ORIGINAL PAPER

THE $\left(\frac{G'}{G}\right)$ METHOD TO DERIVE EXACT SOLUTIONS FOR THE TIME-DELAYED BURGERS-FISHER EQUATION

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Abstract. *Time-delayed equations play a vital role in various fields of science and engineering. In our study, exact solutions for time-delayed Burgers-Fisher equations are derived by adapting* $\left(\frac{G'}{G}\right)$*- expansion approach. The existence of time-delayed constants will be discussed in two sections of the nonlinear equations. The findings demonstrate the effectiveness and potency of the* $\left(\frac{G'}{G}\right)$*- expansion method in resolving nonlinear time-delayed nonlinear equations that arise in mathematical physics. The results are presented graphically using Mathematica software.*

Keywords: *Time-delay; traveling wave solutions;* $\left(\frac{G'}{G}\right)$*- expansion method; Burger-Fisher equation; exact solutions.*

Mathematics Subject Classification: *35C07; 35C08; 35Q53.*

1. INTRODUCTION

Conduction and diffusion, as dominant heat-transfer processes, interact to characterize a variety of nonlinear phenomena, including physical, biochemical, and biological phenomena. The continuous motion of reacting particles can be seen as a flow delay in each solute gradient. Time-delay is hence a crucial component of convection-diffusion systems. The generalized time-delayed Burgers-Fisher equation is given by:

$$\tau u_{tt} + (1-\tau f_u) u_t = u_{xx} - p u^s u_x + f(u), \qquad f(u) = qu(1-u^s)\,, \tag{1}$$

where p, q and s are constants, τ is a time-delayed constant, and Hyunsoo et al. al [1-3] provided the results for the following equation

$$\tau u_{tt} + u_t + p u^s u_x - u_{xx} = 0\,.$$

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The classical Burgers equation will come from equation (1) when $q = \tau = 0$ and $p = s = 1$. To obtain solitary solutions to any nonlinear PDEs, there are many methods like the $\left(\frac{G'}{G}\right)$-expansion method [4], tanh-coth method [5], the Homotopy Perturbation Method [6-7], Exponential -function method [8], the Jacobi Elliptic Functions Method [9], the Adomian Decomposition Method [10], etc. Wang and his companions [11] introduced a new direct technique called the $\left(\frac{G'}{G}\right)$-expansion method to find travelling wave solutions of nonlinear evolution equations. The concept of the $\left(\frac{G'}{G}\right)$- expansion method is that the solutions of a nonlinear equation can be derived by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (LODE) as follows:

$$G'' + G'\lambda + \mu G = 0,$$

where $G' = \frac{dG(\xi)}{d\xi}$, $\xi = k(x - \omega t)$.

The degree of the polynomial is determined by manipulating the homogeneous balance between the highest-order derivatives and nonlinear terms that appear in a PDE. After applying the $\left(\frac{G'}{G}\right)$- expansion procedure, a related set of equations must be solved to get the coefficients of that polynomial. Later, Sheng Zhang and his companions [12] found a generalized $\left(\frac{G'}{G}\right)$-expansion method to solve variable coefficient and high-dimensional equations.

If we input $q = 0$ in Equation (1), we get:

$$\tau u_{tt} + u_t + u^s u_x - u_{xx} = 0.$$

In our work, the numerical solutions of the time-delayed Burgers equation given below can be derived using the technique of $\left(\frac{G'}{G}\right)$-expansion method

$$\tau u_{tt} + u_t + u^{s+1} u_x - u_{xx} = 0, \tag{2}$$

for various values of $\tau, k,$ and s. Equation (2) has received significant attention in the development of Population Growth Models, Forest Fires [13], etc.

2. ALGORITHM OF THE $\left(\frac{G'}{G}\right)$ EXPANSION METHOD

2.1. MATERIALS

The partial differential equation (PDE):

$$P(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0, \tag{3}$$

can be transformed into an ordinary differential equation (ODE):

$$Q(u, -k\omega u', ku', -k^2u'', k^2\omega^2u'', \ldots\ldots) = 0, \tag{4}$$

by taking $\xi = k(x-\omega t)$. By the $\left(\frac{G'}{G}\right)$ expansion technique, the ODE (4) can be composed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G = G(\xi)$ satisfies:

$$G'' + G'\lambda + \mu G = 0. \tag{6}$$

Here, $a_1, a_2, \ldots\ldots a_m, \lambda$ and μ are constants to be found later, with $a_m \neq 0$. The decision authority m will be obtained by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in ODE (4). By inputting Equation (5) into Equation (4) and using the second-order LODE (6), ordering all terms with the same order of $\left(\frac{G'}{G}\right)$ all together, the left-hand part of Equation (4) is reformed into a polynomial in $\left(\frac{G'}{G}\right)$.

Equating every coefficient of that polynomial to zero, we will find a set of algebraic equations for $a_1, a_2, \ldots\ldots a_m, \lambda, k, \omega$ and μ. We can obtain different values for these variables by solving this set of algebraic equations. Substituting those values and the general solutions of (6) into (5), we can obtain the exact solutions of (3).

3. RESULTS AND DISCUSSION

3.1. EXACT RESULTS TO THE TIME - DELAYED BURGERS EQUATION

To obtain the solution for Equation (2), we consider the transformation $u = U(\xi), \xi = k(x - \omega t)$, where k and ω are constants to be determined later. We obtained the time-delayed Burgers' equation (2) in the upcoming nonlinear ODE of the form:

$$kU^{1+s}U' - k\omega U' + (\tau\omega^2 - 1)k^2U'' = 0. \tag{7}$$

Balancing $U^{1+s}U'$ with U'' in equation (7) gives $m = \frac{1}{s+1}$. To get a solution, we input the transformation $U = v^{\frac{1}{s+1}}$ to swap equation (7) as

$$(1+s)v^2v' - (1+s)\omega vv' - (\tau\omega^2 - 1)ks(v')^2 + (\tau\omega^2 - 1)(1+s)kvv'' = 0. \tag{8}$$

Suppose that the solution of ODE (8) can be written as a polynomial in as follows:

$$v(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{9}$$

where $G = G(\xi)$ satisfies the second-order LODE in the form

$$G''(\xi) + G'(\xi)\lambda + \mu G(\xi) = 0, \tag{10}$$

thereby we get $G(\xi) = e^{\frac{1}{2}(-\lambda-\sqrt{\lambda^2-4\mu})\xi}C[1] + e^{\frac{1}{2}(-\lambda+\sqrt{\lambda^2-4\mu})\xi}C[2]$ where C[1] and C[2] are constants.

By using equations (9) and (10), it is derived that

$$v^2 = a_m\left(\frac{G'}{G}\right)^{2m} + \ldots.., \tag{11}$$

$$v' = -ma_m\left(\frac{G'}{G}\right)^{m+1} + \ldots.., \tag{12}$$

$$v'' = m(m+1)a_m\left(\frac{G'}{G}\right)^{m+2} + \ldots... \tag{13}$$

To find the parameter m, we will balance the highest-order linear terms in Equation (8) against the highest-order nonlinear terms. Considering the homogeneous balance between $v^{''}$ and v^2 in (8), based on (11) and (13), we yield that $2m+2=3m$. So, we can write (9) as

$$v = a_1\left(\frac{G'}{G}\right) + a_0, \tag{14}$$

where $a_1 \neq 0$.

By using Equations (9) and (10), it is derived that

$$v^2 = a_1^{\,2}\left(\frac{G'}{G}\right)^2 + 2a_1a_0\left(\frac{G'}{G}\right) + a_0^{\,2}, \tag{15}$$

$$v' = -a_1\mu - a_1\lambda\left(\frac{G'}{G}\right) - a_1\left(\frac{G'}{G}\right)^2, \tag{16}$$

$$v^{''} = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + a_1\lambda\mu, \tag{17}$$

$$\begin{aligned} v^{'2} = \mu^2 a_1^{\,2} + 2\lambda\mu a_1^{\,2}\left(\frac{G'}{G}\right) + (\lambda^2 a_1^{\,2} + 2\mu a_1^{\,2})\left(\frac{G'}{G}\right)^2 \\ + (2\lambda a_1^{\,2})\left(\frac{G'}{G}\right)^3 + a_1^{\,2}\left(\frac{G'}{G}\right)^4. \end{aligned} \tag{18}$$

By substituting (14) - (18) in (8), we get *a* polynomial in $\left(\frac{G'}{G}\right)$. Setting each coefficient of this polynomial to zero yields a set of equations for a_1, a_0 and k as follows:

$$\begin{gathered} -k(1+s)\lambda a_0 + (1+s)\omega a_0 + k(1+s)\lambda\tau\omega^2 a_0 - (1+s)a_0^{\,2} + ks\mu(1-\tau\omega^2)a_1 = 0, \\ 2k(1+s)(-1+\tau\omega^2)a_0 + (-k\lambda + \omega + \omega s + k\lambda\tau\omega^2)a_1 - 2(1+s)a_1a_0 = 0, \\ k(2+s)(-1+\tau\omega^2)a_1 - (1+s)a_1^{\,2} = 0. \end{gathered} \tag{19}$$

Solving the above equations using Mathematica, we will get the following:

$$\begin{gathered} k = \pm\frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}, \\ a_0 = \frac{(2+s)(\pm\lambda + \sqrt{\lambda^2-4\mu})\omega}{2(\sqrt{\lambda^2-4\mu})}, \\ a_1 = \pm\frac{(2+s)\omega}{\sqrt{\lambda^2-4\mu}}, \end{gathered} \tag{20}$$

where ω, τ, s, μ *and* λ are constants.

By inputting (20) in (14), we get,

$$v(\xi) = -\frac{(2+s)\omega}{\sqrt{\lambda^2-4\mu}}\left(\frac{G^{'}}{G}\right) + \frac{(2+s)(-\lambda+\sqrt{\lambda^2-4\mu})\omega}{2(\sqrt{\lambda^2-4\mu})} \tag{21}$$

where $\xi = -\frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}(x-\omega t)$ and

$$v(\xi) = \frac{(2+s)\omega}{\sqrt{\lambda^2-4\mu}}\left(\frac{G^{'}}{G}\right) + \frac{(2+s)(\lambda+\sqrt{\lambda^2-4\mu})\omega}{2(\sqrt{\lambda^2-4\mu})}, \tag{22}$$

where $\xi = \frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}(x-\omega t)$.

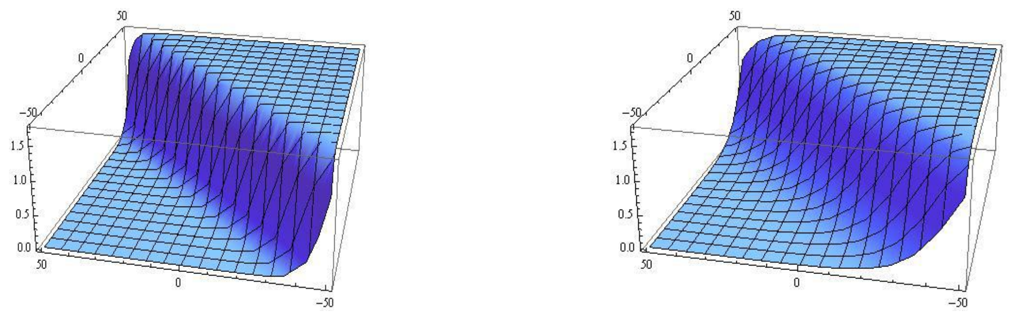
By putting the general solutions of (10) in (21) and (22), we get the solutions of the time-delayed Burgers' equation (Equation (2).

$$U_1(x,t) = \left(\frac{(2+s)\omega C[1]}{C[1] + e^{\sqrt{\lambda^2-4\mu}\left(-\frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}(x-\omega t)\right)}C[2]}\right)^{\frac{1}{1+s}}, \tag{23}$$

$$U_2(x,t) = \left(\frac{e^{\sqrt{\lambda^2-4\mu}\left(\frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}(x-\omega t)\right)}(2+s)\omega C[2]}{C[1] + e^{\sqrt{\lambda^2-4\mu}\left(\frac{(1+s)\omega}{(-1+\tau\omega^2)\sqrt{\lambda^2-4\mu}}(x-\omega t)\right)}C[2]}\right)^{\frac{1}{1+s}}, \tag{24}$$

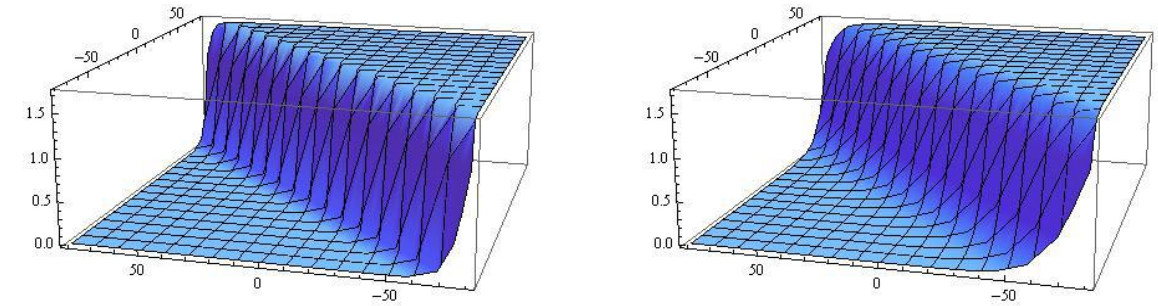
where $C[1]$ and $C[2]$ are arbitrary constants.

The effects of time-delay solutions (23) and (24) are shown in Figures 1 and 2 for the various values of τ respectively.



When ω=1, C[1]=0.1, C[2]=0.1, λ=3, μ=1, s=1, τ=5 When ω=1, C[1]=0.1, C[2]=0.1, λ=3, μ=1, s=1, τ=10

Figure 1. Graphical representation of waves for the time-delay solutions (23).



When ω=1, C[1]=0.1, C[2]=0.1, λ=3, μ=1, s=1, τ=5 When ω=1, C[1]=0.1, C[2]=0.1, λ=3, μ=1, s=1, τ=10

Figure 2. Graphical illustration of waves for the time delay of solutions (24).

3.2. RESULTS FOR THE TIME-DELAYED BURGERS EQUATION WHEN $\tau = 1$

More travelling wave results for the time-delayed Burgers-Fisher equation are obtained in this section. When $p = s = q = \tau = 1$, Equation (1) reduces to the form

$$u_{tt} + 2uu_t - u_{xx} + uu_x - u + u^2 = 0\,. \tag{25}$$

where $\tau = 1$is a time-delayed constant. By transforming through $u = U(\xi), \xi = k(x - \omega t)$, Equation (25) reduces to the form

$$-U + U^2 + kUU' - 2kU\omega U' - k^2U'' + k^2\omega^2U'' = 0. \tag{26}$$

To look for the travelling wave solutions of Equation (25), we balance U'' and UU' in (26), which gives $M = 1$. Now we can write (5) as

$$U(\xi) = a_1\left(\frac{G'}{G}\right) + a_0, \quad a_1 \neq 0. \tag{27}$$

By substituting (14)-(17) and (27) into (26) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (26) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero yields a set of equations for a_0, a_1, k and ω .

Solving the system of algebraic equations using Mathematica gives

$$\begin{aligned} k &= \pm\frac{4}{3\sqrt{\lambda^2 - 4\mu}}, \\ \omega &= \frac{5}{4}, \end{aligned} \tag{28}$$

$$a_0 = \frac{1}{2}\left(1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}}\right),$$

$$a_1 = \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}},$$

where a_0, a_1, k and ω are arbitrary constants.

By using (28), Expansion (27) can be written as

$$U(\xi) = \pm \frac{1}{\sqrt{\lambda^2 - 4\mu}}\left(\frac{G'}{G}\right) + \frac{1}{2}\left(1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}}\right), \tag{29}$$

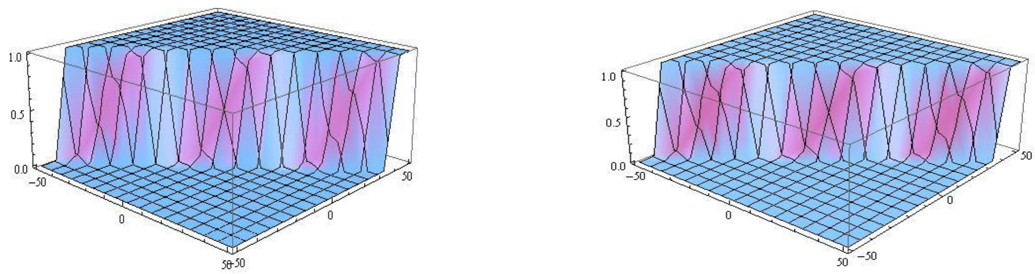
where $\xi = \pm \frac{4}{3\sqrt{\lambda^2 - 4\mu}}\left(x - \frac{5}{4}t\right)$.

Substituting the solutions of (6) in (29), we obtain solutions of the time-delayed Equation (26).

$$U_1(x,t) = \frac{C[1]}{C[1] + e^{\sqrt{\lambda^2 - 4\mu}\frac{4}{3\sqrt{\lambda^2 - 4\mu}}(x - \frac{5}{4}t)} C[2]}, \tag{30}$$

$$U_2(x,t) = \frac{e^{\sqrt{\lambda^2 - 4\mu}\left(-\frac{4}{3\sqrt{\lambda^2 - 4\mu}}\left(x - \frac{5}{4}t\right)\right)} C[2]}{C[1] + e^{\sqrt{\lambda^2 - 4\mu}\left(-\frac{4}{3\sqrt{\lambda^2 - 4\mu}}\left(x - \frac{5}{4}t\right)\right)} C[2]}, \tag{31}$$

where $\xi = \pm \frac{4}{3\sqrt{\lambda^2 - 4\mu}}\left(x - \frac{5}{4}t\right)$ and $C[1]$ and $C[2]$ are arbitrary constants.



When λ=3, μ=1, C[1]=0.1, C[2]=0.1, τ=1 When λ=3, μ=1, C[1]=0.1, C[2]=0.1, τ=1

Figure 3. Displays the wave graph for the time-delayed solutions (30) and (31).

3.3. DISCUSSION

At the end of our results, we have discovered some new solutions for the time-delayed Burger-Fisher equation. Some modified equations were employed to discover additional solutions in our article. The - expansion method was effectively used, yielding some new solutions. The mathematical graph might be extended to identify large-wave solutions for our time-delayed nonlinear equations.

The time-delayed nonlinear Burger-Fisher partial differential equation was analytically studied by using the $\left(\frac{G'}{G}\right)$-expansion technique. Several exact travelling-wave solutions were formally derived using this solid method. The results clearly demonstrate the efficiency of the method. Moreover, the method can greatly reduce the computational workload compared to other existing techniques. This method changes the problem from solving nonlinear partial differential equations to solving a separable ordinary differential equation. The results are entirely new for the different values of time delay τ .

4. CONCLUSION

In our study, we have used the $\left(\frac{G'}{G}\right)$ - expansion method to find accurate wave solutions to several time-delayed nonlinear partial differential equations. Mathematica has been used to validate all solutions plotted and presented in this study. Our effective methods can be used to solve broader classes of time-delayed equations and determine their solutions. More time delays might be used in the future to determine various solutions for particle motion.

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