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SPECIAL DUAL CATENARY CURVES AND THEIR CHARACTERIZATIONS

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Abstract. *This study investigates special dual catenary curves by taking the domain as the set of real numbers. The dual curves are constructed to satisfy the orthogonality condition* $\langle \gamma'(s), {\gamma^*}'(s) \rangle = 0$. *Assuming that* γ *represents a catenary curve according to the given definition, the involute and evolute of the catenary are analyzed. Based on these examinations, the corresponding dual curves and their geometric characterizations are explored in detail.*

Keywords: *Dual space; dual catenary curve, involute-evolute curve.*
Mathematics Subject Classification: *53A10; 53A35.*

1. INTRODUCTION

The concept of dual numbers was first introduced by W. K. Clifford (1873), who recognized their potential as a powerful tool for geometric analysis [1]. Later, Kotelnikov (1895) extended their use to mechanics, leading to important developments in the study of rigid body motion. Since then, dual numbers have found extensive applications in various fields, including screw theory, kinematic modelling of planar and spatial mechanisms, and the analysis of inertial forces. The introduction of the dual inertia operator enabled the three-dimensional formulation of rigid body dynamics relative to an arbitrary point [2,3]. In line geometry, lines in three-dimensional Euclidean space are represented by six Plücker coordinates—three for direction and three for moment. Building upon this, Eduard Study (1903) employed dual numbers and dual vectors to explore the geometry and kinematics of lines, establishing a one-to-one correspondence between oriented lines in Euclidean space and points on the unit dual sphere [4]. Later, Hacısalihoğlu (1977) investigated the Study transformation of a circle, contributing further to the understanding of this correspondence [5]. According to the Study correspondence principle, every curve drawn on the unit dual sphere generates a ruled surface. This principle has inspired numerous studies on ruled surfaces and dual curves in recent decades [6-12].

The catenary is a classical curve that has been the subject of extensive research in variational calculus. It arises in two principal contexts: first, as the solution to the physical problem describing the shape of a hanging chain, and second, as the generating curve of a surface of revolution with minimal surface area. In the two-dimensional Euclidean space $(R^2, <,>)$, let $\beta = \beta(s)$ denote a curve satisfying the relation $k(s) = \alpha \frac{<n,u>}{<\beta,u>}$ for a constant unit vector u and a real number α. The curve $\beta(s)$ is called the α -catenary, where k and n are the curvature of $\beta(s)$ and the unit normal vector field, respectively. When the notion of a catenary is extended to hypersurfaces, the corresponding equation takes the form $nH =$

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$\frac{\alpha}{u\sqrt{1+|\nabla u|^2}}$, where H denotes the mean curvature of the hypersurface. This surface is known as the n-dimensional analog of the catenary or or as a singular minimal surface when $\alpha = 1$. The equation describes the equilibrium configuration of the potential energy functional $\int_\Omega u^\alpha\sqrt{1+|\nabla u|^2}$ which represents the energy of a surface under gravitational forces, and serves as the Euler–Lagrange equation associated with E given as under gravitational forces and is the Euler equation of potential energy E [13-15]. Historically, this problem traces back to the foundational work of Lagrange and Poisson, who studied the equations modelling heavy surfaces in a vertical gravitational field. In the particular case $\alpha = 1$, the singular minimal surface corresponds to a surface that minimizes its center of gravity. Such equations have been a central topic of study in geometric analysis since the 1980s [16-18].

This study investigates the extension of classical catenary curves within the framework of dual numbers. The main objective is to define and analyze special types of dual catenary curves and to explore their geometric properties. Section 2 presents the theoretical background together with the materials and methods used in the study, including the main mathematical concepts and formulations. Section 3 focuses on the construction of special dual catenary curves and examines their involute and evolute forms to understand their geometric behaviour. This section also presents the discussion of the results. Section 4 provides the discussion of the results and concluding remarks, highlighting the significance of the findings and potential directions for further research.

2. MATERIALS AND METHODS

2.1. MATERIALS

Let R^3 be a Euclidean 3-space with the canonical metric $\langle\cdot,\cdot\rangle$ and (x, y, z) denote the rectangular coordinates of R^3. Consider a smooth immersion $X: M^2 \to R^3_+(v)$ of an oriented surface M^2 into the half-space $R^3_+(v) = \{q \in R^3 \mid \langle q, v\rangle > 0\}$, where v is a given unit vector in R^3. The potential α-energy of X in the direction v is defined as

$$E(X) = \int_{M^2} \langle X(q), v\rangle^\alpha \, dM^2, \quad q \in M^2,$$

where dM^2 denotes the measure on M^2 induced by the metric tensor of R^3. If X is the critical point of the energy E, then the mean curvature H of M^2 verifies

$$2H = \alpha\frac{\langle N, v\rangle}{\langle X, v\rangle}, \tag{1.1}$$

where N is the unit normal to M^2 (*see* [13,14]). A surface M is called a singular minimal surface if the mean curvature H satisfies (1.1); we also say that M is an α-minimal surface if we want to emphasise the value of α. Some particular cases of α -minimal surfaces are: if $\alpha = 0$, then M is a minimal surface; if $\alpha = 1$, then M is the two-dimensional analogue of the catenary; if $\alpha = -2$, then M is a minimal surface in 3-dimensional hyperbolic space H^3 when this space is viewed in the upper half-space model. Throughout the paper, we assume $\alpha \neq 0$ unless otherwise stated. The one-dimensional case corresponds to fixing a unit vector $w \in R^2$ and considering a planar curve $\beta: I \to R^2, \beta = \beta(s)$, whose curvature κ satisfies

$$\kappa(s)=\alpha\frac{<N(s),w>}{<\beta(s),w>},s\in I, \tag{1.2}$$

where $N(s)$ is the principal unit normal vector of β, which are referred to as α -catenaries. After a change of coordinates, we suppose $w=(0,1)$ and β is given locally by $\beta(s)=(s,f(s))$ for some positive function $f:I\subset R\rightarrow R^{+}$. Let us determine the tangent vector, normal vector, and the curvature of the catenary. The unit tangent vector $T(s)$ and the unit normal vector $N(s)$ are given by

$$T(s)=\left(\frac{1}{\sqrt{1+f'^2}},\frac{f'}{\sqrt{1+f'^2}}\right),N(s)=\left(-\frac{f'}{\sqrt{1+f'^2}},\frac{1}{\sqrt{1+f'^2}}\right).$$

Using the relation $T'(s)=\|\beta'(s)\|\kappa(s)N(s)$, where $\kappa(s)$ denotes the curvature, we compute: $T'(s)=\left(-\frac{f'f'\prime}{\left(1+f'^2\right)^{\frac{3}{2}}},\frac{f''}{\left(1+f'^2\right)^{\frac{3}{2}}}\right)$. Substituting this expression into the above relation yields:

$$\left(-\frac{f'f''}{\left(1+f'^2\right)^{\frac{3}{2}}},\frac{f''}{\left(1+f'^2\right)^{\frac{3}{2}}}\right)=\kappa(s)\sqrt{1+f'^2}\left(-\frac{f'}{\sqrt{1+f'^2}},\frac{1}{\sqrt{1+f'^2}}\right).$$

By equating the components, the curvature is obtained as $\kappa(s)=\frac{f'\prime}{\left(1+f'^2\right)^{\frac{3}{2}}}\cdot$ Thus, the equation (1.2) can be rewritten in the case $(\alpha=1)$ as

$$\frac{f''(s)}{1+f'(s)^2}=\frac{1}{f(s)}, \tag{1.3}$$

[13,14,17,18]. Let $D=R\times R=\{\bar{x}=(x,x^*):x,x^*\in R\}$ denote the set of pairs. An equality and two operations are defined as follows:
1) $\bar{x}=\bar{y}$ if and only if $x=y$ and $x^*=y^*$.
2) Given $\bar{x}=(x,x^*)$ and $\bar{y}=(y,y^*)$, we define the following operations:

$$+_D:D\times D\rightarrow D,x+_Dy=(x+y,x^*+y^*),$$
$$._D:D\times D\rightarrow D,x._D\,y=(xy,xy^*+yx^*).$$

Let $D=R\times R=\{x=(x,x^*)\mid x,x^*\in R\}$, where equality and two inner operations $+_D$ and $._D$ are considered, then D is called the dual numbers system, and an element $\bar{x}=(x,x^*)$ is called a dual number. Here, x is called the real part of $\bar{x}$, and x^* is called the dual part of $\bar{x}$. For simplicity, throughout this work, we will use $+$ and . instead of $+_D$ and $._D$operations. In this case, the set of all dual numbers is given $D=\{\bar{x}=x+\varepsilon x^*\mid x,x^*\in R,\varepsilon^2=0\}$. Thus, the sum and product of dual numbers $\bar{x}=x+\varepsilon x^*$ and $\bar{y}=y+\varepsilon y^*$ are defined as:

$$(x+\varepsilon x^*)+(y+\varepsilon y^*)=(x+y)+\varepsilon(x^*+y^*);$$
$$(x+\varepsilon x^*)\cdot(y+\varepsilon y^*)=xy+\varepsilon(xy^*+x^*y).$$

Additionally, for these dual numbers, if $y\neq0$, then the division is defined as follows

$$\frac{\bar{x}}{\bar{y}} = \frac{x}{y} + \varepsilon\left(\frac{x^* y - x y^*}{y^2}\right).$$

The scalar and the vector products of any dual vectors $\vec{\bar{x}} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{\bar{y}} = \vec{y} + \varepsilon\vec{y}^*$ are defined respectively by

$$\langle\vec{\bar{x}}, \vec{\bar{y}}\rangle = \langle\vec{x}, \vec{y}\rangle + \varepsilon(\langle\vec{x}, \vec{y}^*\rangle + \langle\vec{x}^*, \vec{y}\rangle)$$

and

$$\vec{\bar{x}} \wedge \vec{\bar{y}} = \vec{x} \wedge \vec{y} + \varepsilon(\vec{x} \wedge \vec{y}^* + \vec{x}^* \vec{y}).$$

Here $\vec{\bar{x}} = \vec{x}_i + \varepsilon\vec{x}_i^*$ and $\vec{\bar{y}} = \vec{y}_i + \varepsilon\vec{y}_i^* \in D^3$, for $i = 1,2,3$. The norm of a dual vector $\bar{x}$ is given by

$$\|\vec{\bar{x}}\| = \|\vec{x}\| + \varepsilon\frac{\langle\vec{x}, \vec{x}^*\rangle}{\|\vec{x}\|}.$$

Let a unit speed regular curve C and C_i be given. For $\forall\, s \epsilon I$, then the curve C_i is called the involute of the curve C, if the tangent at the point $C(s)$ to the curve C passes through the tangent at the point $C_i(s)$ to the curve C_i and $\langle T_i(s), T(s)\rangle = 0$. The distance between corresponding points of the involute curve in E^3 is $d\big(C(s), C_i(s)\big) = |k - s|, k = sbt$. Let a unit speed regular curve C and C_e be given. For $\forall\, s \in I$, then the curve C_e is called the evolute of the curve C if the tangent at the point $C_e(s)$ to the curve C_e passes through the tangent at the point C to the curve C and $\langle T_e(s), T(s)\rangle = 0$ [19].

2.2. METHODS

This study investigates the characterization of special dual catenary curves in the dual plane D^2. The approach begins by considering dual curves whose domain is the set of real numbers and which satisfy the orthogonality condition $\langle\gamma'(s), \gamma^{*\prime}(s)\rangle = 0$ where γ and γ^* denote the real and dual parts, respectively. A dual catenary is defined as a dual curve $\bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ whose real part γ is a classical catenary. Under the assumptions $\|\gamma^{*\prime}\| = 1$ and $\langle\gamma'(s), \gamma^{*\prime}(s)\rangle = 0$, a system of differential equations is derived to determine the components of the dual part. Solving these equations yields an explicit integral expression for the dual catenary. From these results, the tangent vector, normal vector, and dual curvature of the curve are determined. The curvature of the dual catenary is expressed as a dual quantity consisting of a real and an infinitesimal component, which characterizes the geometric behavior of the curve in D^2. Furthermore, the analysis is extended to special cases governed by the Emden--Fowler type equation for $\alpha = 1$, leading to curvature formulas dependent on a constant parameter c

In addition, two related constructions are investigated:

Dual Involute Catenary: The involute of a dual catenary is obtained using the standard involute transformation applied to the real part, and then extended to the dual framework. The general expression of the dual involute curve is derived, followed by simplified forms for specific catenaries such as $\gamma(\mathrm{s}) = (s, coshs)$.

Dual Evolute Catenary: Similarly, the evolute of a dual catenary is defined using the normal vector and curvature of the real catenary. The resulting dual evolute expression incorporates both position and curvature information of the generating curve. Simplified cases for classical catenaries are presented to demonstrate the construction.

For each case, the tangent and normal vectors, together with the dual curvature functions, are determined analytically. These results provide the fundamental geometric characteristics of dual catenaries and their associated involute and evolute curves.

3. RESULTS AND DISCUSSION

3.1. RESULTS

This section focuses on special dual catenary curves. The analysis focuses on the case where the domain of the dual function is taken to be the set of real numbers. We consider dual curves satisfying the condition $\langle \gamma'(s), {\gamma^*}'(s)\rangle = 0$. Then, the curve γ is assumed to be a catenary curve according to the above definition. Let the real part $\bar{\gamma}$ be a catenary curve. By examining its involute and evolute, we explore corresponding dual curves and their properties.

Definition 3.1. *Let $\bar{\gamma}: I \subseteq R \to D^2, \bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ be a dual curve defined in the D^2, with $(\langle \gamma'(s), {\gamma^*}'(s)\rangle = 0)$, where γ is the real part and γ^* is the dual part*. If the real part γ is a catenary, then $\bar{\gamma}$ is called a dual catenary.
Assume that s is an arc parameter for $\gamma^*(s) = \big(x(s), y(s)\big)$ but not for $\gamma(s) = \big(s, f(s)\big)$. Using the conditions $\|\gamma^*(s)'\| = 1$ and $\langle \gamma'(s), {\gamma^*}'(s)\rangle = 0$, we obtain the differential equations:

$$x' + f'y' = 0 \quad \text{and} \quad x'^2 + y'^2 = 1$$

Substituting $x' = -f'y'$ into the second equation gives: $(-f'y')^2 + y'^2 = 1$. Solving for y', we have

$$y'^2 = \frac{1}{(f')^2 + 1} \Rightarrow y = \pm \int \frac{1}{\sqrt{(f')^2 + 1}} ds$$

Now substitute $y' = \pm \frac{1}{\sqrt{(f')^2+1}}$ into the expression for x':

$$x = \mp \int \frac{f'}{\sqrt{(f')^2 + 1}} ds.$$

Choosing the positive sign yields, then we have $x = -\int \frac{f'}{\sqrt{(f')^2+1}} ds$. The following proposition can be stated.

Proposition 3.1. Let $\bar{\gamma}: I \subseteq R \to D^2, \bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ be a dual catenary defined in the D^2, $\|{\gamma^*}'(s)\| = 1$ and $\langle \gamma'(s), {\gamma^*}'(s)\rangle = 0$. The dual catenary is written by

$$\bar{\gamma}(s) = \big(s, f(s)\big) + \varepsilon\left(-\int \frac{f'}{\sqrt{(f')^2 + 1}} ds, \int \frac{1}{\sqrt{(f')^2 + 1}} ds\right). \tag{3.1}$$

The derivative in the tangent direction of $\bar{\gamma}$ is defined as

$$\overline{\gamma'}(s) = \gamma'(s) + \varepsilon\gamma^{*\prime}(s) = \big(1, f'(s)\big) + \varepsilon\left(-\frac{f'}{\sqrt{(f')^2+1}}, \frac{1}{\sqrt{(f')^2+1}}\right).$$

Since the norm is $\|\gamma^{*\prime}(s)\| = 1$, the tangent vector of the dual catenary is

$$\overline{T}(s) = \left(\frac{1}{\sqrt{1+f'^2}}, \frac{f'}{\sqrt{1+f'^2}}\right) + \varepsilon\left(-\frac{f'}{1+f'^2}, \frac{1}{1+f'^2}\right). \tag{3.2}$$

Theorem 3.1. *Let $\bar{\gamma}: I \subseteq R \to D^2$ be a dual catenary defined by $\bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$, where $\|\gamma^{*\prime}(s)\| = 1$ and $\langle\gamma'(s), \gamma^{*\prime}(s)\rangle = 0$.* Then the unit normal $\overline{N}$ is given by

$$\overline{N} = (-f', 1) + \varepsilon\left(\frac{-1}{\sqrt{(f')^2+1}}, -\frac{f'}{\sqrt{(f')^2+1}}\right). \tag{3.3}$$

The curvature $\bar{\kappa}$ of $\bar{\gamma}$ is given by

$$\bar{\kappa} = \frac{f''}{(1+f'^2)^{3/2}} - \varepsilon\frac{f'f''}{(1+f'^2)^2}. \tag{3.4}$$

Proof: The transformation from the tangent vector to the normal vector of a curve can be interpreted as a rotation. In this context, the normal vector is obtained by a rotation of $\pi/2$ from the tangent vector. Thus, the unit normal vector becomes

$$\overline{N} = \left(-\frac{f'}{\sqrt{1+f'^2}}, \frac{1}{\sqrt{1+f'^2}}\right) + \varepsilon\left(-\frac{1}{1+f'^2}, \frac{-f'}{1+f'^2}\right).$$

The curvature $\bar{\kappa}$ of $\bar{\gamma}$ is given by $\bar{\kappa} = \frac{1}{\|\overline{\gamma'}(s)\|}\langle\overline{T}', \overline{N}\rangle$. Substituting into the expression, we arrive at

$$\bar{\kappa} = \frac{1}{\sqrt{1+f'^2}}\left\langle\left(-\frac{f'f''}{(1+f'^2)^{3/2}}, \frac{f''}{(1+f'^2)^{3/2}}\right) + \varepsilon\left(\frac{f''(-1+f'^2)}{(1+f'^2)^2}, \frac{-2f'f''}{(1+f'^2)^2}\right),\right.$$
$$\left.\left(-\frac{f'}{\sqrt{1+f'^2}}, \frac{1}{\sqrt{1+f'^2}}\right) + \varepsilon\left(-\frac{1}{1+f'^2}, \frac{-f'}{1+f'^2}\right)\right\rangle.$$

This computation leads to the expression in equation (3.4).

Theorem 3.2. *Let $\bar{\gamma}: I \subseteq R \to D^2, \bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ be a dual curve, such that $\|\gamma^{*\prime}(s)\| = 1$ and $\langle\gamma'(s), \gamma^{*\prime}(s)\rangle = 0$.*

i) If the dual curve $\bar{\gamma}$ is a dual catenary, then its curvature is

$$\bar{\kappa} = \frac{1}{f\sqrt{1+f'^2}} - \varepsilon\frac{f'}{f(1+f'^2)},\ f > 0. \tag{3.5}$$

ii) When considering the Embden-Fowler type equation, for $\alpha = 1$, the curvature of the dual curve $\bar{\gamma}$ is

$$\bar{\kappa} = \frac{1}{cf} \mp \varepsilon c\sqrt{c^2f^2-1},\ f > 0. \tag{3.6}$$

where c is constant.

Remark 3.1. Let $\gamma(s)$ be a catenary curve with the parametrization $\gamma(s) = (s, a\cosh(s/a)), a \neq 0$. Substituting this into Equation (3.1), the corresponding dual catenary curve is obtained as:

$$\bar{\gamma}(s) = (s, a\cosh(s/a)) + \varepsilon(-a\,ln(\cosh(s/a)) + c_1, a\,arctan(\sinh(s/a)) + c_2),$$

where c_1 and c_2 are integration constants. Then the unit tangent vector becomes

$$\bar{T}(s) = \left(\frac{1}{\cosh(s/a)}, \tanh(s/a)\right) + \varepsilon\left(-\frac{\sinh(s/a)}{\cosh^2(s/a)}, \frac{1}{a\cosh^2(s/a)}\right).$$

The unit normal vector $\bar{N}$ is then given by

$$\bar{N} = \left(-\tanh(s/a), \frac{1}{\cosh(s/a)}\right) + \varepsilon\left(-\frac{1}{a\cosh^2(s/a)}, -\frac{\cosh(s/a)}{\cosh^2(s/a)}\right).$$

The curvature $\bar{\kappa}$ of of the dual catenary curve $\bar{\gamma}$ is defined by

$$\bar{\kappa} = \frac{1}{\|\bar{\gamma}'(s)\|}\langle \bar{T}', \bar{N}\rangle.$$

Explicitly, it is given by

$$\bar{\kappa} = \frac{\sinh^2(s/a)}{a\cosh^4(s/a)} + \frac{1}{\cosh^4(s/a)} + \varepsilon\left(\frac{\sinh(s/a)}{a\cosh^4(s/a)} - \frac{\sinh(s/a)}{\cosh^5(s/a)} + \frac{\sinh(s/a)}{a\cosh^3(s/a)} - \frac{2\sinh^3(s/a)}{a\cosh^5(s/a)} - \frac{1}{2\cosh^3(s/a)\sinh(s/a)}\right).$$

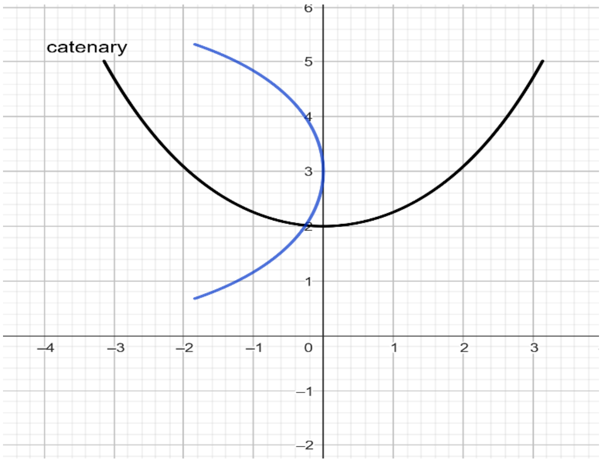

Figure 3.1. The dual curve $\bar{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ with $\bar{\tau} = 0$. The curve γ^* is a curve (blue) and γ is a catenary (black).

Theorem 3.3. Let $\bar{\gamma}_i: I \subseteq R \to D^2, \bar{\gamma}_i(s) = \gamma_i(s) + \varepsilon\gamma^*(s)$ be a dual curve defined in the D^2, $\langle\gamma'(s), \gamma^{*\prime}(s)\rangle = 0$. If γ_i is the involute of the catenary curve γ, then the dual involute catenary is obtained by

$$\bar{\gamma}_i = \big(s, f(s)\big) + \varepsilon\left(s + \frac{\int_0^s \sqrt{1+f'^2}dt}{\sqrt{1+f'^2}}, f + \frac{f' \int_0^s \sqrt{1+f'^2}dt}{\sqrt{1+f'^2}}\right). \tag{3.7}$$

When considering the Embden-Fowler type equation, for $\alpha = 1$, the dual curve $\bar{\gamma}_i$ is

$$\bar{\gamma}_i = \big(s, f(s)\big) + \varepsilon\left(s + \frac{\int_0^s f(t)dt}{f(s)}, f \pm \frac{\sqrt{c^2 f^2 - 1}\int_0^s f(t)dt}{f(s)}\right) \tag{3.8}$$

where c is constant.

Proof: Let γ be a catenary curve with zero curvature at every point. If the arc length parameter function of γ is $t(s)$, and its unit tangent vector is T_γ, then an involute curve of the catenary is defined as

$$\gamma_i = \gamma + t(s)T_\gamma, t(s) = \int_0^s \sqrt{1+f'^2(t)}\, dt,$$

$$\gamma_i = \big(s, f(s)\big) + \frac{\int_0^s \sqrt{1+f'^2(t)}\, dt}{\sqrt{1+f'^2(t)}}\big(1, f'(s)\big).$$

Then, we have (3.7). When considering the Embden-Fowler type equation for case $\alpha = 1$ $\left(f'(s) = \pm\sqrt{c^2 f^{2\alpha}(s) - 1}\right)$, we get (3.8). The tangent vector of the dual involute catenary is then given by

$$\bar{\gamma}_i' = \left(2 - \frac{f'f'' \int_0^s \sqrt{1+f'^2}dt}{(1+f'^2)^{3/2}}, 2f' + \frac{f'' \int_0^s \sqrt{1+f'^2}dt}{\sqrt{1+f'^2}} - \frac{f'f'' \int_0^s \sqrt{1+f'^2}dt}{1+f'^2}\right) + \varepsilon\big(1, f'(s)\big). \tag{3.9}$$

In particular, if the catenary is taken as $\gamma(s) = (s, \cosh(s))$, we can give the remarks the following:

Remark 3.2. Let $\bar{\gamma}_i : I \subseteq R \to D^2, \bar{\gamma}_i(s) = \gamma_i(s) + \varepsilon\gamma^*(s)$ be the dual involute catenary defined in the D^2, $\langle \gamma_i'(s), \gamma^{*\prime}(s)\rangle = 0$. If γ_i is the involute of the catenary curve $\gamma(s) = (s, \cosh(s))$, then the dual curve is obtained by

$$\bar{\gamma}_i = (s, \cosh(s)) + \varepsilon\left(s - \tanh(s), \frac{1}{\cosh(s)}\right). \tag{3.10}$$

Remark 3.3. Let $\bar{\gamma}_i : I \subseteq R \to D^2, \bar{\gamma}_i(s) = (s, \cosh(s)) + \varepsilon\left(s - \tanh(s), \frac{1}{\cosh(s)}\right)$, be the dual involute catenary in the D^2. The curvature of the dual involute catenary $\bar{\gamma}_i$ is

$$\bar{\kappa}_i = \frac{1}{\cosh^2(s)} + \varepsilon\left(-\frac{\sinh^2(s)}{\cosh^4(s)} - \frac{\sinh(s)\left(2 - \sinh^2(s)\left(\sinh(s)+1\right)\right)}{\cosh^6(s)}\right), s \neq 0. \tag{3.11}$$

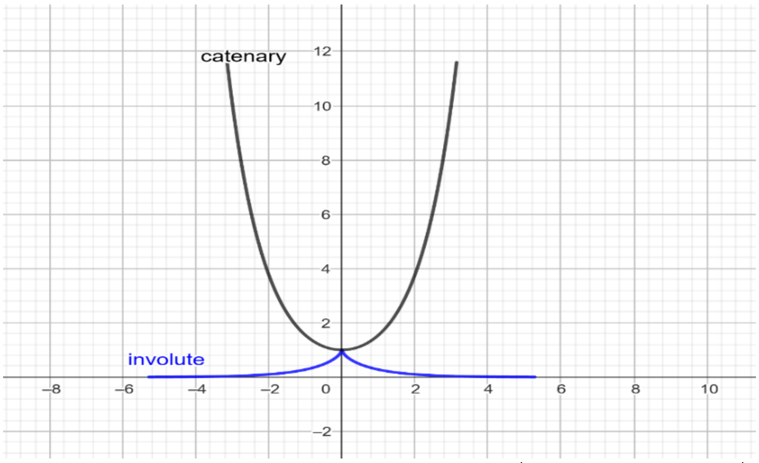

Figure 3.2. The dual involute catenary $\overline{\gamma}_i = (s, \cosh(s)) + \varepsilon\left(s - \tanh(s), \frac{1}{\cosh(s)}\right)$ with $\overline{\tau} = 0$. The involute of the catenary curve (blue) and catenary (black).

Theorem 3.4. Let $\overline{\gamma}_e : I \subseteq R \to D^2, \overline{\gamma}_e(s) = \gamma(s) + \varepsilon\gamma_e(s)$ be a dual curve defined in the D^2, $\langle \gamma'(s), {\gamma^*}'(s) \rangle = 0$. If γ_e is the evolute of the catenary curve γ, then the dual evolute catenary is obtained by

$$\overline{\gamma}_e = \left(s - \frac{f'(1+f'^2)}{f''}, f + \frac{1+f'^2}{f''}\right) + \varepsilon\big(s, f(s)\big). \tag{3.12}$$

When considering the Embden-Fowler type equation, for $\alpha = 1$, the dual evolute catenary is obtained by

$$\overline{\gamma}_e = \left(s - \frac{1+f'^2}{c^2}, f + \frac{1+f'^2}{c^2 f'}\right) + \varepsilon\big(s, f(s)\big), c \neq 0. \tag{3.13}$$

where c is constant.

Proof: Let γ be a catenary curve with zero curvature at every point. An evolute curve of the catenary is defined as

$$\gamma_e = \gamma + \frac{1}{\kappa_\gamma} N_\gamma,$$

$$\gamma_e = \big(s, f(s)\big) + \frac{(1+f'^2)^{(3/2)}}{f''}\left(\frac{-f'}{\sqrt{(f')^2+1}}, \frac{1}{\sqrt{(f')^2+1}}\right),$$

$$\gamma_e = \left(s - \frac{-f'(1+f'^2)}{f''}, f + \frac{1+f'^2}{f''}\right).$$

Then, we have (3.12). When considering the Embden-Fowler type equation for $\alpha = 1$ $\left(f'(s) = \pm\sqrt{c^2 f^{2\alpha}(s) - 1}\right)$, we get (3.13).

The tangent vector of the dual evolute catenary is

$$\overline{\gamma}_e' = \big(1, f'(s)\big) + \varepsilon\left(-3f'^2 + \frac{f'f'''(1+f'^2)}{f''^2}, 3f' + \frac{(1+f'^2)f'''}{f''^2}\right). \tag{3.14}$$

Specially, if we get the catenary as $\gamma(s) = (s, \cosh(s))$, we can give the results the following:

Remark 3.4. Let $\bar{\gamma}_e : I \subseteq R \to D^2, \bar{\gamma}_e(s) = \gamma(s) + \varepsilon\gamma_e(s)$ be the dual evolute catenary defined in D^2, $\langle \gamma_e'(s), \gamma^{*\prime}(s)\rangle = 0$. If γ_e is the evolute of the catenary curve $\gamma(s) = (s, \cosh(s))$, then the dual curve is obtained by

$$\bar{\gamma}_e = (s, \cosh(s)) + \varepsilon(s - \sinh(s)\cosh(s), 2\cosh(s)). \tag{3.15}$$

Remark 3.5. Let $\overline{\gamma_e}(s) = (s, \cosh(s)) + \varepsilon(s - \sinh(s)\cosh(s), 2\cosh(s))$ be the dual evolute catenary defined in D^2. The curvature of the dual evolute catenary $\bar{\gamma}_e$ is

$$\bar{\kappa}_e = \frac{1}{\cosh^2(s)}\varepsilon\left(\frac{2}{\cosh^2(s)} \pm \frac{2\sinh^2(s) + 4\sinh^2(s)\cosh^5(s) + 2\sinh^4(s)\cosh^3(s)}{\cosh^7(s)}\right. \tag{3.16}$$

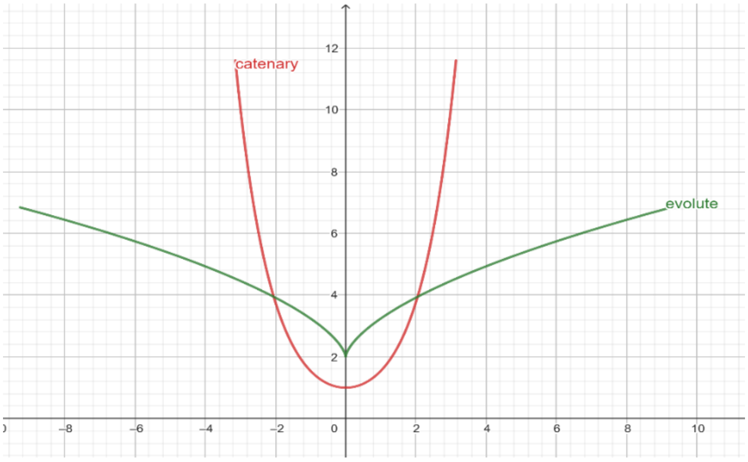

Figure 3.3. The dual evolute catenary $\overline{\gamma}_e = (s, \cosh(s)) + \varepsilon(s - \sinh(s)\cosh(s), 2\cosh(s))$ with $\bar{\tau} = 0$. The evolute of the catenary curve (green) and catenary (red).

3.2. DISCUSSION

The catenary is one of the classical curves in differential geometry, known for describing the shape of a uniform chain hanging under its own weight [20]. Its mathematical representation $\gamma(s) = (s, \cosh(s))$ characterizes a curve of minimal potential energy and appears naturally in the study of minimal and rotational surfaces. Extending this curve into the dual number framework allows infinitesimal geometric variations while preserving its fundamental properties. The obtained results show that the dual catenary maintains the essential geometric behavior of the classical catenary, while the dual component introduces additional information about infinitesimal deformations. The condition $\langle \gamma'(s), \gamma^{*\prime}(s)\rangle = 0$ ensures geometric harmony between the real and dual parts, similar to orthogonality conditions used on dual helices and ruled surfaces. Therefore, the dual catenary can be interpreted as a natural generalization of the classical curve into dual geometry.

The derived expressions for the dual tangent, normal, and curvature vectors illustrate how the dual term modifies the local geometry of the real catenary without altering its global structure. Furthermore, the construction of the dual involute and dual evolute catenaries follows the same geometric logic as the corresponding relations in Euclidean space [21,22] These results indicate that classical geometric relationships can be consistently extended into the dual framework. Finally, the example $\gamma(s) = (s, \cosh(s))$ confirms that the theoretical relations are consistent and geometrically meaningful. The obtained dual involute and dual evolute curves behave similarly to those of classical catenaries while incorporating the infinitesimal effects of the dual component.

4. CONCLUSIONS

In this work, the classical catenary curve has been extended to the dual number framework, and the notion of the dual catenary has been introduced together with its involute and evolute counterparts. The geometric condition $\langle \gamma'(s), \gamma^{*\prime}(s) \rangle = 0$ ensures a consistent relationship between the real and dual parts of the curve. Analytical expressions for the dual tangent, normal, and curvature vectors have been derived, describing how the dual component influences the local geometry. The obtained results indicate that the dual catenary preserves the essential geometric properties of the classical catenary while incorporating infinitesimal information through its dual structure. The example of the curve $\gamma(s) = (s, \cosh(s))$ verifies the validity of the theoretical expressions and illustrates their geometric interpretation. In conclusion, the study demonstrates that the fundamental characteristics of catenary curves can be consistently generalized to the dual setting, providing a useful framework for further research on dual curves and surfaces in differential geometry.

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