ORIGINAL PAPER

ON COPPER-RIEMANNIAN STRUCTURES

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Abstract. *The main goal of this study is to investigate copper-Riemannian structures. An integrability condition for a copper-Riemannian structure is investigated via the Tachibana operator, which is applied to pure tensor fields. Afterwards, the twin copper-Riemannian metric is studied. The copper structure is investigated on the tangent bundle of a Riemannian manifold, equipped with the vertical-rescaled metric.*

Keywords: *Decomposable tensor; copper structure; vertical rescaled metric.*
Mathematics Subject Classification: *53C15; 58D17.*

1. INTRODUCTION

Let M be a smooth manifold of dimension n. We denote by $\mathfrak{I}_s^r(M)$ the $F(M)-$module of all smooth tensor fields of type (r,s) on the manifold M; that is, tensors of contravariant degree r and covariant degree s, where $F(M)$ denotes the algebra of smooth functions on M. Throughout this work, manifolds, tensor fields, and connections are assumed to be differentiable and smooth, unless otherwise specified.

A manifold endowed with specific differential-geometric structures plays an important role in differential geometry. Yano introduced the concept of an $f-$structure, which is a tensor field of type $(1,1)$ of constant rank on a differentiable manifold M and ensures the equality $f^3+f=0$ [1]. The almost complex and almost contact structures are derived from the $f-$structure. Extending the concept of an $f-$structure, Goldberg and Yano introduced the notion of polynomial structure of degree d on manifold M. The polynomial structure satisfies the equality given in Equation (1)

$$Q(f)=f^d+a_d f^{d-1}+...+a_2 f+a_1 I=0 \tag{1}$$

where I is the identity field and $a_1,a_2,...,a_d$ are real numbers [2].

Spinadel introduced the metallic means family, which includes important metallic ratios such as the golden mean, silver mean, bronze mean, and copper mean [3,4]. The concept of metallic means in the context of manifolds, particularly Riemannian manifolds, has garnered significant attention in differential geometry. All of the metallic means family members are positive quadratic irrationals $\sigma_{p,q}=\dfrac{p+\sqrt[2]{p^2+4q}}{2}$, which are the solutions of a quadratic equation $x^2-px-q=0$. Inspired by the metallic means family, Hretcanu and

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Crasmareanu introduced the metallic structure on a manifold M which is determined by a tensor field Ω of type $(1,1)$ on manifold M satisfying $\Omega^2 = p\Omega + qI,\ \ p,q \in \mathbb{R}$ [5]. A metallic structure is an example of polynomial structures. Metallic structures, including golden, silver, bronze, and copper structures, have recently become important topics in differential geometry and have been studied by many geometers [6-12]. The copper mean, a specific type of metallic mean, is an intriguing concept that can be explored in differential geometry, particularly in relation to manifolds. The notion of a copper mean $\omega = 2$ which is a positive root of the equation $x^2 - x - 2 = 0$, taking $p = 1$ and $q = 2$ [3-5]. A copper structure on manifold M which is determined by a tensor field Ω of type $(1,1)$ on manifold M satisfying $\Omega^2 = \Omega + 2I$ [5].

Let (M,g) be a Riemannian manifold endowed with the copper structure Ω such that

$$g(\Omega Y, Z) = g(Y, \Omega Z) \tag{2}$$

for all Y, $Z \in \Im_0^1(M)$. Equivalently to Equation (2), we obtain Equation (3):

$$g(\Omega Y, \Omega Z) = g(\Omega^2 Y, Z) = g(\Omega Y + 2Y, Z) = g(\Omega Y, Z) + 2g(Y, Z). \tag{3}$$

The Riemannian metric in Equation (2) is called Ω-compatible, and the Riemannian manifold (M,Ω,g) is called a copper-Riemannian manifold. (g,Ω) is a copper-Riemannian structure on M [5]. The Riemannian metric (2) is also referred to as a pure metric [13,14].

Let Ω be a tensor field of type $(1,1)$ on manifold M. A tensor field k of type $(0,r)$ is called a pure tensor field according to Ω if

$$k(\Omega Z_1, Z_2, ..., Z_r) = k(Z_1, \Omega Z_2, ..., Z_r) = ... = k(Z_1, Z_2, ..., \Omega Z_r)$$

for any $Z_1, Z_2, ..., Z_r \in \Im_0^1(M)$. The Tachibana operator ϕ_Ω is applied to a tensor field of type $(0,r)$ as follow

$$\begin{aligned}(\phi_\Omega k)(Y, Z_1, ..., Z_r) = (\Omega Y)k(Z_1, ..., Z_r) - Yk(\Omega Z_1, ..., Z_r) \\ + \sum_{\lambda=1}^{r} k(Z_1, ..., (L_{Z_\lambda}\Omega)Y, ..., Z_r)\end{aligned} \tag{4}$$

for any $Y, Z_1, Z_2, ..., Z_r \in \Im_0^1(M)$, where L_Z denotes the Lie differentiation according to Z [13,14].

2. MAIN RESULTS

2.1. LOCALLY DECOMPOSABLE COPPER-RIEMANNIAN MANIFOLD

Let (M,g,Ω) be a copper-Riemannian manifold with a copper structure Ω. A polynomial structure is integrable if and only if there exists a torsion-free linear connection ∇

with respect to which the structure tensor is covariantly constant [15]. The copper structure Ω, which is a polynomial structure, is integrable if and only if there exists a torsion-free linear connection ∇ with respect to which the copper structure Ω is covariantly constant. Another condition for the integrability of a copper structure on Riemannian manifolds can be formulated using the Tachibana operator ϕ.

Theorem 1. *Let M be a copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g*. Then Ω is integrable if $\phi_\Omega g = 0$.

Proof: By using the purity of g and the condition $\nabla g = 0$, we have

$$g\left(Y,(\nabla_W\Omega)Z\right) = g\left((\nabla_W\Omega)Y,Z\right) \tag{5}$$

for all $Y,Z,W \in \Im_0^1(M)$, where ∇ denotes the operator of the Riemannian covariant derivative with respect to g.

By using Equation (5) and $[Y,Z] = \nabla_Y Z - \nabla_Z Y$, we can write Equation (4) in a different form as follows:

$$(\phi_\Omega g)(Y,Z,W) = -g\left((\nabla_Y\Omega)Z,W\right) + g\left((\nabla_Z\Omega)Y,W\right) + g\left(Z,(\nabla_W\Omega)Y\right). \tag{6}$$

By replacing Y and W, we have

$$(\phi_\Omega g)(W,Z,Y) = -g\left((\nabla_W\Omega)Z,Y\right) + g\left((\nabla_Z\Omega)W,Y\right) + g\left(Z,(\nabla_Y\Omega)W\right). \tag{7}$$

If we add Equations (6) and (7), we obtain

$$(\phi_\Omega g)(Y,Z,W) + (\phi_\Omega g)(W,Z,Y) = 2g\left(Y,(\nabla_Z\Omega)W\right). \tag{8}$$

By setting $\phi_\Omega g = 0$ in Equation (8), we obtain $\nabla\Omega = 0$. Hence, the proof is concluded.

Corollary 1. Let M be a copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g. The condition $\phi_\Omega g = 0$ is equivalent to $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection of g.

Let M be an $n-$dimensional Riemannian manifold equipped with a positive definite Riemannian metric g. If there exists a non-trivial tensor field P of type $(1,1)$ on M such that

$$P^2 = I$$

and

$$g(PY,PZ) = g(Y,Z)$$

for all $Y,Z \in \Im_0^1(M)$, then P is called an almost product structure, and the triple (M,g,P) is referred to as an almost product Riemannian manifold [16]. We aim to establish the relationships between almost product structures and copper structures on the manifold M.

Proposition 1. Every copper structure Ω on M induces two almost product structures on this manifold:

$$P = \pm\left(\frac{2}{3}\Omega - \frac{1}{3}I\right). \tag{9}$$

Conversely, every almost product structure P on manifold M induces two copper structures on the manifold M, given as follows:

$$\Omega_1 = \frac{1}{2}I + \frac{3}{2}P, \qquad \Omega_2 = \frac{1}{2}I - \frac{3}{2}P. \tag{10}$$

Proof: Let P be an almost product structure on a Riemannian manifold (M,g), i.e. $P^2 = I$. Then each of the structures $\Omega_1 = \frac{1}{2}I + \frac{3}{2}P$ and $\Omega_2 = \frac{1}{2}I - \frac{3}{2}P$ given in Equation (10), obtained from the almost product structure P, is a copper structure. In fact,

$$\begin{aligned}
\Omega_1^2 &= (\frac{1}{2}I + \frac{3}{2}P)^2 \\
&= \frac{1}{4}I + \frac{3}{2}PI + \frac{9}{4}P^2 \\
&= \frac{1}{4}I + \frac{3}{2}P + \frac{9}{4}I \\
&= \frac{10}{4}I + \frac{3}{2}P \\
&= \frac{10}{4}I + \frac{3}{2}.\frac{2}{3}\Omega_1 - \frac{3}{2}.\frac{1}{3}I \\
&= \Omega_1 + 2I.
\end{aligned}$$

Similarly, the equation $\Omega_2^2 - \Omega_2 - 2I = 0$ obtained for a copper structure Ω_2. Conversely, let Ω be a copper structure on a Riemannian manifold (M,g). Then the structure $P = \frac{2}{3}\Omega - \frac{1}{3}I$ induced by the copper structure Ω is an almost product structure. In fact,

$$P^2 = \frac{4}{9}\Omega^2 - 2.\frac{2}{3}.\frac{1}{3}\Omega I + \frac{1}{9}I^2 = \frac{4}{9}\Omega^2 - \frac{4}{9}\Omega + \frac{1}{9}I = \frac{4}{9}(\Omega^2 - \Omega) + \frac{1}{9}I = \frac{4}{9}2I + \frac{1}{9}I = \frac{9I}{9} = I.$$

Hence, the proof is concluded. A Riemannian metric g is pure according to a copper structure if and only if the Riemannian metric g is pure according to the corresponding almost product structure P. The relation between the Tachibana operators $\phi_P g$ and $\phi_\Omega g$ is given as

$$\phi_P g = \frac{2}{3}\phi_\Omega g\,. \tag{11}$$

In [17], Salimov et al. demonstrated that on an almost-product Riemannian manifold equipped with a pure metric g, the vanishing of the tensor $\phi_P g = 0$ implies the integrability of the almost-product structure P. Therefore, in light of Theorem 1 and Equation (11), it follows that:

Proposition 2. Let (M,Ω,g) be a copper-Riemannian manifold and P its corresponding almost product structure. The copper structure Ω is integrable if $\phi_P g = 0$.

A copper-Riemannian manifold (M,Ω,g) is referred to as a locally copper-Riemannian manifold if the copper structure Ω is integrable. If the metric g of a locally copper-Riemannian manifold takes the form

$$ds^2 = g_{cb}\left(\xi^a\right)d\xi^c d\xi^b + g_{zy}\left(\xi^x\right)d\xi^z d\xi^y, \quad a,b,c = 1,...,m, \quad x,y,z = m+1,...,n$$

that is g_{cb} are functions of ξ^a only, $g_{cy} = 0$ and g_{zy} are functions of ξ^x only, then we call the manifold M a locally decomposable copper-Riemannian manifold. On the other hand, we say that the locally product Riemannian manifold with corresponding product P is a locally decomposable Riemannian manifold if and only if P is covariantly constant with respect to the Levi-Civita connection of g [16,18].

Since $\phi_P g = 0$ is equivalent to $\nabla P = 0$ [17], we obtain the following proposition:

Proposition 3. Let M be a copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g. The manifold M is a locally decomposable copper-Riemannian manifold if and only if $\phi_P g = 0$, where P is the corresponding almost product structure.

The twin copper-Riemannian metric is defined by Equation (12):

$$G(Z,W) = g(\Omega Z, W) \tag{12}$$

for all $Z,W \in \Im_0^1(M)$. The purity of the twin copper-Riemannian metric G with respect to the copper structure Ω is represented by

$$\begin{aligned}
G(\Omega Z,W) &= g(\Omega^2 Z,W)\\
&= g\left(\Omega Z + 2Z, W\right)\\
&= g\left(\Omega Z, W\right) + g\left(2Z, W\right)\\
&= g\left(Z, \Omega W\right) + g\left(Z, 2W\right)\\
&= g\left(Z, \Omega W + 2W\right)\\
&= g(Z, \Omega^2 W)\\
&= G\left(Z, \Omega W\right).
\end{aligned}$$

It is easily seen that the twin copper Riemannian metric G is pure according to the copper structure Ω. The vanishing of the Nijenhuis tensor of the almost product structure P is also known to give the integrability of the almost product structure P [19]. The relation between the Nijenhuis tensor N_P and N_Ω is obtained as

$$N_P(Y,Z)=\frac{4}{9}N_\Omega(Y,Z). \tag{13}$$

If ϕ_Ω -operator is applied to the twin copper-Riemannian metric G, the equation

$$(\phi_\Omega G)(Y,Z,W)=(\phi_\Omega g)(Y,\Omega Z,W)+g(N_\Omega,(Y,Z),W) \tag{14}$$

is obtained where N_Ω is the Nijenhuis tensor constructed from Ω. Therefore, in light of Proposition 3, Equation (11), Equation (13) and Equation (14), we have

Proposition 4. Let M be a locally decomposable copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g. Then, $\phi_\Omega g=0$ is equivalent to $\phi_\Omega G=0$ if $N_\Omega=0$.

If a pure tensor k satisfies the condition $\phi_\varphi k=0$, then it is referred to as a $\phi-$tensor. If a tensor field φ of type $(1,1)$ is a complex structure, then a $\phi-$tensor is an analytic tensor. If φ is a product structure, i.e. an almost product structure such that its Nijenhuis tensor vanishes, then a $\phi-$tensor is a decomposable tensor [19].

Theorem 2. *Let M be a locally decomposable copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g*. The Riemannian curvature tensor field R is a $\phi-$ tensor field.

Proof: The Riemannian curvature tensor field R of the Riemannian metric g is pure according to the copper structure Ω, as expressed in Equation (15):

$$R(\Omega Z_1,Z_2,Z_3,Z_4)=R(Z_1,\Omega Z_2,Z_3,Z_4)=R(Z_1,Z_2,\Omega Z_3,Z_4)=R(Z_1,Z_2,Z_3,\Omega Z_4) \tag{15}$$

for all $Z_1,Z_2,Z_3,Z_4\in\Im_0^1(M)$. Applying the Tachibana operator to the Riemannian curvature R of type $(0,4)$ as in Equation (4), we write

$$(\phi_\Omega R)(Y,Z_1,Z_2,Z_3,Z_4)=(\nabla_{\Omega Y}R)(Z_1,Z_2,Z_3,Z_4)-(\nabla_Y R)(\Omega Z_1,Z_2,Z_3,Z_4). \tag{16}$$

Considering the Riemannian curvature tensor R to be pure and applying Bianchi's 2nd identity to (16), we obtain Equation (17):

$$(\phi_\Omega R)(Y,Z_1,Z_2,Z_3,Z_4)=0. \tag{17}$$

Since the interim calculations are similar to those in the proof of [20], we omit the standard calculations. By (9) and (16), we can find, in a similar way to Equation (11),

$$\phi_P R=\frac{2}{3}\phi_\Omega R \tag{18}$$

where Ω is the copper structure and P is its corresponding almost product structure. Based on Theorem 2 and Equation (18), we obtain the following proposition.

Proposition 5. Let M be a locally decomposable copper-Riemannian manifold endowed with a copper structure Ω and a Riemannian metric g. The Riemannian curvature tensor field is a decomposable tensor field.

2.2. COPPER STRUCTURE ON THE TANGENT BUNDLE

Let (M,g) be a $n-$dimensional Riemannian manifold and $T(M)$ be its tangent bundle with $2n-$dimension. The bundle projection $\pi : T(M) \to M$ establishes the natural bundle structure of the tangent bundle $T(M)$ over the manifold M. For any point ξ of $T(M)$, the correspondence $\xi \to p$ determines the bundle projection $\pi : T(M) \to M$, $\pi(\xi) = \pi(p,u) = p$. A system of local coordinates $(U, x^j)_{j=1,\dots n}$ in manifold M, $U \subset M$, induces a system of local coordinates $\left(\pi^{-1}(U), x^j, x^{\bar{j}} = u^j\right)_{j=1,\dots,n,\ \bar{j}=n+1,\dots,2n}$ in the tangent bundle $T(M)$, where (u^j) are the Cartesian coordinates in each tangent space $T_p(M)$ at $p \in M$ according to the natural base [21].

Let $Y = Y^j \dfrac{\partial}{\partial x^j}$ be the local expression in U of a vector field Y. The local expressions of the vertical lift ${}^V Y$ and horizontal lift ${}^H Y$ of Y are given in Equation (19):

$$\begin{aligned} {}^V Y &= Y^j \frac{\partial}{\partial u^j} \\ {}^H Y &= Y^j \frac{\partial}{\partial x^j} - Y^j u^i \Gamma^k_{ji} \frac{\partial}{\partial u^k} \end{aligned} \tag{19}$$

where Γ^k_{ji} are the coefficients of the Riemannian connection ∇ of g [21].

In reference [22], the authors define a new class of a naturally metric on the tangent bundle $T(M)$ and denoted by $G^f_{(p,u)}$ where f is some strictly positive smooth function in M. The vertical rescaled metric $G^f_{(p,u)}$ on the tangent bundle $T(M)$ is defined by

$$\begin{aligned} G^f_{(p,u)}({}^V Y, {}^V Z) &= f(p)\, g_p(Y,Z) \\ G^f_{(p,u)}({}^V Y, {}^H Z) &= 0 \\ G^f_{(p,u)}({}^H Y, {}^H Z) &= g_p(Y,Z) \end{aligned} \tag{20}$$

for any $Y, Z \in \Im^1_0(M)$ on manifold M. For $f = 1$, the metric G^f is exactly the Sasaki metric.

We define a copper structure $\tilde{\Omega}$ on $T(M)$

$$\begin{cases} \tilde{\Omega}\,{}^{H}Y = \frac{1}{2}\,{}^{H}Y + \frac{3}{2}\,{}^{V}Y \\ \tilde{\Omega}\,{}^{V}Y = \frac{1}{2}\,{}^{V}Y + \frac{3}{2}\,{}^{H}Y \end{cases} \tag{21}$$

which implies $\tilde{\Omega}^2 - \tilde{\Omega} - 2I = 0$.

Theorem 3. *Let (M,g) be a Riemannian manifold and $T(M)$ be its tangent bundle endowed with the vertical rescaled metric $G^{f}_{(p,u)}$ and copper structure $\tilde{\Omega}$ defined by (21).* The triple $\left(T(M), \tilde{\Omega}, G^{f}_{(p,u)}\right)$ is a copper-Riemannian manifold if and only if $f(p) = 1$.

Proof: The tensor K is defined by Equation (22):

$$K\left(\tilde{Y},\tilde{Z}\right) = G^{f}_{(p,u)}\left(\tilde{\Omega}\tilde{Y},\tilde{Z}\right) - G^{f}_{(p,u)}\left(\tilde{Y},\tilde{\Omega}\tilde{Z}\right) \tag{22}$$

for any $\tilde{Y},\tilde{Z} \in \Im^{1}_{0}\left(T(M)\right)$. For all vector fields $\tilde{Y}$ and $\tilde{Z}$ which are of the form ${}^{V}Y, {}^{V}Z$ and ${}^{H}Y, {}^{H}Z$, we must have $K\left(\tilde{Y},\tilde{Z}\right) = 0$, i.e. $G^{f}_{(p,u)}$ is pure with respect to the copper structure $\tilde{\Omega}$.
From (20) and (21), we obtain

$$\begin{aligned} &K\left({}^{H}Y, {}^{H}Z\right) = 0 \\ &K\left({}^{V}Y, {}^{H}Z\right) = \frac{3}{2} g_p(Y,Z) - \frac{3}{2} f(p) g_p(Y,Z) \\ &K\left({}^{H}Y, {}^{V}Z\right) = \frac{3}{2} f(p) g_p(Y,Z) - \frac{3}{2} g_p(Y,Z) \\ &K\left({}^{V}Y, {}^{V}Z\right) = 0. \end{aligned}$$

From the equations above, we say that $G^{f}_{(p,u)}$ is pure with respect to $\tilde{\Omega}$ if and only if $f(p) = 1$. Thus, the proof is completed. Now we apply Tachibana operator $\phi_{\tilde{\Omega}}$ to the vertical rescaled metric $G^{f}_{(p,u)}$. For this, first we state the following proposition.

Proposition 6. Let (M,g) be a pseudo-Riemannian manifold, ∇ be a Levi-Civita connection and R be a curvature tensor. Then the Lie bracket on the tangent bundle TM satisfies the following identities:

i. $\left[{}^{V}Y, {}^{V}Z\right] = 0$

ii. $\left[{}^{H}Y, {}^{V}Z\right] = {}^{V}\left(\nabla_Y Z\right)$

iii. $\left[{}^{H}Y, {}^{H}Z\right] = {}^{H}[Y,Z] - {}^{V}\left(R(Y,Z)u\right)$

for all $Y,Z \in \Im^{1}_{0}(M)$ and $\xi = (p,u) \in T(M)$ [23] .

Using Proposition 6 and the several properties of lifts on tangent bundle $T(M)$ that

${}^{V}Y\,{}^{V}\big(g(Z,W)\big)=0$ and ${}^{H}Y\,{}^{V}\big(g(Z,W)\big)={}^{V}\big(Yg(Z,W)\big)$, we calculate

$$\begin{aligned}\left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left(\tilde{Y},\tilde{Z},\tilde{W}\right) &= \left(\tilde{\Omega}\tilde{Y}\right)\left(G^{f}_{(p,u)}\left(\tilde{Z},\tilde{W}\right)\right)-\tilde{Y}\left(G^{f}_{(p,u)}\left(\tilde{\Omega}\tilde{Z},\tilde{W}\right)\right)\\ &\quad+G^{f}_{(p,u)}\left(\left(L_{\tilde{Z}}\tilde{\Omega}\right)\tilde{Y},\tilde{W}\right)+G^{f}_{(p,u)}\left(\tilde{Z},\left(L_{\tilde{W}}\tilde{\Omega}\right)\tilde{Y}\right)\end{aligned}\tag{23}$$

for all $\tilde{Y},\tilde{Z},\tilde{W}\in\Im^{1}_{0}\left(T(M)\right)$. From Equation (23), we obtain

$$\begin{aligned}\left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{V}Y,{}^{V}Z,{}^{V}W\right) &= 0\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{V}Y,{}^{V}Z,{}^{H}W\right) &= -\frac{3}{2}f(p)g_{p}\left(Z,R(W,Y)u\right)\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{V}Y,{}^{H}Z,{}^{V}W\right) &= -\frac{3}{2}f(p)g_{p}\left(R(Z,Y)u,W\right)\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{V}Y,{}^{H}Z,{}^{H}W\right) &= 0\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{H}Y,{}^{V}Z,{}^{V}W\right) &= 0\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{H}Y,{}^{V}Z,{}^{H}W\right) &= -\frac{3}{2}g_{p}\left(Z,\nabla_{Y}W\right)+\frac{3}{2}f(p)g_{p}\left(Z,\nabla_{Y}W\right)\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{H}Y,{}^{H}Z,{}^{V}W\right) &= -\frac{3}{2}f(p)g_{p}\left(Z,\nabla_{Y}W\right)+\frac{3}{2}g_{p}\left(Z,\nabla_{Y}W\right)\\ \left(\phi_{\tilde{\Omega}}G^{f}_{(p,u)}\right)\left({}^{H}Y,{}^{H}Z,{}^{H}W\right) &= \frac{3}{2}g_{p}\left(R(Z,Y)_{u},W\right)+\frac{3}{2}g_{p}\left(Z,R(W,Y)_{u}\right).\end{aligned}\tag{24}$$

So, from Theorem 3, Proposition 3, Equation (11) and Equation (24), we obtain the following theorem.

Theorem 4. *Let (M,g) be a Riemannian manifold and let $T(M)$ be its tangent bundle endowed with the vertical rescaled metric $G^{f}_{(p,u)}$ and the copper structure $\tilde{\Omega}$ defined by (21).* The triple $\left(T(M),\tilde{\Omega},G^{f}_{(p,u)}\right)$ is a locally decomposable copper-Riemannian manifold if and only if the Riemannian manifold is locally flat and $f(p)=1$.

3. CONCLUSION

An integrability condition for a copper-Riemannian structure is investigated via the Tachibana operator on a copper-Riemannian manifold. The relationship between the almost-product structure and the copper structure is established on a Riemannian manifold. The necessary and sufficient condition for the copper-Riemannian manifold to be locally decomposable is obtained. It is shown that the Riemannian curvature tensor field is an $\phi-$ tensor field on the locally decomposable copper-Riemannian manifold. A copper structure is obtained on the tangent bundle of a Riemannian manifold. The necessary and sufficient condition for the tangent bundle endowed with the vertical rescaled metric and the copper structure to be a locally decomposable copper-Riemannian manifold is obtained.

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