ORIGINAL PAPER

TETRANACCI GRAPHS

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Abstract. *Graphs are fundamental structures widely used across diverse scientific fields, including chemistry, biology, network theory, and the social sciences. In recent years, the study of graph-theoretic models involving specific integer sequences has attracted considerable interest, owing to their rich combinatorial and algebraic characteristics. Among these sequences, the Tetranacci numbers stand out for their notable connections to continued fractions, quadratic fields, and certain classes of Diophantine equations. In this study, we introduce a new family of graphs, termed "Tetranacci graphs", whose degree sequences are composed of n consecutive Tetranacci numbers. Using the graph invariant $\Omega(D)$ — a tool that provides critical insights into the structural features of graphs such as realizability, connectivity, and cyclic components— we explore the conditions under which such sequences yield valid graphical realizations. We derive necessary and sufficient criteria for the realizability of these sequences for any positive integer n, and we classify all possible graphical forms (Tetranacci graphs) explicitly for the cases $1 \leq n \leq 4$. Our analysis is further extended to offer a general structural characterization for the case n ≥5. These fundamental criteria establish a comprehensive framework for studying Tetranacci graphs, opening new avenues for research into the interplay between number theory and graph realizability.*

Keywords: *Tetranacci number; Tetranacci graph; degree sequence; Ω invariant; realizability.*

Mathematics Subject Classification: *05C07; 05C75; 05C38; 11B39.*

1. INTRODUCTION AND MOTIVATION

Let $G = (V, E)$ be a graph with vertex set V and edge set E, such that the number of vertices is $|V(G)| = n$ and the number of edges is $|E(G)| = m$. These are referred to as the *graph's order and size*, respectively. For each vertex $v \in V(G)$, the degree of v is denoted by d_v. The smallest and largest vertex degrees in G are denoted by δ and Δ, respectively. A graph is said to be *connected* if every pair of vertices is joined by a path; otherwise, it is called *disconnected*.

An edge that begins and ends at the same vertex is referred to as a *loop*, whereas *multiple edges* are defined as two or more edges connecting the same pair of vertices. A graph is called *simple* if it contains neither loops nor multiple edges.

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A degree sequence can be denoted as $D = \left\{d_1^{(a_1)}, d_2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$, where each a_i represents the non-negative multiplicity of the corresponding degree d_i. Alternatively, the sequence may be written in the form $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$, allowing for some multiplicities a_i to be zero.

Let $D = \{d_1, d_2, \cdots, d_n\}$ be a sequence of non-negative integers arranged in non-decreasing order. If there exists a graph G whose vertex degrees match the elements of D, then D is said to be *realizable*, and the graph G is referred to as a *realization* of D. One widely used method for determining whether a sequence is realizable is the Havel–Hakimi algorithm, as detailed in [1].

A degree sequence is said to be realizable if there exists at least one graph whose degree sequence matches it. In general, the number of such possible realizations can be large, and no closed-form formula is currently available to enumerate them. Among the most effective and widely accepted tools for checking the realizability of degree sequences are the Havel–Hakimi algorithm and the Erdős–Gallai theorem (see [2-3]). In addition to these methods, various other techniques may also be employed for this purpose.

The Fibonacci sequence is one of the most familiar and widely studied integer sequences. It is denoted by $(F_r)_{r\geq 0}$ and defined by the recurrence relation $F_0 = 0, F_1 = 1$, $F_r = F_{r-1} + F_{r-2}$ for all $r \geq 2$. A natural extension of this sequence involves summing more preceding terms. For any integer $k \geq 2$, we define the "*k-generalized Fibonacci sequence*", denoted by $\left(F_r^{(k)}\right)_{r\geq -(k-2)}$, via the recurrence

$$F_r^{(k)} = F_{r-1}^{(k)} + F_{r-2}^{(k)} + \cdots + F_{r-k}^{(k)},$$

valid for all $r \geq 2$, with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$, $F_1^{(k)} = 1$. In particular,

$$F_1^{(k)} = 1, F_2^{(k)} = 1, F_3^{(k)} = 2, F_4^{(k)} = 3, \dots, F_{k+1}^{(k)} = 2^{k-1},$$

and which also satisfy the identity $F_r^{(k)} = 2^{r-2}$, for $2 \leq r \leq k+1$. Moreover, the sequence satisfies the three-term recurrence relation

$$F_r^{(k)} = 2F_{r-1}^{(k)} - F_{r-k-1}^{(k)} \text{ for all } n \geq 3.$$

This family of sequences generalizes the classical Fibonacci numbers: each choice of k yields a distinct sequence where every term is formed by summing the previous k terms. Table 1 provides the first few values of these sequences for small values of k and $r \geq 1$. Here, 1 and 0 denote odd and even terms, respectively.

Table 1. The first few values of k-Fibonacci numbers

k	Name	First non-zero terms	Parity (mod 2)
2	Fibonacci Numbers	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, $\cdots$	1, 1, 0, repeated
3	Tribonacci Numbers	1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, $\cdots$	1, 1, 0, 0 repeated
4	Tetranacci Numbers	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, $\cdots$	1, 1, 0, 0, 0 repeated

Recently, there has been growing interest in incorporating special number sequences—such as the Fibonacci and Tribonacci numbers—into graph theory by designing graphs whose degree sequences are derived from consecutive elements of these sequences, see [4-5].

Motivated by earlier work on Fibonacci and Tribonacci graphs, we propose a new class of graphs, termed Tetranacci graphs, whose degree sequences consist of n consecutive Tetranacci numbers. In this study, the Tetranacci numbers will be denoted by T_r. These numbers are defined by the recurrence relation $T_r = T_{r-1} + T_{r-2} + T_{r-3} + T_{r-4}$ for $r \geq 4$, with initial terms $T_0 = 0, T_1 = 1, T_2 = 1,$ and $T_3 = 2$. These graphs offer a novel perspective on the interaction between number theory and graph theory. To investigate their structural characteristics, we utilize the graph invariant $\Omega(D)$, which serves as a tool for examining realizability, connectivity, the presence of cycles, loops, pendant edges, and bridges. We establish the criteria under which a sequence D of n successive Tetranacci numbers forms a realizable degree sequence and provide a complete enumeration of all such realizations for small values of n. Furthermore, we generalize these findings to broader cases, shedding light on the intricate relationship between Tetranacci sequences and graph-theoretic structures.

Observation 1.1. The Tetranacci sequence follows the pattern: odd, odd, even, even, even, odd, odd, even, even, even, ... In other words, if the index r of a Tetranacci number T_r is 1 or 2 *modulo* 5, then T_r is odd; otherwise, it is even.

2. Ω INVARIANT

We consider the graph invariant $\Omega(G)$ $\left(\text{or } \Omega(D)\right)$, when referring to a degree sequence, which was defined in [6] for analyzing the realizability and structural characteristics of graphs. Below, we briefly recall its definition and key properties as developed in the literature [7-10].

Let *T* be a tree. Then, the number of pendant (i.e., degree one) vertices, denoted by a_1, is given by

$$a_1 = 2 + a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta, \tag{1}$$

where a_i corresponds to the number of vertices of degree *i*. It should be noted that Equation (1) is equivalent to

$$a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2. \tag{2}$$

Based on this equation, the invariant $\Omega(D)$ was proposed in [6] and is defined as follows:

Definition 2.1. *Let* $D = \left\{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\right\}$ *be a realizable degree sequence with its realization being the graph G*. The $\Omega(D)$ is introduced in terms of *D* and is defined as

$$\Omega(D) = a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i$$

Several key properties of the Ω invariant are detailed in [6]. In this section, we emphasize some of the most significant aspects.

The relation presented here serves as a fundamental tool for calculating $\Omega(G)$ for a given graph *G* and is instrumental in numerous proofs involving the Ω invariant:
Theorem 2.1. [6] For any graph *G*,

$$\Omega(G) = 2(m - n).$$

This implies that the Ω invariant always takes even values. Therefore, if $\Omega(D)$ is found to be odd for a set *D* of non-negative integers, it immediately follows that *D* is not realizable, thereby providing a novel criterion for realizability.

Next, we consider the parameter *r*, which represents both the number of independent cycles in a connected graph *G* and the number of bounded faces (regions enclosed by edges, excluding the exterior face) in its planar embedding. This classical graph-theoretic quantity, known as the cyclomatic number of *G*, can be expressed in terms of the Ω invariant as follows:

Theorem 2.2. [6] *Let* $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. If *D* is realizable as a connected planar graph *G*, then the number *r* of enclosed regions (faces) in *G* is determined by the formula

$$r = \frac{\Omega(G)}{2} + 1.$$

This result has a range of applications and can be naturally extended to disconnected graphs. Specifically, for a graph with *c* connected components, the generalization of Theorem 2.2 is obtained by replacing the term $+1$ on the right-hand side with $+c$.

Corollary 2.1. [6] Let $D = \{1^{(a_1)}, 2^{(a_2)}, \cdots, \Delta^{(a_\Delta)}\}$. If *D* is realizable as a connected planar graph *G*, then the number *r* of faces of *G* is determined by the formula

$$r = \frac{\Omega(G)}{2} + c.$$

In [7], the following formula is presented to calculate the maximum possible number of components in any realization of a given degree sequence:

$$c_{max} = \sum_{d_i\ even} a_i + \frac{1}{2} \sum_{d_i\ odd} a_i. \tag{3}$$

This formula will be instrumental in our analysis as we establish an upper bound on the number of components corresponding to a specified degree sequence.

In this study, we examine specific families of graphs containing at most four vertices. The scenarios involving graphs with one or two vertices have already been explored in [7], where they were used to tackle an extremal problem concerning the maximum possible number of components among all realizations of a prescribed degree sequence. Throughout the paper, we use the notation L_q to represent a graph consisting of a single vertex equipped with *q* loops. Figure 1 provides various instances of such L_q graphs.

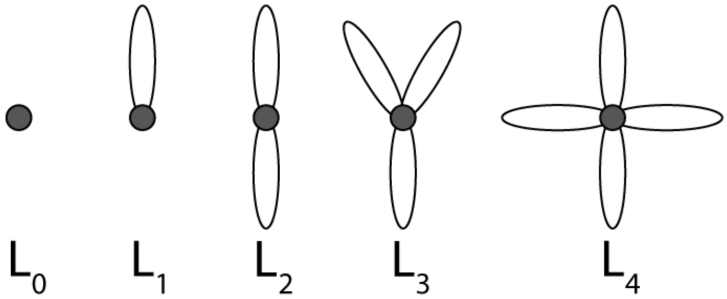

Figure 1. Some L_q graphs.

The second graph family, denoted by $B_{r,s}$ for natural numbers r and s, consists of a single edge connecting two vertices, where one vertex is incident to r loops and the other to s loops. A variety of illustrative examples of such $B_{r,s}$ graphs are displayed in Figure 2.

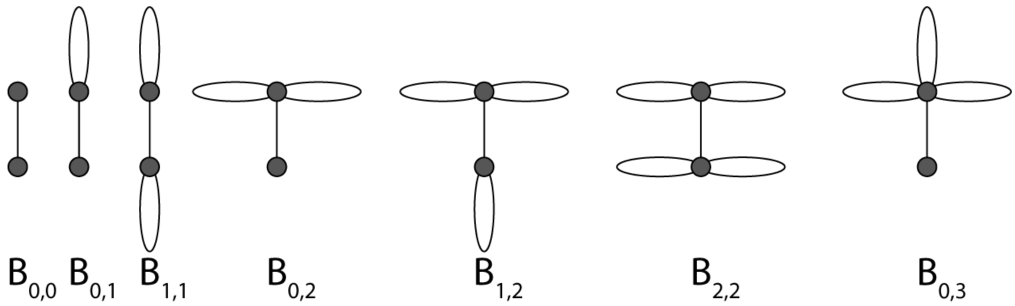

Figure 2. Some $B_{r,s}$ graphs.

The third category of graphs, denoted by $T_{x,y,z}$, refers to a connected structure composed of three vertices $u, v,$ and w arranged along a path $P_3 = \{u, v, w\}$. The degrees of these vertices are given by $2x + 1, 2y + 2,$ and $2z + 1$, respectively, for some integers $x, y,$ and z. In this configuration, x loops are incident to vertex u, y loops to v, and z loops to w. Various examples of $T_{x,y,z}$ graphs are illustrated in Figure 3.

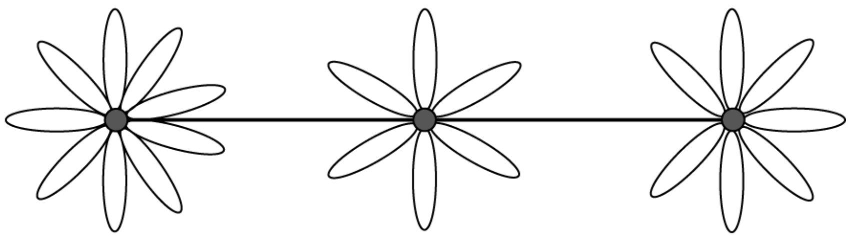

Figure 3. $T_{9,6,7}$ graph.

The fourth and final graph class, denoted by $Q_{x,y,z,t}$, consists of a connected graph with four vertices u, v, w and l arranged along a path $P_4 = \{u, v, w, l\}$. The degrees of these vertices are given by $2x + 1, 2y + 2, 2z + 2$ and $2t + 1$, respectively, where x, y, z and t are integers. In this structure, x loops are attached to u, y to v, z to w, and t to l. Several instances of such $Q_{x,y,z,t}$ graphs are shown in Figure 4.

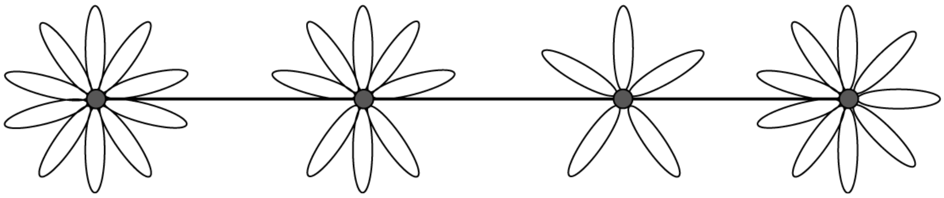

Figure 4. $Q_{10,8,5,9}$ graph.

For non-negative integers *r* and *k*, the notation $r[k]$ refers to a construction in which *k* loops are attached to a newly introduced vertex, and this vertex is connected to one of the *r* existing loops that are already incident to other vertices. More generally, the notation $r[k_1, k_2, \cdots, k_t]$ describes a configuration where k_1 loops are added to a new vertex connected to a loop of degree *r*, k_2 loops to another such vertex, and so on, continuing this process for *t* distinct new vertices. In contrast, the expression $r[k_1; k_2; \cdots; k_t]$ indicates that all loops $k_1, k_2, \cdots, k_t$ are connected to the same loop of degree *r*, but each through a different newly introduced vertex. Examples corresponding to the notations $B_{2[3],2}, B_{2[3,1],2}$, and $B_{2[3;1],2}$ are depicted in Figure 5.

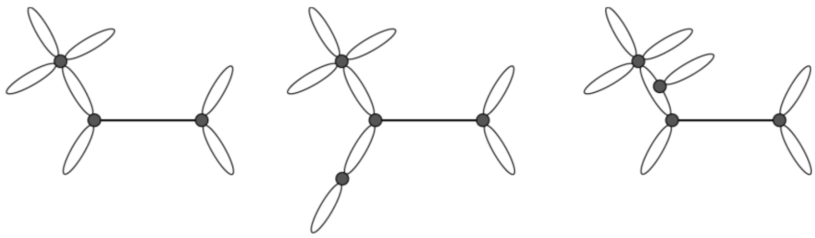

Figure 5. The graphs $B_{2[3],2}, B_{2[3,1],2}$, and $B_{2[3;1],2}$.

3. EXISTENCE CONDITIONS FOR TETRANACCI GRAPHS

In this section, we explore whether a graph can exist whose degree sequence consists of *n* consecutive Tetranacci numbers. Specifically, we aim to determine which such *n*-tuples of consecutive Tetranacci numbers are realizable as the degree sequence of a graph. To address this question, we use the invariant Ω together with key results from [6], extending the framework to accommodate arbitrary values of *n*.

3.1. TETRANACCI GRAPHS OF ORDER 1

We start our analysis with Tetranacci graphs of order 1, each consisting of a single vertex whose degree matches a Tetranacci number. This case serves as the foundational and most straightforward instance in our investigation.

Theorem 3.1. *A graph of order 1 is a Tetranacci graph if and only if its size equals* $\frac{T_{5k-2}}{2}, \frac{T_{5k-1}}{2}$ *or* $\frac{T_{5k}}{2}$ *for some* $k \in \mathbb{Z}^+$. In this case, the graphs are $L_{\frac{T_{5k-2}}{2}}$, $L_{\frac{T_{5k-1}}{2}}$ or $L_{\frac{T_{5k}}{2}}$.

Proof: (⇒) Let G be a tetranacci graph consisting of a single vertex v. Since G is a tetranacci graph, the degree of v is given by $d_v = T_r$ for some positive integer r. The sum of the vertex

degrees, which in this case is simply d_v, must be an even number. Consequently, T_r must also be even, which happens only if r is divisible by 5 or congruent to -2 or -1 *modulo* 5.
($\Leftarrow$) Consider a graph of order 1, with a single vertex v and size $\frac{T_{5k}}{2}$. Since there is only one vertex, all edges must be loops connected to v. As a result, the degree of v is twice $\frac{T_{5k}}{2}$, which simplifies to T_{5k}. This confirms that the graph follows the tetranacci structure, making it a tetranacci graph. Furthermore, the proof proceeds in a similar manner when the size associated with vertex v is taken to be $\frac{T_{5k-2}}{2}$ or $\frac{T_{5k-1}}{2}$.

For this scenario, the degree sequence of a tetranacci graph of order 1 should be given by either $D_1 = \left\{T_{5k-2}^{(1)}\right\}, D_2 = \left\{T_{5k-1}^{(1)}\right\}$, or $D_3 = \left\{T_{5k}^{(1)}\right\}$. Consequently, its realization corresponds to the graph $L_{\frac{T_{5k-2}}{2}}$, $L_{\frac{T_{5k-1}}{2}}$, or $L_{\frac{T_{5k}}{2}}$ where $\Omega\left(L_{\frac{T_{5k-2}}{2}}\right) = T_{5k-2} - 2, \Omega\left(L_{\frac{T_{5k-1}}{2}}\right) = T_{5k-1} - 2$, or $\Omega\left(L_{\frac{T_{5k}}{2}}\right) = T_{5k} - 2$. This implies that all tetranacci graphs of order 1 can have 1, 2, 4, 28, 54, or 104 faces, meaning they take the forms $L_1, L_2, L_4, L_{28}, \cdots L_{\frac{T_{5k-2}}{2}}, L_{\frac{T_{5k-1}}{2}} \cdots$. Thus, we have established this result.

Corollary 3.1. A graph consisting of a single vertex is a tetranacci graph if and only if it has $L_{\frac{T_{5k-2}}{2}}, L_{\frac{T_{5k-1}}{2}}$, or $L_{\frac{T_{5k}}{2}}$ faces for some $k \in \mathbb{Z}^+$.

3.2. TETRANACCI GRAPHS OF ORDER 2

We now investigate the situation where the vertex degrees of the graph correspond to two consecutive tetranacci numbers. Specifically, our goal is to identify which pairs of consecutive tetranacci numbers can be realizable as graph degree sequences. According to the parity rule stated in Observation 1.1, the degree sequence of such a graph may consist of two consecutive odd tetranacci numbers or two consecutive even tetranacci numbers. In particular, the vertex degrees of such a graph must be either T_{5k-4} and T_{5k-3}, or T_{5k-2} and T_{5k-1}, or T_{5k-1} and T_{5k}, for some $k \in \mathbb{Z}^+$. In general, we have established the following result:

Theorem 3.2. *A graph with exactly two vertices can be classified as a tetranacci graph if and only if its degree sequence is either*

$$D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}\right\}, \text{ or } D_2 = \left\{T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}, \text{ or } D_3 = \left\{T_{5k-1}^{(1)}, T_{5k}^{(1)}\right\},$$

for some $k \in \mathbb{Z}^+$.

Under these conditions, the value of Ω for D_i's are, $i = 1,2,3$,

$$\Omega(D_1) = (T_{5k-4} - 2) \cdot 1 + (T_{5k-3} - 2) \cdot 1 = T_{5k-4} + T_{5k-3} - 4,$$

or

$$\Omega(D_2) = (T_{5k-2} - 2) \cdot 1 + (T_{5k-1} - 2) \cdot 1 = T_{5k-2} + T_{5k-1} - 4,$$

or

$$\Omega(D_3) = (T_{5k-1} - 2) \cdot 1 + (T_{5k} - 2) \cdot 1 = T_{5k-1} + T_{5k} - 4,$$

respectively.

Case 1. First, let $D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}\right\}$. In this case, Ω of D_1 is the upper bound on the number of components

$$c_{max} = \sum_{d_i\, even} a_i + \frac{1}{2}\sum_{d_i\, odd} a_i = \frac{1}{2}(1+1) = 1.$$

When considering the specific scenario for $k = 1$, in which $T_1 = 1$ and $T_2 = 1$, the only possible connected realization is$B_{0,0}$, as shown in Figure 6. This realization has

$$r = \frac{T_1 + T_2 - 4}{2} + 1 = \frac{1+1-4}{2} + 1 = 0$$

faces (loops), there is no face.

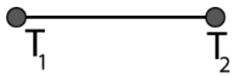

Figure 6. The graph $B_{0,0}$.

Let us now examine $D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}\right\}$ for $k \geq 2$. In light of the preceding discussion, identifying the existence of the unique connected realization shown in Figure 7 is straightforward. This realization is labeled as $B_{\frac{T_{5k-4}-1}{2}, \frac{T_{5k-3}-1}{2}}$. In this graph, the number of faces (loops) is

$$r = \frac{T_{5k-4} + T_{5k-3} - 4}{2} + 1 = \frac{T_{5k-4} + T_{5k-3} - 2}{2}.$$

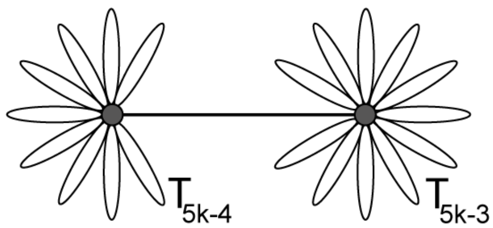

Figure 7. The graph $B_{\frac{T_{5k-4}-1}{2}, \frac{T_{5k-3}-1}{2}}$.

Case 2. Let $D_2 = \left\{T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}$. In this case, $\Omega(D_2)$, the upper bound on the number of components is given by

$$c_{max} = \sum_{d_i\, even} a_i + \frac{1}{2}\sum_{d_i\, odd} a_i = 1 + 1 = 2.$$

Therefore, there are two realizations: one connected and the other disconnected. When considering the specific case $k = 1$, in which $T_3 = 2$ and $T_4 = 4$, the only possible connected realization is $L_{2[0]}$, and the disconnected realization is $L_1 \cup L_2$, as shown in Figure 8. These realizations have

$$r_1 = \frac{T_3+T_4-4}{2} + 1 = \frac{2+4-4}{2} + 1 = 2 \text{ and } r_2 = \frac{T_3+T_4-4}{2} + 2 = \frac{2+4-4}{2} + 2 = 3$$

faces (loops), respectively.

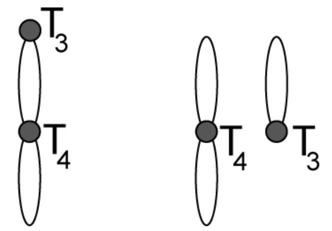

Figure 8. The graphs $L_{2[0]}$ and $L_1 \cup L_2$.

Now, consider $D_2 = \left\{T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}$ for $k \geq 2$. The only possible connected realization is $L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k-1}}{2}\right]} = L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k-2}}{2}\right]}$, and the disconnected realization, which consists of two components, is $L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}}$, as illustrated in Figure 9. These realizations have

$$r_1 = \frac{T_{5k-2}+T_{5k-1}-4}{2} + 1 = \frac{T_{5k-2}+T_{5k-1}-4}{2} - 1 \text{ and } r_2 = \frac{T_{5k-2}+T_{5k-1}-4}{2} + 2 = \frac{T_{5k-2}+T_{5k-1}}{2}$$

faces (loops), respectively.

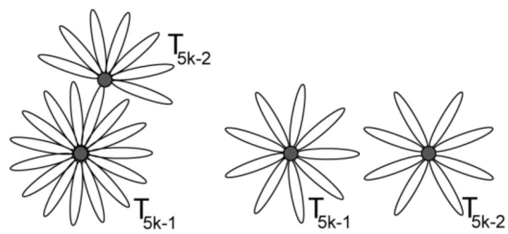

Figure 9. The graphs $L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k-1}}{2}\right]}$ and $L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}}$.

Case 3. Let $D_3 = \left\{T_{5k-1}^{(1)}, T_{5k}^{(1)}\right\}$. Similar to the approach followed in Case 2, for any positive integer $k \in \mathbb{Z}^+$, by Eq. (3), the maximum number of components is $c_{max} = 2$. Accordingly, there are two realizations: one connected and one disconnected. The connected realization is $L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k}}{2}\right]} = L_{\frac{T_{5k}}{2}\left[\frac{T_{5k-1}}{2}\right]}$, and the disconnected realization consists of two components: $L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k}}{2}}$. These realizations have

$$r_1 = \frac{T_{5k-1}+T_{5k}-4}{2} + 1 = \frac{T_{5k-1}+T_{5k}}{2} - 1 \text{ and } r_2 = \frac{T_{5k-1}+T_{5k}-4}{2} + 2 = \frac{T_{5k-1}+T_{5k}}{2}$$

faces (loops), respectively.

3.3. TETRANACCI GRAPHS OF ORDER 3

We now investigate the case where the vertex degrees of a graph correspond to three consecutive Tetranacci numbers. Specifically, our goal is to determine which such triples can be realized as graph degree sequences. According to the parity rule stated in Observation 1.1, the degree sequence of such a graph may consist either of two consecutive odd and one even tetranacci numbers, or of three successive even tetranacci numbers. In particular, the vertex degrees must be either T_{5k-4}, T_{5k-3} and T_{5k-2}, or T_{5k-2}, T_{5k-1} and T_{5k}, or T_{5k}, T_{5k+1} and T_{5k+2}, for some $k \in \mathbb{Z}^+$. In other words, a degree sequence composed of three consecutive tetranacci numbers can be realized only if it corresponds to one of the specific sequences:

$$D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}\right\}, \text{ or } D_2 = \left\{T_{5k-2}^{(1)}, T_{5k-1}^{(1)}, T_{5k}^{(1)}\right\}, \text{ or } D_3 = \left\{T_{5k}^{(1)}, T_{5k+1}^{(1)}, T_{5k+2}^{(1)}\right\}.$$

Under these conditions, the value of Ω for D_i's are, $i = 1,2,3$,

$$\begin{aligned}\Omega(D_1) &= (T_{5k-4} - 2) \cdot 1 + (T_{5k-3} - 2) \cdot 1 + (T_{5k-2} - 2) \cdot 1 \\ &= T_{5k-4} + T_{5k-3} + T_{5k-2} - 6 \\ &= T_{5k} - T_{5k-1} - 6,\end{aligned}$$

$$\begin{aligned}\Omega(D_2) &= (T_{5k-2} - 2) \cdot 1 + (T_{5k-1} - 2) \cdot 1 + (T_{5k} - 2) \cdot 1 \\ &= T_{5k-2} + T_{5k-1} + T_{5k} - 6 \\ &= T_{5k+2} - T_{5k+1} - 6,\end{aligned}$$

and

$$\begin{aligned}\Omega(D_3) &= (T_{5k} - 2) \cdot 1 + (T_{5k+1} - 2) \cdot 1 + (T_{5k+2} - 2) \cdot 1 \\ &= T_{5k} + T_{5k+1} + T_{5k+2} - 6 \\ &= T_{5k+4} - T_{5k+3} - 6.\end{aligned}$$

respectively.

Since $\Omega(D_i)$ are even for all $k \in \mathbb{Z}^+$, Theorem 2.1 ensures that D_i are always realizable. Now, let us compute the maximum number of components. In any of the realizations considered above, the following situations arise by Eq. (3): since the first and third degree sequences contain two odd and one even number, the maximum number of components is $c_{max} = 2$ in these cases. On the other hand, since all terms in the second-degree sequence are even, the maximum number of components becomes $c_{max} = 3$.

Case 1. Let $D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}\right\}$. In this case, the upper bound on the number of components, as given by Eq. (3), is $c_{max} = 2$. This implies that there are two possible realizations: one connected and one disconnected. For the specific case $k = 1$, in which $T_1 = 1$, $T_2 = 1$, and $T_3 = 2$, the only possible connected realization is $T_{0,0,0}$, and the disconnected realization is $L_1 \cup B_{0,0}$, as shown in Figure 10.

Figure 10. The graphs $T_{0,0,0}$ and $L_1 \cup B_{0,0}$.

These realizations have

$$r_1 = \frac{T_1 + T_2 + T_3 - 6}{2} + 1 = \frac{1+1+2-6}{2} + 1 = 0$$

and

$$r_2 = \frac{T_1 + T_2 + T_3 - 6}{2} + 2 = \frac{1+1+2-6}{2} + 2 = 1$$

faces (loops), respectively. That is, the connected realization has no faces, while the disconnected realization has one face.

Now, consider $D_1 = \left\{T^{(1)}_{5k-4}, T^{(1)}_{5k-3}, T^{(1)}_{5k-2}\right\}$ for $k \geq 2$. There are three possible connected realizations:

$$T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}}, \text{ and } B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right]}.$$

There is only one possible disconnected realization, which consists of two components:

$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}},$$

as illustrated in Figure 11.

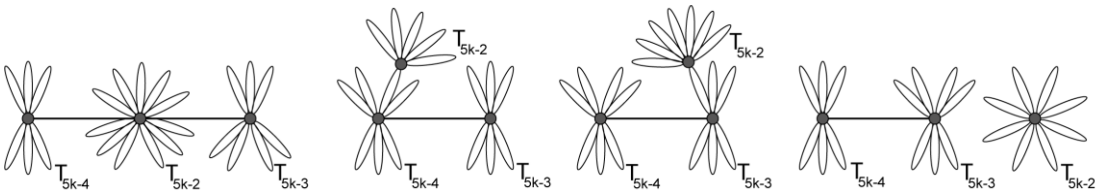

Figure 11. The graphs.

$$T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right]} \text{ and }$$
$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}}$$

These realizations contain either

$$r_1 = \frac{T_{5k-4} + T_{5k-3} + T_{5k-2} - 6}{2} + 1 = \frac{T_{5k} - T_{5k-1}}{2} - 2$$

or

$$r_2 = \frac{T_{5k-4} + T_{5k-3} + T_{5k-2} - 6}{2} + 2 = \frac{T_{5k} - T_{5k-1}}{2} - 1$$

faces (loops), respectively.

Case 2. Let $D_2 = \left\{T^{(1)}_{5k-2}, T^{(1)}_{5k-1}, T^{(1)}_{5k}\right\}$. In this case, the maximum number of components in any realization is $c_{max} = 3$, by using Eq. (3). This implies that there are three distinct realizations: one connected and two disconnected. These realizations correspond to the following numbers of loops (or faces):

$$r_1 = \frac{\Omega(D_2)}{2} + 1 = \frac{T_{5k-2} + T_{5k-1} + T_{5k} - 6}{2} + 1 = \frac{T_{5k+2} - T_{5k+1} - 6}{2} + 1$$
$$= \frac{T_{5k+2} - T_{5k+1}}{2} - 2,$$

$$r_2 = \frac{\Omega(D_2)}{2} + 2 = \frac{T_{5k-2} + T_{5k-1} + T_{5k} - 6}{2} + 2 = \frac{T_{5k+2} - T_{5k+1} - 6}{2} + 2$$
$$= \frac{T_{5k+2} - T_{5k+1}}{2} - 1,$$

$$r_3 = \frac{\Omega(D_3)}{2} + 3 = \frac{T_{5k-2} + T_{5k-1} + T_{5k} - 6}{2} + 3 = \frac{T_{5k+2} - T_{5k+1} - 6}{2} + 3 = \frac{T_{5k+2} - T_{5k+1}}{2},$$

where $k \geq 1$. The connected realization is given by the graph $L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k}-2}{2}\right]\right]}$.

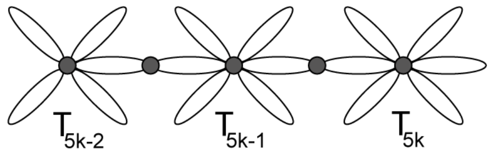

Figure 12: The graph $L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k}-2}{2}\right]\right]}$.

The disconnected realizations consist of either two or three components. In the case of two components, possible realizations include:

$$L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k}-2}{2}\right]}, L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k}-2}{2}\right]}, \text{ and } L_{\frac{T_{5k}}{2}} \cup L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k-2}-2}{2}\right]}.$$

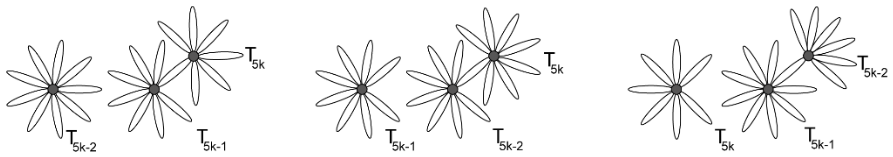

Figure 13: The graphs $L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k}-2}{2}\right]}$, $L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k}-2}{2}\right]}$, and $L_{\frac{T_{5k}}{2}} \cup L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k-2}-2}{2}\right]}$.

Finally, in the case of three disconnected components, the realization is

$$L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k}}{2}}.$$

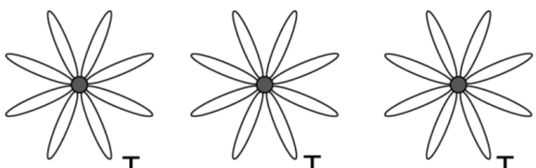

Figure 14. The graph $L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k}}{2}}$.

Case 3. Let $D_3 = \left\{T_{5k}^{(1)}, T_{5k+1}^{(1)}, T_{5k+2}^{(1)}\right\}$. Similar to the approach followed in Case 1, for any positive integer $k \in \mathbb{Z}^+$, the maximum number of components is $c_{max} = 2$. Accordingly, there are two types of realizations: one with three connected components and one with disconnected components. The connected realizations are

$$B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}},\ B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2}\right]},\text{ and } T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}}.$$

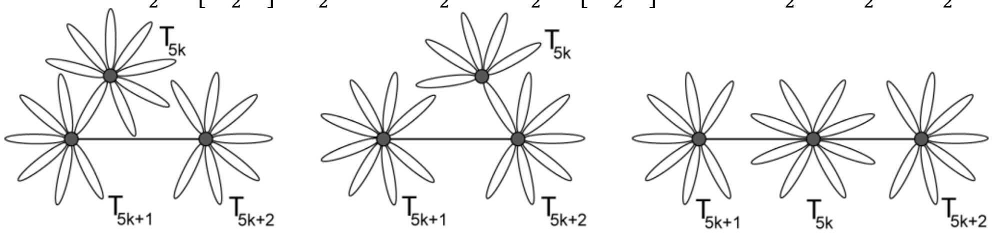

Figure 15. The graphs $B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}}$, $B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2}\right]}$, and $T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}}$.

The disconnected realization with two components is

$$L_{\frac{T_{5k}}{2}} \cup B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}}$$

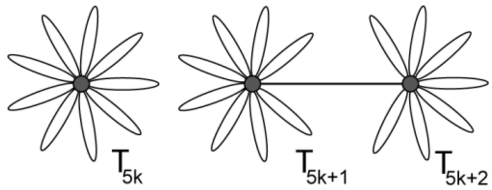

Figure 16. The graph $L_{\frac{T_{5k}}{2}} \cup B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}}$.

The connected realizations have

$$r_1 = \frac{\Omega(D_3)}{2} + 1 = \frac{T_{5k} + T_{5k+1} + T_{5k+2} - 6}{2} + 1 = \frac{T_{5k+4} - T_{5k+3} - 6}{2} + 1$$
$$= \frac{T_{5k+4} - T_{5k+3}}{2} - 2$$

faces (loops), whereas the disconnected realizations have

$$r_2 = \frac{\Omega(D_3)}{2} + 2 = \frac{T_{5k} + T_{5k+1} + T_{5k+2} - 6}{2} + 2 = \frac{T_{5k+4} - T_{5k+3} - 6}{2} + 2$$
$$= \frac{T_{5k+4} - T_{5k+3}}{2} - 1,$$

faces.

In general, we establish the following result:

Theorem 3.3. *A graph with exactly three vertices can be classified as a Tetranacci graph if and only if its degree sequence is one of the following:*

$$D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}\right\}, \text{ or } D_2 = \left\{T_{5k-2}^{(1)}, T_{5k-1}^{(1)}, T_{5k}^{(1)}\right\}, \text{ or } D_3 = \left\{T_{5k}^{(1)}, T_{5k+1}^{(1)}, T_{5k+2}^{(1)}\right\},$$

where $k \in \mathbb{Z}^+$.

3.4. TETRANACCI GRAPHS OF ORDER 4

In the final step, we consider the case when $n = 4$. Our objective is to determine whether a degree sequence composed of four consecutive tetranacci numbers can be realized. To ensure the sum of the sequence is even, the four consecutive tetranacci numbers must be selected as either $T_{5k-4}, T_{5k-3}, T_{5k-2}$, and T_{5k-1}, or $T_{5k-1}, T_{5k}, T_{5k+1}$, and T_{5k+2} for $k \geq 1$. That is, for a degree sequence D consisting of four consecutive tetranacci numbers to be realizable, it must be either

$$D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}, \text{ or } D_2 = \left\{T_{5k-1}^{(1)}, T_{5k}^{(1)}, T_{5k+1}^{(1)}, T_{5k+2}^{(1)}\right\}.$$

Under these conditions, the value of Ω for D_1 and D_2 are computed as follows:

$$\begin{aligned}\Omega(D_1) &= (T_{5k-4} - 2) \cdot 1 + (T_{5k-3} - 2) \cdot 1 + (T_{5k-2} - 2) \cdot 1 + (T_{5k-1} - 2) \cdot 1 \\ &= T_{5k-4} + T_{5k-3} + T_{5k-2} + T_{5k-1} - 8 \\ &= T_{5k} - 8,\end{aligned}$$

and

$$\begin{aligned}\Omega(D_2) &= (T_{5k-1} - 2) \cdot 1 + (T_{5k} - 2) \cdot 1 + (T_{5k+1} - 2) \cdot 1 + (T_{5k+2} - 2) \cdot 1 \\ &= T_{5k-1} + T_{5k} + T_{5k+1} + T_{5k+2} - 8 \\ &= T_{5k+3} - 8.\end{aligned}$$

respectively.

Since $\Omega(D_i), i = 1,2$, are even for all $k \in \mathbb{Z}^+$, Theorem 2.1 ensures that D_i's are always realizable. We now determine the maximum number of components. In each of the realizations described above, the following observation holds since the degree sequences include two even and two odd numbers, the maximum number of components is $c_{max} = 3$.

Case 1. Let $D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}$. For the specific case $k = 1$, in which $T_1 = 1$, $T_2 = 1, T_3 = 2$, and $T_4 = 4$, the connected realizations are $T_{0,1[0],0}, Q_{0,0,1,0}$, and $Q_{0,1,0,0}$, while the disconnected realizations include $L_{1[2]} \cup B_{0,0}$, $L_2 \cup T_{0,0,0}$, and $L_1 \cup T_{0,1,0}$. The only realization consisting of three disconnected components is $L_1 \cup L_2 \cup B_{0,0}$, as shown in Figure 17.

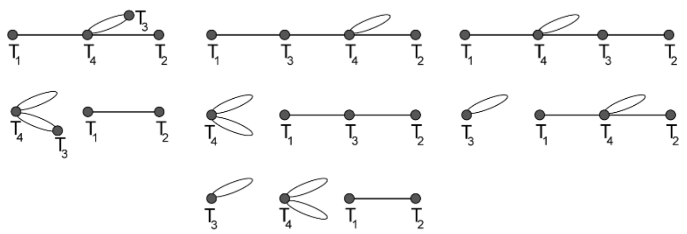

Figure 17. The graphs $T_{0,1[0],0}, Q_{0,0,1,0}, Q_{0,1,0,0}, L_{1[2]} \cup B_{0,0}$, $L_2 \cup T_{0,0,0}$ and $L_1 \cup L_2 \cup B_{0,0}$.

The number of faces (or loops) in these realisations is calculated as:

$$r_1 = \frac{\Omega(D_1)}{2} + 1 = \frac{T_1 + T_2 + T_3 + T_4 - 8}{2} + 1 = \frac{T_5 - 8}{2} + 1 = \frac{8 - 8}{2} + 1 = 1,$$

or

$$r_2 = \frac{\Omega(D_1)}{2} + 2 = \frac{T_1 + T_2 + T_3 + T_4 - 8}{2} + 2 = \frac{T_5 - 8}{2} + 2 = \frac{8 - 8}{2} + 2 = 2,$$

or

$$r_3 = \frac{\Omega(D_1)}{2} + 3 = \frac{T_1 + T_2 + T_3 + T_4 - 8}{2} + 3 = \frac{T_5 - 8}{2} + 3 = \frac{8 - 8}{2} + 3 = 3.$$

Now, consider $D_1 = \left\{T_{5k-4}^{(1)}, T_{5k-3}^{(1)}, T_{5k-2}^{(1)}, T_{5k-1}^{(1)}\right\}$ for $k \geq 2$. There are twenty possible connected realizations:

$$B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right]\right],\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k-2}-2}{2}\right]\right],\frac{T_{5k-3}-1}{2}},$$
$$B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2},\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right]\right]},$$
$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2},\frac{T_{5k-1}-2}{2}\right]},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k-2}-2}{2}\right]\right]},$$
$$B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2};\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-1}-2}{2};\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}},$$
$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2};\frac{T_{5k-1}-2}{2}\right]},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-1}-2}{2};\frac{T_{5k-2}-2}{2}\right]},$$
$$B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-1}-2}{2},\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-1}-2}{2},\frac{T_{5k-2}-2}{2}\right]},$$
$$T_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}},\ T_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-1}-2}{2},\frac{T_{5k-3}-1}{2}},$$
$$T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k-3}-1}{2}},\ T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}},$$
$$T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right]},\ T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right]},$$
$$Q_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k-3}-1}{2}},\ Q_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}}.$$

There are eight possible disconnected realizations consisting of two components, and only one disconnected realization consisting of three components:

$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}\left[\frac{T_{5k-1}-2}{2}\right]},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right],\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}},$$
$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-2}-2}{2}\right]} \cup L_{\frac{T_{5k-1}}{2}},\ B_{\frac{T_{5k-4}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}},$$
$$B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right]} \cup L_{\frac{T_{5k-2}}{2}},\ T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-2}-2}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}},$$
$$T_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}},\ B_{\frac{T_{5k-4}-1}{2},\frac{T_{5k-3}-1}{2}} \cup L_{\frac{T_{5k-2}}{2}} \cup L_{\frac{T_{5k-1}}{2}}.$$

These realizations contain either

$$r_1 = \frac{\Omega(D_1)}{2} + 1 = \frac{T_{5k-4} + T_{5k-3} + T_{5k-2} + T_{5k-1} - 8}{2} + 1 = \frac{T_{5k} - 8}{2} + 1 = \frac{T_{5k}}{2} - 3,$$

or

$$r_2 = \frac{\Omega(D_1)}{2} + 2 = \frac{T_{5k-4} + T_{5k-3} + T_{5k-2} + T_{5k-1} - 8}{2} + 2 = \frac{T_{5k} - 8}{2} + 2 = \frac{T_{5k}}{2} - 2,$$

or

$$r_3 = \frac{\Omega(D_1)}{2} + 3 = \frac{T_{5k-4} + T_{5k-3} + T_{5k-2} + T_{5k-1} - 8}{2} + 3 = \frac{T_{5k} - 8}{2} + 3 = \frac{T_{5k}}{2} - 1$$

faces (loops), respectively.

Case 2. Let $D_2 = \left\{T_{5k-1}^{(1)}, T_{5k}^{(1)}, T_{5k+1}^{(1)}, T_{5k+2}^{(1)}\right\}$. Now, consider realizations having these vertex degrees for $k \geq 1$. There are twenty possible connected realizations:

$$B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k}-2}{2}\right]\right],\frac{T_{5k+2}-1}{2}}, B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right]\right],\frac{T_{5k+2}-1}{2}},$$
$$B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k-1}-2}{2},\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}}, B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right]\right]},$$
$$B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k-1}-2}{2},\frac{T_{5k}-2}{2}\right]}, B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k}-2}{2}\right]\right]},$$
$$B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k-1}-2}{2};\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}}, B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2};\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k+2}-1}{2}},$$
$$B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k-1}-2}{2};\frac{T_{5k}-2}{2}\right]}, B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2};\frac{T_{5k-1}-2}{2}\right]},$$
$$B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2},\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k+2}-1}{2}}, B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2},\frac{T_{5k-1}-2}{2}\right]},$$
$$T_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2}\right],\frac{T_{5k-1}-2}{2},\frac{T_{5k+2}-1}{2}}, T_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}},$$
$$T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k-1}-2}{2}\left[\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}}, T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k+2}-1}{2}},$$
$$T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2}\right]}, T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right]},$$
$$Q_{\frac{T_{5k+1}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}}, Q_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k+2}-1}{2}}.$$

There are eight possible disconnected realizations consisting of two components, and only one disconnected realization consisting of three components:

$$B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}\left[\frac{T_{5k}-2}{2}\right]}, B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right],\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k}}{2}},$$
$$B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k-1}-2}{2}\right]} \cup L_{\frac{T_{5k}}{2}}, B_{\frac{T_{5k+1}-1}{2}\left[\frac{T_{5k}-2}{2}\right],\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}},$$
$$B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}\left[\frac{T_{5k}-2}{2}\right]} \cup L_{\frac{T_{5k-1}}{2}}, T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k-1}-2}{2},\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k}}{2}},$$
$$T_{\frac{T_{5k+1}-1}{2},\frac{T_{5k}-2}{2},\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}}, B_{\frac{T_{5k+1}-1}{2},\frac{T_{5k+2}-1}{2}} \cup L_{\frac{T_{5k-1}}{2}} \cup L_{\frac{T_{5k}}{2}}.$$

These realizations contain either

$$r_1 = \frac{\Omega(D_2)}{2} + 1 = \frac{T_{5k-1} + T_{5k} + T_{5k+1} + T_{5k+2} - 8}{2} + 1 = \frac{T_{5k+3} - 8}{2} + 1 = \frac{T_{5k+3}}{2} - 3,$$

or

$$r_2 = \frac{\Omega(D_2)}{2} + 2 = \frac{T_{5k-1} + T_{5k} + T_{5k+1} + T_{5k+2} - 8}{2} + 2 = \frac{T_{5k+3} - 8}{2} + 2 = \frac{T_{5k+3}}{2} - 2,$$

or

$$r_3 = \frac{\Omega(D_2)}{2} + 3 = \frac{T_{5k-1} + T_{5k} + T_{5k+1} + T_{5k+2} - 8}{2} + 3 = \frac{T_{5k+3} - 8}{2} + 3 = \frac{T_{5k+3}}{2} - 1,$$

faces (i.e., loops), respectively.

In general, we establish the following result:

Theorem 3.4. *A graph with exactly four vertices can be classified as a tetranacci graph if and only if its degree sequence is one of the following*:

$$D_1 = \left\{T^{(1)}_{5k-4}, T^{(1)}_{5k-3}, T^{(1)}_{5k-2}, T^{(1)}_{5k-1}\right\} \text{ or } D_2 = \left\{T^{(1)}_{5k-1}, T^{(1)}_{5k}, T^{(1)}_{5k+1}, T^{(1)}_{5k+2}\right\},$$

where $k \in \mathbb{Z}^+$.

3.5. TETRANACCI GRAPHS OF ORDER $\mathrm{n} \geq 5$

As established in Observation 1.1, the Tetranacci sequence follows a periodic parity pattern of length five: odd, odd, even, even, even. This implies that every block of five consecutive Tetranacci numbers contains exactly two odd and three even terms. As a result, the sum of any five consecutive Tetranacci numbers is always even. By Theorem 2.1, this parity property ensures that for any positive integer l, if we consider a graph of order $n = 5l$ whose degree sequence consists of n consecutive Tetranacci numbers is realizable. This observation leads directly to the following corollaries.

Corollary 3.2. Let T_r denote the r-th Tetranacci number. Then for any integer r, the sum

$$T_r + T_{r+1} + T_{r+2} + T_{r+3} + T_{r+4}$$

is even.

Corollary 3.3. Let $n = 5l$ for some positive integer l. Then any degree sequence consisting of n consecutive Tetranacci numbers is realizable, regardless of the starting index.

In subsections 3.1–3.4, we established the conditions under which sets of $n = 1,2,3,$ and 4 consecutive tetranacci numbers can be realizable and provided a generalization for sets of length $n = 5l$ in Corollary 3.3. Four cases remain where the length of the set is $n = 5l + 1, n = 5l + 2, n = 5l + 3,$ or $n = 5l + 4$. A similar discussion leads to the following generalizations:

Corollary 3.4.
i) A set $\{\mathrm{T_r}, \mathrm{T_{r+1}}, \mathrm{T_{r+2}}, \cdots, \mathrm{T_{r+n-1}}\}$ consisting of $\mathrm{n} = 5\mathrm{l} + 1$ consecutive Tetranacci numbers is realizable if and only if $\mathrm{r} \not\equiv 1, 2 \pmod 5$.
ii) A set $\{\mathrm{T_r}, \mathrm{T_{r+1}}, \mathrm{T_{r+2}}, \cdots, \mathrm{T_{r+n-1}}\}$ consisting of $\mathrm{n} = 5\mathrm{l} + 2$ consecutive Tetranacci numbers is realizable if and only if $\mathrm{r} \not\equiv 2 \pmod 5$.
iii) A set $\{\mathrm{T_r}, \mathrm{T_{r+1}}, \mathrm{T_{r+2}}, \cdots, \mathrm{T_{r+n-1}}\}$ consisting of $\mathrm{n} = 5\mathrm{l} + 3$ consecutive Tetranacci numbers is realizable if and only if $\mathrm{r} \not\equiv 2, 4 \pmod 5$.
iv) A set $\{\mathrm{T_r}, \mathrm{T_{r+1}}, \mathrm{T_{r+2}}, \cdots, \mathrm{T_{r+n-1}}\}$ consisting of $\mathrm{n} = 5\mathrm{l} + 4$ consecutive Tetranacci numbers is realizable if and only if $\mathrm{r} \not\equiv 2, 3 \pmod 5$.

CONCLUSIONS

In this paper, we introduced Tetranacci graphs, whose degree sequences consist of consecutive Tetranacci numbers. Using the graph invariant, we established necessary and sufficient conditions for the realizability of these sequences. We also completely classified all Tetranacci graphs and obtained a general structural characterization.

These results provide a foundation for the study of graphs generated by Tetranacci numbers and highlight a new connection between graph realizability and linear recurrence sequences. Future work may focus on analogous graph classes arising from other recursive sequences and on further structural properties of Tetranacci graphs.

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