said to be totally geodesic if $\sigma(X, Y) = 0$, for each $X, Y \in \Gamma(TM)$ and M is said to be a minimal submanifold if $H = 0$.

For any $X \in TM$, we can write

$$\phi X = TX + NX \tag{15}$$

where TX and NX are the tangential and normal components of ϕX on M, respectively. Similarly, for any $V \in T^{\perp} M$, we have

$$\phi V = tV + nV \tag{16}$$

where tV and nV are the tangential and normal components of ϕV on M, respectively. The covariant derivatives of projection morphisms in (11) and (12) are defined as

$$\begin{gathered}(\overline{\nabla}_X T) Y = \nabla_X TY - T \nabla_X Y, (\overline{\nabla}_X N) Y = \nabla_X^{\perp} NY - N \nabla_X Y, \\ (\overline{\nabla}_X t) V = \nabla_X tV - t \nabla_X V, (\overline{\nabla}_X n) V = \nabla_X^{\perp} nV - n \nabla_X V\end{gathered} \tag{17}$$

for any $X, Y \in \Gamma(TM)$ and $V \in T^{\perp}M$.

Definition 2.1. *A submanifold M of an almost contact metric manifold $\overline{M}$ is said to be semi-slant [12], if there exist two orthogonal complementary distributions D and D^{θ} on M such that*

$$TM = D \oplus D^{\theta} \oplus < \xi >$$

where D is invariant and D^{θ} is slant with slant angle θ. In this case, the angle θ is called the semi-slant angle.

Definition 2.2. *A submanifold M of an almost contact metric manifold $\overline{M}$ is said to be semi-slant [26], [27] if there exist two orthogonal complementary distributions D^{θ} and $D^{\perp}$ on M such that*

$$TM = D^{\theta} \oplus D^{\perp} \oplus < \xi >$$

where D^{θ} is slant with slant angle θ and $D^{\perp}$ is anti-invariant. In this case, the angle θ is called the hemi-slant angle.

3. QUASI HEMI-SLANT SUBMANIFOLDS OF QUASI-SASAKIAN MANIFOLDS WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

Definition 3.1. *A submanifold M of a quasi-Sasakian manifold $\overline{M}$ is called a quasi-hemi-slant submanifold if there exists a distribution D, D^{θ} and $D^{\perp}$ on M such that*
(i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D^{\theta} \oplus D^{\perp} \oplus < \xi > \tag{18}$$

(ii) The distribution D is ϕ invariant, i.e. $\phi D = D$.
(iii) For any non-zero vector field $X \in (D^{\theta})_p, p \in M$, the angle θ between JX and $(D^{\theta})_p$ is constant and independent of the choice of point p and X in $(D^{\theta})_p$.
(iv) The distribution $D^{\perp}$ is ϕ anti-invariant, i.e. $\phi D^{\perp} \subseteq T^{\perp}M$.

In this case, we call θ quasi-hemi-slant angle of M. Suppose the dimension of distributions D, D^{θ} and $D^{\perp}$ are n_1, n_2 , n_3 respectively. Then we can easily see the following cases
(i) If n_1=0, M is a hemi-slant submanifold.
(ii) If n_2=0, M is a semi-invariant submanifold.
(iii) If n_3=0, M is a semi-slant submanifold.

We say that a quasi-hemi-slant submanifold M is proper if $D \neq 0, D^{\perp} \neq 0$ and $\theta \neq 0, \pi/2$. This means that the notion of quasi-hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, and semi-slant submanifolds, and also, they are examples of quasi-hemi-slant submanifolds.

Remark 3.2. The definition can be generalized by taking $TM = D \oplus D^{\theta_1} \oplus D^{\theta_2} \dots\dots\dots \oplus D^{\theta_k} \oplus D^{\perp} \oplus < \xi >$. Hence, we can define multi-slant submanifolds, quasi-multi-slant submanifolds, quasi-hemi-multi-slant submanifolds, etc. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\overline{M}$. We denote the projection of $X \in \Gamma(\mathrm{TM})$on the distributions D, D^{θ} and $D^{\perp}$ by P, Q and R respectively. Then we can write

$$X = PX + QX + RX + \eta(X).\xi \tag{19}$$

Now we put,

$$\phi X = TX + NX \tag{20}$$

where TX and NX are the tangential and normal components of ϕX on M. Using (19) and (20), we have

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX \tag{21}$$

Since, $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}N$, we have $NPX = 0$ and $TRX = 0$. So, we get

$$\phi X = TPX + TQX + NQX + NRX \tag{22}$$

Then for any $X \in \Gamma(\mathrm{TM})$, it is easy to write

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX$$

So, from (22), we have the following decomposition

$$\phi(TM) = D \oplus TD^{\theta} \oplus ND^{\theta} \oplus ND^{\perp} \tag{23}$$

where, $'\oplus'$ denotes orthogonal direct sum. Since $ND^{\theta} \subset T^{\perp}M$ and $D^{\perp} \subset T^{\perp}M$, we have

$$T^{\perp}M = ND^{\theta} \oplus ND^{\perp} \oplus \mu \tag{24}$$

where μ is the orthogonal complement of $ND^{\theta} \oplus ND^{\perp}$ in$\Gamma(T^{\perp}\mathrm{M})$and it is invariant with respect to ϕ. For any non-zero vector field $V \in \Gamma(T^{\perp}\mathrm{M})$. We put

$$\phi T = tV + nV \tag{25}$$

where, $tV \in \Gamma(D^{\theta} \oplus D^{\perp})$ and $V \in \Gamma(\mu)$.

Proposition 3.3. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection, then for any $X \in \Gamma(TM)$, we have

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - t\sigma(X,Y) = \eta(Y)AX - g(AX,Y)\xi - \eta(Y)\phi X,$$
$$\sigma(X,TY) + \nabla_X^{\perp} NY - N\nabla_X Y - \eta\sigma(X,Y) = 0$$

and

$$TD = D, TD^{\theta} = D^{\theta}, TD^{\perp} = 0, tND^{\theta} = D^{\theta}, tND^{\perp} = D^{\perp}$$

Proof: By using Equations (7), (10), and (11) and equating tangential and normal components.

Proposition 3.4. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then the endomorphism T and N, t and n in the tangent bundle of M, satisfy the following identities,
(i) $T^2 + tn = -I + \eta \otimes \xi$ *on* TM,
(ii) $NT + nN = 0$on TM
(iii) $Nt + n^2 = -I$ on TM,

(iv) $Tt + tn = 0$ on $T^{\perp}M$
where I is the identity.

Proof: By using Equations (20) and (25), and using $\phi^2 = -I + \eta \otimes \xi$, then on equating tangential and normal components.

Lemma 3.5. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection, then
(i)$T^2X = cos^2\theta X$,
(ii) $g(TX, TY) = cos^2\theta g(X, Y)$
(iii) $g(NX, NY) = sin^2\theta g(X, Y)$
for any $X, Y \in D^{\theta}$.

Proof: The proof follows similar steps as in Proposition 3.4. [22]

Proposition 3.6. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection, then

$$(\bar{\nabla}_X T)Y = A_{NY}X + t\sigma(X, Y) + \eta(Y)AX - g(AX, Y)\xi - (Y)\phi X,$$
$$(\bar{\nabla}_X N)Y = n\sigma(X, Y) - \sigma(X, TY)$$

and

$$(\bar{\nabla}_X \eta)V = -\sigma(X, tY) - NA_V X$$

for any $X, Y \in \Gamma(\mathrm{TM})$ and $V \in \Gamma(T^{\perp}\mathrm{M})$.

Proof: By using Equations (7), (10), (11), (15), (16), and (17), and equating tangential and normal components.

Lemma 3.7. Let M be a quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection, then

$$A_{\phi X}Y = A_{\phi Y}X - T([Y, X]) + g(AY, X)\xi - g(AX, Y)\xi$$

and

$$\nabla^{\perp}_X \phi Y = \nabla^{\perp}_Y \phi X + N([X, Y])$$

for all $X, Y \in D^{\perp}$.

Proof: For all $X, Y \in D^{\perp}$ and using covariant differentiation in Equation (7), we have

$$\bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) = \eta(Y)AX - g(AX, Y)\xi - \eta(Y)\phi X$$

By using (10) and (11), we have

$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y + \sigma(X, Y)) = -g(AX, Y)\xi$$
$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y) - \phi(\sigma(X, Y)) = -g(AX, Y)\xi$$

By using (15) and (16), we have

$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - T(\nabla_X Y) - N(\nabla_X Y) - t\big(\sigma(X, Y)\big) - n(\sigma(X, Y)) = -g(AX, Y)\xi$$
$$A_{\phi Y}X + T(\nabla_X Y) - t\big(\sigma(X, Y)\big) = g(AX, Y)\xi \tag{26}$$

$$\nabla^{\perp}_X \phi Y = N(\nabla_X Y) + n(\sigma(X, Y)) \tag{27}$$

Interchanging X and Y in (26) and (27), we have

$$A_{\phi X}Y = A_{\phi Y}X - T([Y,X]) + g(AY,X)\xi - g(AX,Y)\xi$$

and

$$\nabla^{\perp}_{X}\phi Y = \nabla^{\perp}_{Y}\phi X + N([X,Y])$$

for all $X, Y \in D^{\perp}$.

4. INTEGRABILITY OF DISTRIBUTIONS

Theorem 4.1. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection, then the distribution $D \oplus < \xi >$ is integrable if and only if*

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(\sigma(Y,TX) - \sigma(X,TY), NQZ + NRZ) \tag{28}$$

for all $X, Y \in \Gamma(\mathrm{D} \oplus < \xi >$ and $Z \in \Gamma\left(D^{\theta} \oplus D^{\perp}\right)$.

Proof: For all $X, Y \in \Gamma(\mathrm{D} \oplus < \xi >$ and Z$\in \Gamma\left(D^{\theta} \oplus D^{\perp}\right)$, we know that

$$g([X,Y],Z) = g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_Y X, Z)$$

Using (3) in the above equation, we have

$$g([X,Y],Z) = g(\phi\overline{\nabla}_X Y, \phi Z) - g(\phi\overline{\nabla}_Y X, \phi Z) \tag{29}$$

Using (7) and differentiating covariantly, we have

$$\phi(\overline{\nabla}_X Y) = \overline{\nabla}_X \phi Y + g(AX,Y)\xi + \eta(Y)\phi X - \eta(Y)AX$$
$$g(\phi(\overline{\nabla}_X Y), \phi Z) = g(\overline{\nabla}_X \phi Y + g(AX,Y)\xi + \eta(Y)\phi X - \eta(Y)AX, \phi Z)$$

As $\phi Z \in T^{\perp}M$, we have

$$g(\phi(\overline{\nabla}_X Y), \phi Z) = g(\overline{\nabla}_X \phi Y, \phi Z)$$

Similarly,

$$g(\phi(\overline{\nabla}_Y X), \phi Z) = g(\overline{\nabla}_Y \phi X, \phi Z)$$

Therefore, from (29), we have

$$g([X,Y],Z) = g(\overline{\nabla}_X \phi Y, \phi Z) - g(\overline{\nabla}_Y \phi X, \phi Z) \tag{30}$$

By using (10) in the above equation, we have

$$g([X,Y],Z) = g(\nabla_X \phi Y + \sigma(X,\phi Y), \phi Z) - g(\nabla_Y \phi X + \sigma(Y,\phi X), \phi Z)$$

By using (15) in the above equation, we have

$$g([X,Y],Z) = g(\nabla_X(TY+NY) + \sigma(X,TY+NY), \phi Z) - g(\nabla_Y(TX+NX) + \sigma(Y,TX+NX), \phi Z)$$
$$g([X,Y],Z) = g(\nabla_X TY + \sigma(X,TY), \phi Z) - g(\nabla_Y TX + \sigma(Y,TX), \phi Z)$$

Let $Z = QZ + RZ$, then by using (20) in the above equation, we have

$$g([X,Y],Z) = g(\nabla_X TY + \sigma(X,TY), TQZ + TRZ + NQZ + NRZ)$$
$$-g(\nabla_Y TX + \sigma(Y,TX), TQZ + TRZ + NQZ + NRZ)$$

As $\phi D^\perp \in T^\perp M$, so $TRZ = 0$, then we have

$$g([X,Y],Z) = g(\nabla_X TY, TQZ) + g(\sigma(X,TY), NQZ + NRZ)$$
$$-g(\nabla_Y TX, TQZ) - g(\sigma(Y,TX), NQZ + NRZ)$$
$$g([X,Y],Z) = g(\nabla_X TY - \nabla_Y TX, TQZ) + g(\sigma(X,TY) - \sigma(Y,TX), NQZ + NRZ)$$

As D $\oplus < \xi >$ is integrable, we have

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(\sigma(Y,TX) - \sigma(X,TY), NQZ + NRZ)$$

which is the required result.

Theorem 4.2. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then the slant distribution $D^\theta \oplus < \xi >$ is integrable if and only if*

$$g(A_{NTZ}Y - A_{NTY}Z, W) = g(A_{NZ}Y - A_{NY}Z, TPW) + g\left(\nabla^\perp_Z NY - \nabla^\perp_Y NZ, NRW\right) \tag{31}$$

for all $Y, Z \in \Gamma(D^\theta \oplus < \xi >)$ and $W \in \Gamma\left(D \oplus D^\perp\right)$.

Proof: For any$Y, Z \in \Gamma(D^\theta + < \xi >)$ and $W = PW + RW \in \Gamma(\mathrm{D} + D^\perp)$, we know that

$$g([Y,Z],W) = g(\bar{\nabla}_Y Z, W) - g(\bar{\nabla}_Z Y, W)$$

By using (3) and (7) and differentiating covariantly, we have

$$g([Y,Z],W) = g(\bar{\nabla}_Y \phi Z, \phi W) - g(\bar{\nabla}_Z \phi Y, \phi W) \tag{32}$$

Using (15) in (32), we have

$$g([Y,Z],W) = g(\bar{\nabla}_Y (TZ + NZ), \phi W) - g(\bar{\nabla}_Z (TY + NY), \phi W)$$
$$g([Y,Z],W) = g(\bar{\nabla}_Y TZ, \phi W) + g(\bar{\nabla}_Y NZ, \phi W) - g(\bar{\nabla}_Z TY, \phi W) - g(\bar{\nabla}_Z NY, \phi W)$$
$$g([Y,Z],W) = -g(\bar{\nabla}_Y \phi TZ, W) + g(\bar{\nabla}_Z \phi TY, W) + g(\bar{\nabla}_Y NZ, \phi W) - g(\bar{\nabla}_Z NY, \phi W)$$

By using (11) and (15) in the above equation, we have

$$g([Y,Z],W) = -g(\bar{\nabla}_Y T(TZ) + N(TZ), W) + g(\bar{\nabla}_Z T(TY) + N(TY), W)$$
$$+g\left(-A_{NZ}Y + \nabla^\perp_Y NZ, \phi W\right) + g\left(-A_{NY}Z + \nabla^\perp_Z NY, \phi W\right)$$
$$g([Y,Z],W) = -g(\bar{\nabla}_Y T^2 Z, W) - g(\bar{\nabla}_Y NTZ, W) + g(\bar{\nabla}_Z T^2 Y, W)$$
$$+g(\bar{\nabla}_Z NTY, W) + g\left(-A_{NZ}Y + \nabla^\perp_Y NZ, \phi W\right) + g\left(-A_{NY}Z + \nabla^\perp_Z NY, \phi W\right)$$
$$g([Y,Z],W) = g(A_{NY}Z, \phi W) - g(A_{NZ}Y, \phi W) + g\left(\nabla^\perp_Y NZ, \phi W\right) - g(\nabla^\perp_Z NY, \phi W)$$
$$-g(\bar{\nabla}_Y T^2 Z, W) + g(\bar{\nabla}_Z T^2 Y, W) + g(\bar{\nabla}_Z NTY, W) - g(\bar{\nabla}_Y NTZ, W)$$

By using (11) in the above equation, we have

$$g([Y,Z],W) = g(A_{NY}Z, \phi W) - g(A_{NZ}Y, \phi W) + g\left(\nabla^\perp_Y NZ, \phi W\right) - g(\nabla^\perp_Z NY, \phi W)$$

$$-g(\overline{\nabla}_Y T^2 Z, W) + g(\overline{\nabla}_Z T^2 Y, W) + g\left(-A_{NTY}Z + \nabla_Z^{\perp} NTY, W\right)$$
$$-g\left(-A_{NTZ}Y + \nabla_Y^{\perp} NTZ, W\right)$$
$$g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y, \phi W) + g\left(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY, \phi W\right)$$
$$-g(\overline{\nabla}_Y T^2 Z - \overline{\nabla}_Z T^2 Y, W) + g(A_{NTZ}Y - A_{NTY}Z, W)$$

By using (20) and Lemma 3.5 in the above equation, we have

$$g([Y,Z],W) = g(A_{NY}Z - A_{NZ}Y, TPW + TRW + NPW + NRW)$$
$$+g\left(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY, TPW + TRW + NPW + NRW\right)$$
$$+cos^2\theta([Y,Z],W) + g(A_{NTZ}Y - A_{NTY}Z, W)(1 - cos^2\theta)([Y,Z],W)$$

As distribution $D \oplus < \xi >$ is integrable

$$g(A_{NTZ}Y - A_{NTY}Z, W) = g(A_{NY}Z - A_{NZ}Y, TPW) + g(\nabla_Y^{\perp} NZ - \nabla_Z^{\perp} NY, NRW)$$

which is the required result.

Theorem 4.3. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection, then the distribution $D^{\perp}$ is integrable, if and only if*

$$g(T[Z,W],TY) = g(N[W,Z],NQY) \tag{33}$$

for all $Z, W \in \Gamma(D^{\perp})$ and $Y \in \Gamma(\mathrm{D} \oplus D^{\theta})$.

Proof: For any$Z, W \in \Gamma(D^{\perp})$ and $Y = PY + QY \in \Gamma(D + D^{\theta})$, we know that

$$g([Z,W],Y) = g(\overline{\nabla}_Z W, Y) - g(\overline{\nabla}_W Z, Y)$$

Using (2) and (7) and differentiating covariantly, we have

$$g([Z,W],Y) = g(\overline{\nabla}_Z \phi W, \phi Y) - g(\overline{\nabla}_W \phi Z, \phi Y)$$

Using (11) in the above equation, we have

$$g([Z,W],Y) = g(-A_{\phi W}Z + \nabla_Z^{\perp}\phi W, \phi Y) - g(-A_{\phi Z}W + \nabla_W^{\perp}\phi Z, \phi Y)$$
$$g([Z,W],Y) = g(A_{\phi Z}W - A_{\phi W}Z, \phi Y) - g(\nabla_W^{\perp}\phi Z - \nabla_Z^{\perp}\phi W, \phi Y)$$

Using (20) in the above equation, we have

$$g([Z,W],Y) = g\left(A_{\phi Z}W - A_{\phi W}Z, TPY + TQY + NPY + NQY\right)$$
$$-g(\nabla_W^{\perp}\phi Z - \nabla_Z^{\perp}\phi W, TPY + TQY + NPY + NQY)$$
$$g([Z,W],Y) = g\left(A_{\phi Z}W - A_{\phi W}Z, TPY + TQY\right) - g(\nabla_W^{\perp}\phi Z - \nabla_Z^{\perp}\phi W, NQY)$$

Using Lemma 3.7 in the above equation, we have

$$g([Z,W],Y) = g(-T[W,Z] + g(AW,Z)\xi - g(AZ,W)\xi, TY) - g(-N[Z,W], NQY)$$
$$g([Z,W],Y) = g(T[Z,W],TY) - g(N[W,Z],NQY)$$
$$g(T[Z,W],TY) = g(N[W,Z],NQY)$$

which is the required result.

5. TOTALLY GEODESIC FOLIATIONS

Theorem 5.1. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then the distribution $D \oplus < \xi >$ defines a totally geodesic foliation on M, if and only if*

$$g(\nabla_X TY, TQZ) = -g(\sigma(X, TY), NQZ + NRZ) \tag{34}$$

$$g(\nabla_X TY, tV) = -g(\sigma(X, TY), nV) \tag{35}$$

for all $X, Y \in \Gamma(\mathrm{D} \oplus < \xi >)$ and $Z \in \Gamma(D^\theta \oplus D^\perp)$ and $V \in \Gamma(T^\perp M)$.

Proof: For any$X, Y \in \Gamma(D \oplus < \xi >)$ and $Z = QZ + RZ \in \Gamma(D^\theta \oplus D^\perp)$ and using (3) and (7), we have

$$g(\overline{\nabla}_X Y, Z) = g(\overline{\nabla}_X \phi Y, \phi Z)$$

Using (15) and $NY = 0$ in the above equation, we have

$$g(\overline{\nabla}_X Y, Z) = g(\overline{\nabla}_X TY, \phi Z)$$

Using (10) and (20), we have

$$\begin{aligned} g(\overline{\nabla}_X Y, Z) &= g(\nabla_X TY + \sigma(X, TY), TQZ + TRZ + NQZ + NRZ\ ) \\ g(\overline{\nabla}_X Y, Z) &= g(\nabla_X TY, TQZ) + g(\sigma(X, TY), NQZ + NRZ\ ) \end{aligned} \tag{36}$$

Now, for any $X, Y \in \Gamma(\mathrm{D} \oplus < \xi >)$, $V \in \Gamma(T^\perp M)$ and using (2.3), (2.7), (3.3) and $NY = 0$, we have

$$g(\overline{\nabla}_X Y, V) = g(\overline{\nabla}_X TY, \phi V)$$

Using (10) and (16), we have

$$\begin{aligned} g(\overline{\nabla}_X Y, V) &= g(\overline{\nabla}_X TY + \sigma(X, TY), tV + nV) \\ g(\overline{\nabla}_X Y, V) &= g(\overline{\nabla}_X TY, tV) + g(\sigma(X, TY), nV) \end{aligned} \tag{37}$$

Since the distribution $\mathrm{D} \oplus < \xi >$ defines a totally geodesic foliation on M, then from (36) and (37), we have

$$\begin{aligned} g(\nabla_X TY, TQZ) &= -g(\sigma(X, TY), NQZ + NRZ) \\ g(\nabla_X TY, tV) &= -g(\sigma(X, TY), nV) \end{aligned}$$

which are the required results?

Theorem 5.2. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then the distribution $D^\perp$ defines a totally geodesic foliation on M, if and only if*

$$g(A_{NZ} Y, TPW + TQW) = g(\nabla^\perp_Y NZ, NQW) \tag{38}$$

$$g(A_{NZ} Y, tV) = g(\nabla^\perp_Y NZ, nV) \tag{39}$$

for all $Y, Z \in \Gamma(D^\perp)$ and $W \in \Gamma(D \oplus D^\theta)$ and $V \in \Gamma(T^\perp M)$.

Proof: For any$Y, Z \in \Gamma(D^{\perp})$ and $W = PW + QW \in \Gamma(D \oplus D^{\theta})$ and using (3) and (7), we have

$$g(\overline{\nabla}_Y Z, W) = g(\overline{\nabla}_Y \phi Z, \phi W)$$

By using (15) and $TZ = 0$ in above equation, we have

$$g(\overline{\nabla}_Y Z, W) = g(\overline{\nabla}_Y NZ, \phi W)$$

Using (11) and (20), we have

$$\begin{gathered} g(\overline{\nabla}_Y Z, W) = g\left(-\mathrm{A}_{NZ}Y + \nabla_Y^{\perp} NZ, TPW + TQW + NPW + NQW\right) \\ g(\overline{\nabla}_Y Z, W) = -g(\mathrm{A}_{NZ}Y, TPW + TQW) + g(\nabla_Y^{\perp} NZ, NQW) \end{gathered} \tag{40}$$

For any $Y, Z \in \Gamma(D^{\perp})$, $V \in \Gamma(T^{\perp}M)$ and by using (2.3), (2.7), (3.3) and $TZ = 0$, we have

$$g(\overline{\nabla}_Y Z, V) = g(\overline{\nabla}_Y NZ, \phi V)$$

By using (11) and (16), we have

$$\begin{gathered} g(\overline{\nabla}_Y Z, V) = g\left(-\mathrm{A}_{NZ}Y + \nabla_Y^{\perp} NZ, tV + nV\right) \\ g(\overline{\nabla}_Y Z, V) = -g\left(\mathrm{A}_{NZ}Y, tV\right) + g\left(\nabla_Y^{\perp} NZ, nV\right) \end{gathered} \tag{41}$$

Since the distribution $D^{\perp}$ defines a totally geodesic foliation on M, then from (40) and (41), we have

$$\begin{gathered} g(\mathrm{A}_{NZ}Y, TPW + TQW) = g\left(\nabla_Y^{\perp} NZ, NQW\right) \\ g(A_{NZ}Y, tV) = g(\nabla_Y^{\perp} NZ, nV) \end{gathered}$$

which are the required results?

Theorem 5.3. *Let M be a proper quasi-hemi-slant submanifold of a quasi-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection, then the distribution $D^{\theta} \oplus < \xi >$ defines a totally geodesic foliation on M, if and only if*

$$g\left(\nabla_Z^{\perp} NW, NRX\right) = g(A_{NW}Z, TPX) - g(A_{NTW}Z, X) \tag{42}$$

$$g(A_{NW}Z, tV) = g\left(\nabla_Z^{\perp} NW, nV\right) - g\left(\nabla_Z^{\perp} NTW, V\right) \tag{43}$$

for all $Z, W \in \Gamma(D^{\theta} \oplus < \xi >)$ and $X \in \Gamma(D \oplus D^{\perp})$ and $V \in \Gamma(T^{\perp}M)$.

Proof: For any$Z, W \in \Gamma(D^{\theta} \oplus < \xi >)$ and $X = PX + RX \in \Gamma(D \oplus D^{\perp})$ and by using (3), (7), and (15), we have

$$\begin{gathered} g(\overline{\nabla}_Z W, X) = g(\overline{\nabla}_Z TW, \phi X) + g(\overline{\nabla}_Z NW, \phi X) \\ g(\overline{\nabla}_Z W, X) = -g(\overline{\nabla}_Z \phi TW, X) + g(\overline{\nabla}_Z NW, \phi X) \end{gathered}$$

By using (11) and (15), we have

$$\begin{gathered} g(\overline{\nabla}_Z W, X) = -g(\overline{\nabla}_Z (T^2 W + NTW), X) + g\left(-A_{NW}Z + \nabla_Z^{\perp} NW, \phi X\right) \\ g(\overline{\nabla}_Z W, X) = -g(\overline{\nabla}_Z T^2 W, X) - g(\overline{\nabla}_Z NTW, X) - g(A_{NW}Z, \phi X) + g(\nabla_Z^{\perp} NW, \phi X) \end{gathered}$$

By using (11) and Lemma 3.5 in the above equation, we have

$$g(\overline{\nabla}_Z W, X) = cos^2\theta g(\overline{\nabla}_Z W, X) - g\big(-A_{NTW}Z + \nabla^{\perp}_Z NTW, X\big) - g(A_{NW}Z, \phi X)$$
$$+g(\nabla^{\perp}_Z NW, \phi X)$$
$$(1 - cos^2\theta)g(\overline{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(\nabla^{\perp}_Z NTW, X) - g(A_{NW}Z, \phi X)$$
$$+g(\nabla^{\perp}_Z NW, \phi X)$$
$$(1 - cos^2\theta)g(\overline{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, \phi X) + g(\nabla^{\perp}_Z NW, \phi X)$$

By using (20) in the above equation, we have

$$(1 - cos^2\theta)g(\overline{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, TPX + TRX + NPX + NRX)$$
$$+g(\nabla^{\perp}_Z NW, TPX + TRX + NPX + NRX)$$

By using the fact that $NPX = 0$, we have

$$(1 - cos^2\theta)g(\overline{\nabla}_Z W, X) = g(A_{NTW}Z, X) - g(A_{NW}Z, TPX) + g(\nabla^{\perp}_Z NW, NRX) \tag{44}$$

Now for any $Z, W \in \Gamma(D^\theta \oplus <\xi>)$, $V \in \Gamma(T^{\perp}M)$ and by using (3), (7) and (15), we have

$$g(\overline{\nabla}_Z W, V) = -g(\overline{\nabla}_Z \phi TW, V) + g(\overline{\nabla}_Z NW, \phi V)$$

By using (11) and (15), we have

$$g(\overline{\nabla}_Z W, V) = -g(\overline{\nabla}_Z (T^2W + NTW), V) + g\big(-A_{NW}Z + \nabla^{\perp}_Z NW, \phi V\big)$$
$$g(\overline{\nabla}_Z W, V) = -g(\overline{\nabla}_Z T^2W, V) - g(\overline{\nabla}_Z NTW, V) + g\big(-A_{NW}Z + \nabla^{\perp}_Z NW, \phi V\big)$$

By using (11), (16) and Lemma 3.5, we have

$$g(\overline{\nabla}_Z W, V) = cos^2\theta g(\overline{\nabla}_Z W, V) - g\big(-A_{NTW}Z + \nabla^{\perp}_Z NTW, V\big) - g(A_{NW}Z, tV + nV)$$
$$+g\big(\nabla^{\perp}_Z NW, tV + nV\big) \tag{45}$$
$$(1 - cos^2\theta)g(\overline{\nabla}_Z W, X) = -g\big(\nabla^{\perp}_Z NTW, V\big) - g(A_{NW}Z, tV) + g(\nabla^{\perp}_Z NW, nV)$$

Since the distribution $D^\theta \oplus <\xi>$ defines a totally geodesic foliation on M, then from (44) and (45), we have

$$g\big(\nabla^{\perp}_Z NW, NRX\big) = g(A_{NW}Z, TPX) - g(A_{NTW}Z, X)$$
$$g(A_{NW}Z, tV) = g\big(\nabla^{\perp}_Z NW, nV\big) - g\big(\nabla^{\perp}_Z NTW, V\big)$$

which are the required results?

6. CONCLUSION

The study introduced and analyzed quasi-hemi-slant submanifolds of quasi-Sasakian manifolds with semi-symmetric non-metric connection, deriving integrability and totally geodesic conditions. The study generalizes various submanifold structures in contact geometry. Future research may examine curvature behaviour, stability, and applications in physics, extending the theory to other manifolds and affine connections to provide broader geometric insights.

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