ORIGINAL PAPER

FACTORABLE SURFACES AND NEW TYPES OF VELAROIDAL SURFACES

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Abstract. *In this study, we consider velaroidal shells (surfaces) to be a special type of factorable surfaces which are obtained by Monge patches. It is accepted as a model for generating an unlimited number of velaroid surfaces, theoretically suitable for forming a self-supporting spatial structure within a rectangular area. The mean and Gaussian curvatures of this type of surface are presented. Furthermore, the values of the generator functions are calculated if these surfaces are flat. As an application, the graphs of these velaroidal surfaces, for which we have made the curvature calculations, are given in the form of tables.*
Keywords: *Monge patch; Factorable surface; Velaroidal surface; Darbazi surface.*
Mathematics Subject Classification: *53C40; 53C42.*

1. INTRODUCTION

In recent years, there has been a significant increase in computer imaging studies that use distance and depth maps as sensor image data. Methods for detecting distance-display features rely on Gaussian and mean curvatures. Therefore, methods have been developed to detect the presence of a 3-dimensional convex surface using Gaussian and mean curvatures, under the assumption that an image surface can be modelled with a Monge patch. The graphical form of the surfaces, called a Monge patch, is used to describe the depth functions on the surface. A Monge surface is generated by moving a planar curve along a given trajectory in Euclidean space. These main lines of curvature on the surface are also called geodesic lines. Monge surfaces constitute an important class of surfaces in differential geometry and geometric modelling. Monge surfaces are named after the mathematician who discovered them in 1807. Monge surfaces, which play an important role in obtaining bivariate surface graphs in analyses, are located in 3D Euclidean space with the parameterization of the form

$$x(u,v) = \big(u, v, h(u,v)\big). \tag{1}$$

One important example arising from this parametrization is the translation surface determined by a height function defined in terms of two planar curves. Translation surfaces are defined by Darboux with the help of the sum of these curves [1]. Since translational

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surfaces are surfaces produced by two curves, they are four-sided surfaces. For this reason, these surfaces are widely used in architecture. The roofs of some buildings are triangular glass surfaces, and when considered as translational surfaces, they become rectangular, making them more economical and useful [2]. Another useful surface provided by the Monge patch is the factorable surfaces. For these surfaces, the height function is expressed as the product of two curves. These surfaces are also referred to as homothetic surfaces in some sources [3-6]. They are also known as velaroidal surfaces in structural mechanics [7]. A velaroidal surface is also called a funicular surface. In the classification of work [8], velaroidal surfaces are transfer surfaces on a rectangular plane formed by the motion generatrix of variable curvature. It is also noted there that only three velaroidal surfaces are known so far - a sinusoidal velaroid, a parabolic velaroid, and an elliptical velaroid. Cultural Centre in Muscat, Oman, is the most interesting velaroidal surface in recent times. This paper is organised as follows: In Sections 2 and 3, we review the mathematical preliminaries for factorable and velaroidal surfaces. In Section 4, we obtain results on the Gaussian and mean curvatures of these surfaces, including sinusoidal, parabolic, and elliptical velaroids, as well as four other types of velaroidal surfaces. Further, we consider some applications of these results on the given tables. The main novelty of this study is the introduction of four generalized classes of velaroidal surfaces along with their curvature characterizations and flatness/minimality conditions. Unlike the classical literature, which focuses mainly on sinusoidal, parabolic, and elliptic velaroids, the present study introduces four additional generalized velaroidal surface families, along with their curvature characterizations.

2. BASIC CONCEPTS

The following basic concepts of differential geometry are given from [9].

Definition 2.1. *For an injective surface patch $x: U \subset R^2 \to R^3$, the unit normal vector field or surface normal N is defined by*

$$N(u,v) = \frac{x_u \times x_v}{\|x_u \times x_v\|}(u,v) \tag{2}$$

at those points $(u,v) \in U$ at which $x_u \times x_v$ does not vanish, i.e., x is regular.

Theorem 2.1. *Let $x: U \subset R^2 \to R^3$ be a regular patch. Then the Gaussian curvature and mean curvature of the surface are given by the formulas*

$$K = \frac{LN - M^2}{EG - F^2}, \tag{3}$$

and

$$H = \frac{LG + NE - 2MF}{2(EG - F^2)}, \tag{4}$$

where

$$E = \langle x_u, x_u \rangle, \qquad F = \langle x_u, x_v \rangle, \quad G = \langle x_v, x_v \rangle, \tag{5}$$

and

$$L = \langle x_{uu}, N \rangle, M = \langle x_{uv}, N \rangle, N = \langle x_{vv}, N \rangle \tag{6}$$

are the coefficients of the first and second fundamental forms of the surface, respectively.

Here, $\langle \cdot,\cdot \rangle$ denotes the standard Euclidean inner product in R^3.

Corollary 2.1. The principal curvatures κ_1 and κ_2 are the roots of the quadratic equation

$$\kappa^2 - 2H\kappa + K = 0. \tag{7}$$

Thus we may choose κ_1 and κ_2 so that

$$\kappa_1 = H + \sqrt{H^2 - K} \text{ and } \kappa_2 = H - \sqrt{H^2 - K}. \tag{8}$$

From (8) follows that

$$K = \kappa_1\kappa_2, H = \frac{\kappa_1 + \kappa_2}{2}. \tag{9}$$

If $K > 0$, the points on the given surface are called elliptical. If $K < 0$, the surface point is called hyperbolic. Finally, if $K = 0$ then the surface point is called parabolic.

3. SPECIAL TYPE OF FACTORABLE SURFACES

The factorable surfaces are precisely defined as

$$x(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big), \tag{10}$$

and

$$x_i(u,v) = f_i(u,v)g_i(u,v), \quad (i = 1,2,3) \tag{11}$$

where $f_i(u,v)$ and $g_i(u,v)$ are smooth functions [10]. In this manner, the surfaces given with the Monge patch

$$x(u,v) = \big(u, v, f(u)g(v)\big) \tag{12}$$

are a special type of factorable surfaces, or, shortly, SF-surfaces. These surfaces are also known as velaroidal surfaces in structural mechanics (see [4] and [11]). Sometimes, a velaroidal surface is called a funicular surface. The velaroidal surfaces are transfer surfaces on a rectangular plane formed by the motion generatrix of variable curvature. Geometrical approaches to funicular shell design and their use in discrete funicular structures are considered by Matthias Rippmann in [12]. It is also noted that the only three velaroid surfaces are known so far - a sinusoidal velaroid, a parabolic velaroid, and an elliptical velaroid. These velaroidal surfaces are written by the following implicit equations:

$$x_3(u,v) = \lambda sin\frac{\pi u}{a} sin\frac{\pi v}{b}, \tag{13}$$

$$x_3(u,v) = \lambda\left(\frac{u^2}{a^2} + \frac{v^2}{b^2} - \frac{u^2}{a^2}\frac{v^2}{b^2}\right), \tag{14}$$

and

$$x_3(u,v) = \pm\left(\lambda^2 - \frac{\lambda^2 - c^2}{a^2}(u^2+v^2) + \frac{\lambda^2 - c^2}{a^4}u^2v^2\right)^{\frac{1}{2}} \tag{15}$$

respectively, where a and b are the dimensions of a flat rectangular contour, λ is the maximum elevation of the surface above the plane $x_3 = 0$ (see [8]).

A sinusoidal velaroid is formed by two families of sinusoidal half-waves lying in mutually perpendicular planes and facing the same direction with their bulges (Figure 1). Each family of sinusoids has the same period. A sinusoidal velaroid is bounded by a flat rectangular contour. The parabolic velaroid is called a “Darbazi surface”.

4. RESULTS

Shell structures create models for three-dimensional curved surfaces. Because of their ease of computation, it is possible to create an infinite number of model examples for the shell designer to use. In fact, different types of shells can be obtained using shell geometry (see [13-15]). The interesting thing is that designers can invent these shell forms by taking inspiration from nature and innovating from peer structures. In the present section, we have considered seven types of velaroidal surfaces (shells). The first three of these are well-known, but we have described four new types of them. We calculated the mean curvature and Gaussian curvature of these surfaces. We also obtained curvature conditions and examined the flat and minimality conditions for them. All curvature computations presented in this study were verified symbolically by using Maple software in order to minimize possible algebraic inconsistencies. We also plotted some models of velaroidal shells using Maple plotting commands.

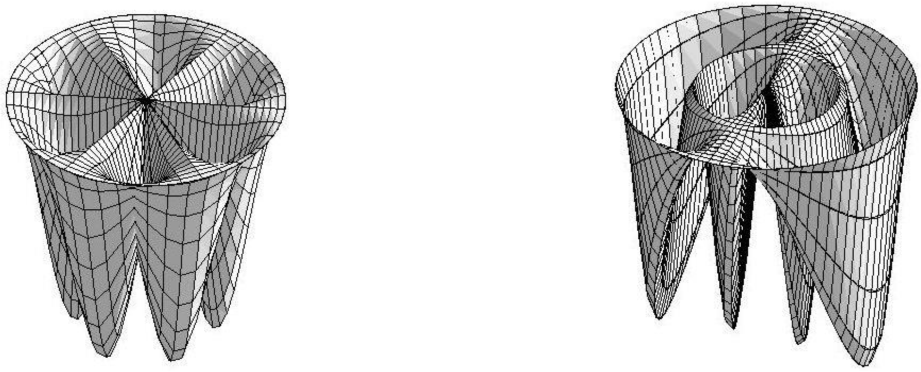
4.1 THE SINUSOIDAL VELAROIDS

In [16] the authors considered two concentric circles with radii r_0 and R, which is taken as the contour curves of the considered surface. Thus, one can obtain the parametric equations of the projected surface in the form:

$$\begin{aligned} x_1(u,v) &= ucosv, \\ x_2(u,v) &= usinv, \\ x_3(u,v) &= \frac{\lambda}{2}(\cos(nv) - 1)\left(cos\left(\frac{2\pi(u-c)}{R - r_0}\right) - 1\right). \end{aligned} \tag{16}$$

They conclude that these types of surfaces may be attributed to velaroidal surfaces. We plot the sinusoidal velaroidal shells with the inner radius $r_0 = 2$ or 3, the outer radius

$R = 4$ and the number of waves $n = 3$ or 2 and $\lambda = 1$, $c = 0$ or 2 in Figure 1-(a) or (b), respectively.



(a) (b)

Figure 1. Sinusoidal velaroidal shells for $n = 3$ or 2 and $\lambda = 1$, $c = 0$ or 2

4.2. THE PARABOLIC VELAROIDS

A parabolic velaroid is also called Shtaerman's surface with plane contours. The parabolic velaroidal surface is described by the following equation [3]:

$$\begin{aligned} x_1(u,v) &= u, \\ x_2(u,v) &= v, \\ x_3(u,v) &= \lambda\left(\frac{u^2}{a^2}-1\right)\left(\frac{v^2}{b^2}-1\right) \end{aligned} \tag{17}$$

where a and b are the measures of the spans and λ is the rise of the surface.

For the geometrical parameters of the model $\lambda = 2$, $a = 1$ or 3 and $b = 3$ we plot the surface for $u, v \in [-3,3]$ in Figure 2-(a) and (b), respectively. We have the following result.

Proposition 4.1. Let M be the parabolic velaroid surface given with the surface patch (17). Then the Gaussian curvature of the parabolic velaroid is

$$K = \frac{4\lambda^2a^2b^2\left(\frac{u^2}{a^2}-1\right)\left(\frac{v^2}{b^2}-1\right) - 16\lambda^2u^2v^2}{a^4b^4(E+G-1)^2} \tag{18}$$

where E and G are the coefficients of the first fundamental form.

Proof: After straightforward but lengthy computations, the coefficients of the first and second fundamental forms are obtained as follows using the equations (5) and (6)

$$\begin{aligned} E &= 1 + \tfrac{4\lambda^2u^2}{a^4}\left(\tfrac{v^2}{b^2}-1\right)^2, \\ F &= \frac{4\lambda^2uv}{a^2b^2}\left(\frac{u^2}{a^2}-1\right)\left(\frac{v^2}{b^2}-1\right), \\ G &= 1 + \tfrac{4\lambda^2v^2}{b^4}\left(\tfrac{u^2}{a^2}-1\right)^2, \end{aligned} \tag{19}$$

and

$$L = \frac{2\lambda}{a^2\sqrt{E+G-1}}\left(\frac{v^2}{b^2} - 1\right),$$
$$M = \frac{4\lambda uv}{a^2b^2\sqrt{E+G-1}}, \tag{20}$$
$$N = \frac{2\lambda}{b^2\sqrt{E+G-1}}\left(\frac{u^2}{a^2} - 1\right)$$

respectively. Then, substituting (19) and (20) into (3), we get the result.

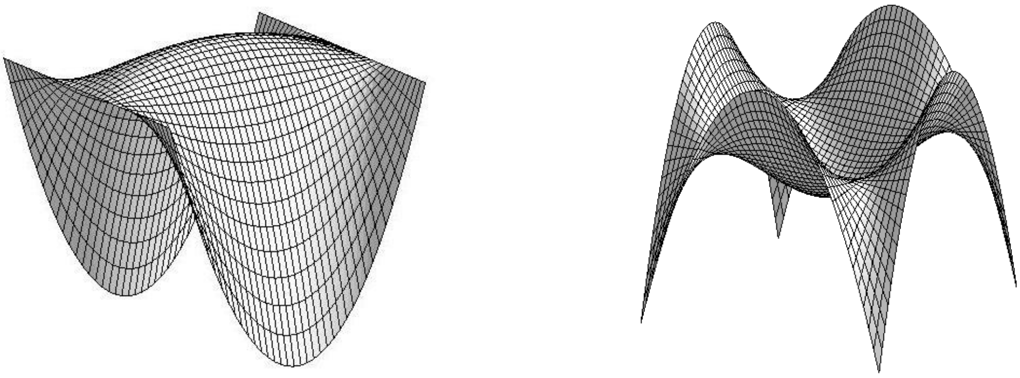

(a) (b)

Figure 2. Parabolic velaroidal surface for $\lambda = 2$, $a = 1$ or 3 and $b = 3$

4.3. THE ELLIPTICAL VELAROIDS

The explicit equation of an elliptical velaroidal formed by semi-ellipses can be considered in the following form using the classical form of elliptic velaroidal surfaces studied in [8,16]:

$$x_1(u,v) = u,$$
$$x_2(u,v) = v, \tag{21}$$
$$x_3(u,v) = \pm\frac{\lambda}{a^2}(a^4 - a^2(u^2+v^2) + u^2v^2)^{\frac{1}{2}}$$

where a is a real constant (Figure 3).

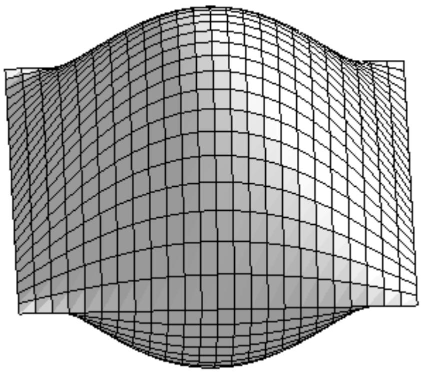


Figure 3. Elliptic velaroidal surface for $u, v \in [-3, 3]$, $\lambda = 4$, and $a = 1$

In the next subsection, we will define special types of velaroidal surfaces to provide guidance for shell designers in their future work.

4.4. THE VELAROIDAL SURFACE OF FIRST KIND

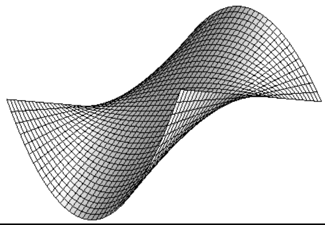
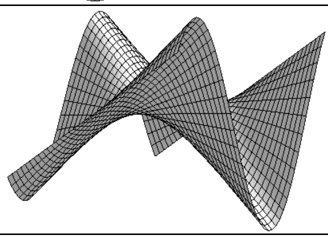
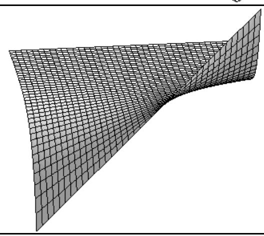
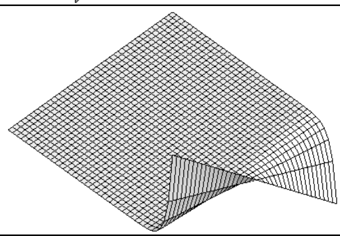
Assume that the generator function is linear of the form $f(u) = c_1 u + c_2$ then the velaroidal surface S_1 will be represented by the parametrization

$$x(u,v) = \big(u, v, (c_1 u + c_2) g(v)\big) \tag{22}$$

where c_1 and c_2 are real constants. We call S_1 the velaroidal surface of first kind (Table 1).

All graphical representations were generated in Maple using the indicated parameter values and rectangular domains.

Table 1. Velaroidal surfaces of first kind.

$\boldsymbol{f(u)}$	$\boldsymbol{g(v)}$	**Velaroidal surfaces**
$3u+1$	$\frac{v^2}{9} - 1$	
⋮	$cosv + sinv$	
⋮	$coshv - sinhv$	
⋮	$3e^{5v}$	

We have the following results;

Proposition 4.2. Let S_1 be the velaroidal surface of first kind given with the surface patch (22). Then the Gaussian curvature and mean curvature of S_1 are given by

$$K = -\frac{c_1^2 (g'(v))^2}{(E+G-1)^2}, \tag{23}$$

and

$$H = \frac{(c_1 u + c_2)\{g''(v)(1 + c_1^2 g^2(v)) - 2c_1^2 g(v)(g'(v))^2\}}{2(E+G-1)^{\frac{3}{2}}} \tag{24}$$

where E and G are the coefficients of the first fundamental form.

Proof: Let S_1 be the velaroidal surface of first kind given with the surface patch (22), then the tangent space of S_1 is spanned by the vector fields $\{x_u, x_v\}$

$$x_u(u,v) = \big(1,0,c_1 g(v)\big),$$
$$x_v(u,v) = \big(0,1,(c_1 u + c_2) g'(v)\big)$$

and the unit normal vector field N of S_1 is calculated as

$$N(u,v) = \frac{(-c_1 g(v), -(c_1 u + c_2) g'(v), 1)}{\sqrt{1 + (c_1 g(v))^2 + ((c_1 u + c_2) g'(v))^2}}.$$

Moreover, taking second partial derivatives of the surface patch, we obtain,

$$x_{uu}(u,v) = (0,0,0),$$
$$x_{uv}(u,v) = (0,0,c_1 g'(v)),$$
$$x_{vv}(u,v) = (0,0,(c_1 u + c_2) g''(v)).$$

So, with respect to these vector fields and after straightforward but lengthy computations, the coefficients of the first and second fundamental forms are obtained as follows using the equations (5) and (6)

$$E = 1 + c_1^2 g^2(v),\ \ F = c_1 g(v) g'(v)(c_1 u + c_2),\ G = 1 + (c_1 u + c_2)^2 (g'(v))^2, \tag{25}$$

and

$$L = 0, \qquad M = \frac{c_1 g'(v)}{\sqrt{E+G-1}}, \qquad N = \frac{(c_1 u + c_2) g''(v)}{\sqrt{E+G-1}} \tag{26}$$

respectively, where $EG - F^2 = E + G - 1$. So, substituting (25) and (26) into (3) and (4), we get the result.

Remark 4.1. Let S_1 be the velaroidal surface of the first kind given with the surface patch (22). If S_1 is a flat surface, then it is a part of a plane. For the non-planar case, since $K < 0$ for every point of the surface, the principal curvatures κ_1 and κ_2 have different signs; hence, every point of the surface is hyperbolic.

Corollary 4.1. Let S_1 be the velaroidal surface of first kind given with the surface patch (22). If S_1 is a minimal surface, then

$$g(v) = \frac{\tan\left(c_1(\lambda v + \mu)\right)}{c_1} \tag{27}$$

holds identically.

Proof: Assume that S_1 is minimal, then the following equation is satisfied:

$$g''(v)(1 + c_1^2 g^2(v)) - 2c_1^2 g(v)(g'(v))^2 = 0. \tag{28}$$

This second-order differential equation has the nontrivial solution (27). For the geometrical parameters of the model $c_1 = 2$, $\lambda = \frac{1}{2}$ and $\mu = \frac{3}{2}$ we plot the minimal velaroidal surface of first kind for $u, v \in [-3,3]$ in Figure 4.

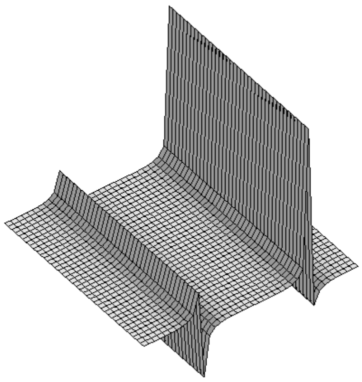


Figure 4. Minimal velaroidal surface of first kind.

4.5. THE VELAROIDAL SURFACE OF SECOND KIND

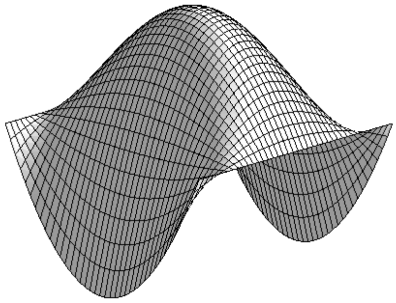
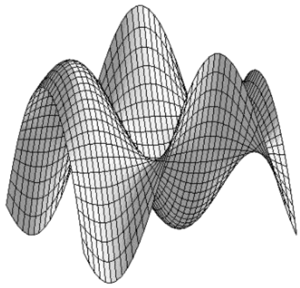
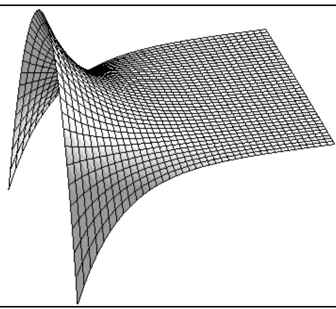
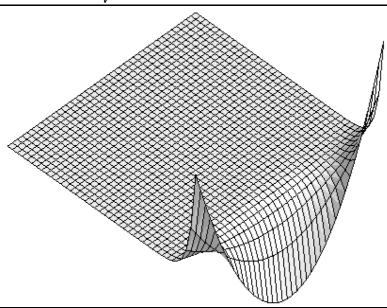
It should be noted that the velaroidal surface of second kind can be regarded as a generalization of the classical parabolic velaroid surface. In particular, the parabolic velaroid is obtained as a special case of the second kind velaroid for a suitable choice of the generator function $g(v)$. Therefore, the second kind extends the family of parabolic-type velaroidal surfaces.

Assume that the generator function is of the form $f(u) = \left(\frac{u^2}{a^2} - 1\right)$ then the velaroidal surface S_2 will be represented by the parametrization

$$x(u,v) = \left(u, v, \left(\frac{u^2}{a^2} - 1\right) g(v)\right) \tag{29}$$

where a is a real constant. We call S_2 the velaroidal surface of second kind (Table 2).

Table 2. Velaroidal surfaces of second kind

$\boldsymbol{f(u)}$	$\boldsymbol{g(v)}$	**Velaroidal surfaces**
$\frac{u^2}{4}-1$	$\frac{v^2}{9}-1$	
$\vdots$	$cosv+sinv$	
$\vdots$	$coshv-sinhv$	
$\vdots$	$3e^{5v}$	

We have the following results:

Proposition 4.3. Let S_2 be the velaroidal surface of second kind given with the surface patch (29). Then the Gaussian curvature and mean curvature of S_2 are given by

$$K=\frac{2g(v)g''(v)(u^2-a^2)-4u^2(g'(v))^2}{a^4(E+G-1)^2}, \tag{30}$$

and

$$H=\frac{g''(v)\left(1+\frac{4u^2g^2(v)}{a^4}\right)(u^2-a^2)+2g(v)\left(1+\frac{(u^2-a^2)^2}{a^4}(g'(v))^2\right)-\frac{8u^2g(v)(g'(\mathrm{v}))^2(u^2-a^2)}{a^4}}{2a^2(E+G-1)^{\frac{3}{2}}} \tag{31}$$

respectively, where E and G are the coefficients of the first fundamental form.

Proof: Let S_2 be the velaroidal surface of second kind given with the surface patch (29), then the tangent space of S_2 is spanned by the vector fields $\{x_u, x_v\}$

$$x_u(u,v) = \left(1,0,\frac{2\mathrm{u}}{a^2}g(v)\right),$$
$$x_v(u,v) = \left(0,1,\left(\frac{u^2}{a^2}-1\right)g'(v)\right)$$

and the unit normal vector field N of S_2 is calculated as

$$N(u,v) = \frac{(-\frac{2\mathrm{u}}{a^2}g(v),\left(\frac{u^2}{a^2}-1\right)g'(v),1)}{\sqrt{1+\frac{4u^2}{a^4}g^2(v)+\left(\frac{u^2}{a^2}-1\right)^2(g'(v))^2}}.$$

Moreover, taking second partial derivatives of the surface patch, we obtain,

$$x_{uu}(u,v) = \left(0,0,\frac{2}{a^2}g(v)\right),$$
$$x_{uv}(u,v) = \left(0,0,\frac{2\mathrm{u}}{a^2}g'(v)\right),$$
$$x_{vv}(u,v) = \left(0,0,\left(\frac{u^2}{a^2}-1\right)g''(v)\right).$$

So, with respect to these vector fields and after straightforward but lengthy computations, the coefficients of the first and second fundamental forms are obtained as follows

$$E = 1+\frac{4u^2g^2(v)}{a^4}, \qquad F = \frac{2ug(v)g'(v)}{a^4}(u^2-a^2), \qquad G = 1+\frac{(g'(v))^2(u^2-a^2)^2}{a^4}, \tag{32}$$

and

$$L = \frac{2g(v)}{a^2\sqrt{E+G-1}}, \qquad M = \frac{2ug'(v)}{a^2\sqrt{E+G-1}}, \quad N = \frac{g''(v)}{a^2\sqrt{E+G-1}}(u^2-a^2) \tag{33}$$

respectively, where $EG - F^2 = E + G - 1$. Then, substituting (32) and (33) into (3) and (4), we get the result.

The obtained differential equation admits separable solutions under appropriate regularity assumptions. Consequently, by fixing one parameter locally, the problem reduces to an ordinary differential equation.

Corollary 4.2. Let S_2 be the velaroidal surface of second kind given with the surface patch (29). If S_2 is a flat surface, then

$$g(v) = \left(\left(1 - \frac{2}{\lambda}\right)(c_1 v + c_2)\right)^{\frac{\lambda}{\lambda - 2}} \tag{34}$$

holds identically, where λ is a real constant, c_1 and c_2 are integral constants.

Proof: Assume that S_2 is a flat surface, then by using (30) we get

$$2g(v)g''(v)(u^2 - a^2) - 4u^2(g'(v))^2 = 0. \tag{35}$$

Rearranging this equation one can get the second order differential equation

$$\lambda g(v)g''(v) - 2(g'(v))^2 = 0$$

where $\lambda = 1 - \frac{a^2}{u_0^2}$, $u_0 \neq 0$ is a real constant. A simple calculation gives that the last differential equation has the nontrivial solution (34).

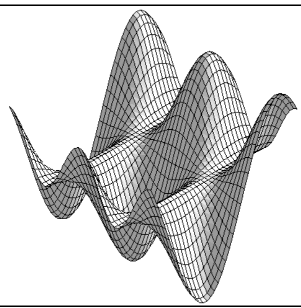
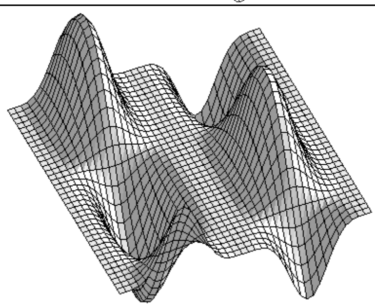
4.6. THE VELAROIDAL SURFACE OF THIRD KIND

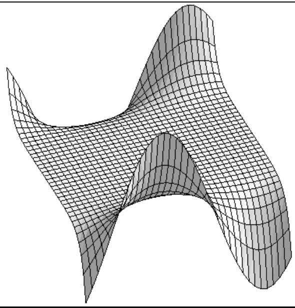
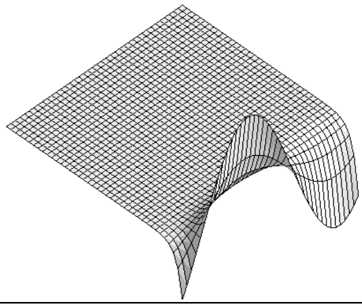
Assume that the generator function is of the form $f(u) = (cosu + sinu)$ then the velaroidal surface S_3 will be represented by the parametrization

$$x(u,v) = \big(u, v, (cosu + sinu)g(v)\big). \tag{36}$$

We call S_3 the velaroidal surface of third kind (Table 3).

Table 3. Velaroidal surfaces of third kind

$\boldsymbol{f(u)}$	$\boldsymbol{g(v)}$	**Velaroidal surfaces**
$cosu + sinu$	$cos^2 v$	
⋮	$sin^5 v$	

$f(u)$	$g(v)$	**Velaroidal surfaces**
⋮	$cosh^2 v - sinh^3 v$	
⋮	$3e^{5v}$	

We have the following results:

Proposition 4.4. Let S_3 be the velaroidal surface of third kind given with the surface patch (36). Then the Gaussian curvature and mean curvature of S_3 are given by

$$K = -\frac{\{g(v)g''(v) - (g'(v))^2\}\sin 2u + g(v)g''(v) + (g'(v))^2}{(E+G-1)^2}, \tag{37}$$

and

$$H = \frac{(cosu + sinu)\{g''(\mathrm{v})(1+g^2(\mathrm{v})(1-\sin 2u)) - 2g(v)(g'(\mathrm{v}))^2(1-\sin 2u) - g(v)(1+(g'(\mathrm{v}))^2(1+\sin 2u))\}}{2(E+G-1)^{\frac{3}{2}}} \tag{38}$$

respectively, where E and G are the coefficients of the first fundamental form.

Proof: Let S_3 be the velaroidal surface of third kind given with the surface patch (36), then the tangent space of S_3 is spanned by the vector fields $\{x_u, x_v\}$

$$x_u(u,v) = \big(1,0,(-\sin u + \cos u)\, g(v)\big),$$
$$x_v(u,v) = \big(0,1,(\cos u + \sin u)\, g'(v)\big)$$

and the unit normal vector field N of S_3 is calculated as

$$N(u,v) = \frac{((\sin u - \cos u)\, g(v), -(\cos u + \sin u)\, g'(v), 1)}{\sqrt{1 + (\sin u - \cos u)^2\, g^2(\mathrm{v}) + (\cos u + \sin u)^2\, (g'(v))^2}}.$$

Moreover, taking second partial derivatives of the surface patch, we obtain,

$$x_{uu}(u,v) = \big(0,0,-(\sin u + \cos u)\, g(v)\big),$$
$$x_{uv}(u,v) = (0,0,(-\sin u + \cos u)\, g'(v)),$$

$$x_{vv}(u,v) = (0,0,(\cos u + \sin u)\, g''(v)).$$

So, with respect to these vector fields and after straightforward but lengthy computations, the coefficients of the first and second fundamental forms are obtained as follows

$$\begin{gathered} E = 1 + g^2(v)(cosu - sinu)^2, \\ F = g(v)g'(v)(cosu - sinu)(cosu + sinu), \\ G = 1 + (g'(v))^2(cosu + sinu)^2, \end{gathered} \tag{39}$$

and

$$\begin{aligned} L &= \frac{-g(v)}{\sqrt{E+G-1}}(cosu + sinu), \\ M &= \frac{g'(v)}{\sqrt{E+G-1}}(cosu - sinu), \\ N &= \frac{g''(v)}{\sqrt{E+G-1}}(cosu + sinu), \end{aligned} \tag{40}$$

respectively, where $EG - F^2 = E + G - 1$. So, substituting (39) and (40) into (3) and (4), we get the desired result.

Corollary 4.3. Let S_3 be the velaroidal surface of third kind given with the surface patch (36). If S_3 is a flat surface, then

$$g(v) = 2^{\lambda+\frac{1}{2}}\left(\frac{c_1 v + c_2}{2\lambda+1}\right)^{\lambda+\frac{1}{2}} \tag{41}$$

holds identically, where λ is a real constant and c_1 and c_2 are integral constants.

Proof: Assume that S_3 is a flat surface, then the following equation is satisfied

$$(g(v)g''(v) - (g'(v))^2)sin2u + g(v)g''(v) + (g'(v))^2 = 0. \tag{42}$$

Rearranging this equation, one can get the second-order differential equation

$$2\lambda(g(v)g''(v) - (g'(v))^2) + g(v)g''(v) + (g'(v))^2 = 0$$

where $\lambda = \frac{sin(2u_0)}{2}$ is a real constant. A simple calculation gives that the last differential equation has the nontrivial solution (41).

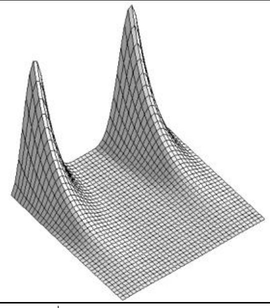
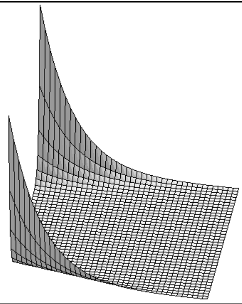
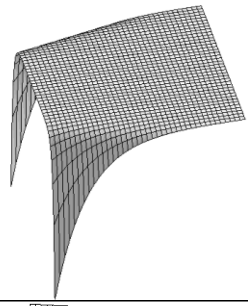
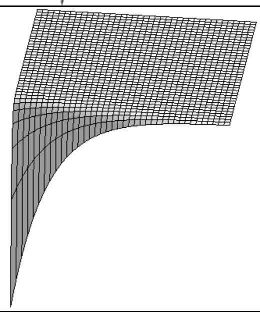
4.7. THE VELAROIDAL SURFACE OF FOURTH KIND

Assume that the generator function is linear of the form $f(u) = (coshu - sinhu)$ then the velaroidal surface S_4 will be represented by the parametrization

$$x(u,v) = \big(u, v, (coshu - sinhu)g(v)\big). \tag{43}$$

We call S_4 the velaroidal surface of fourth kind (Table 4).

Table 4. Velaroidal surfaces of fourth kinds.

$\boldsymbol{f(u)}$	$\boldsymbol{g(v)}$	**Velaroidal surfaces**
$coshu - sinhu$	v^2cos^2v	
$\vdots$	v^3sinh^3v	
$\vdots$	$cosh^2vsinh^2v$	
$\vdots$	$3e^{5v}$	

We have the following results:

Proposition 4.5. Let S_4 be the velaroidal surface of fourth kind given with the surface patch (43). Then the Gaussian curvature and mean curvature of S_4 are given by

$$K = \frac{(g(v)g''(v) - (g'(v))^2)(coshu - sinhu)^2}{(E + G - 1)^2}, \tag{44}$$

and

$$H = \frac{(coshu - sinhu)\{g(v)[1 - (g'(\mathrm{v}))^2 (coshu - sinhu)^2] + g''(\mathrm{v})[1 + g^2(\mathrm{v})(coshu - sinhu)^2]\}}{2(E + G - 1)^{\frac{3}{2}}} \tag{45}$$

respectively, where E and G are the coefficients of the first fundamental form.

Proof: Let S_4 be the velaroidal surface of fourth kind given with the surface patch (43), then the tangent space of S_4 is spanned by the vector fields $\{x_u, x_v\}$

$$x_u(u,v) = \big(1{,}0, (\sinh u - \cosh u)\, g(v)\big),$$
$$x_v(u,v) = \big(0{,}1, (\cosh u - \sinh u)\, g'(v)\big)$$

and the unit normal vector field N of S_4 is calculated as

$$N(u,v) = \frac{((\cosh u - \sinh u)\, g(v), (\sinh u - \cosh u)\, g'(v), 1)}{\sqrt{1 + (\cosh u - \sinh u)^2\, g^2(\mathrm{v}) + (\sinh u - \cosh u)^2\, (g'(v))^2}}.$$

Moreover, taking second partial derivatives of the surface patch, we obtain,

$$x_{uu}(u,v) = \big(0{,}0, (\cosh u - \sinh u)\, g(v)\big),$$
$$x_{uv}(u,v) = (0{,}0, (\sinh u - \cosh u)\, g'(v)),$$
$$x_{vv}(u,v) = (0{,}0, (\cosh u - \sinh u)\, g''(v)).$$

So, with respect to these vector fields, the coefficients of the first and second fundamental forms of the velaroidal surface of fourth kind become

$$\begin{aligned} E &= 1 + g^2(v)(sinhu - coshu)^2, \\ F &= -g(v)g'(v)(sinhu - coshu)^2, \\ G &= 1 + (g'(v))^2(coshu - sinhu)^2, \end{aligned} \tag{46}$$

and

$$\begin{aligned} L &= \frac{g(v)}{\sqrt{E + G - 1}}(coshu - sinhu), \\ M &= \frac{g'(v)}{\sqrt{E + G - 1}}(sinhu - coshu), \\ N &= \frac{g''(v)}{\sqrt{E + G - 1}}(coshu - sinhu), \end{aligned} \tag{47}$$

respectively, where $EG - F^2 = E + G - 1$. Then, substituting these values into (3) and (4), we get the desired result.

Corollary 4.4. Let S_4 be the velaroidal surface of fourth kind given with the surface patch (43). If S_4 is a flat surface, then

$$g(v) = c_1 e^{c_2 v} \tag{48}$$

holds identically, where c_1 and c_2 are real constants.
Proof: Assume that S_4 is a flat surface, then the following equation is satisfied

$$g(v)g''(v) - (g'(v))^2 = 0. \tag{49}$$

This second-order differential equation has the nontrivial solution (48). For the geometrical parameters of the model $c_1 = 3$ and $c_2 = 5$ we plot the flat velaroidal surface of fourth kind (see the last figure in Table 4). All surface visualizations were produced numerically in Maple by selecting representative parameter values that illustrate the geometric behaviour of the corresponding velaroidal family.

5. CONCLUSIONS

The introduced velaroidal surface families provide generalized geometric models that extend several classical shell surfaces used in structural mechanics and architectural geometry. Shell structures are of particular importance to civil engineers. They provide the load-bearing behaviour and lightness of these surfaces, along with material savings. Crustacean structures are frequently used in various fields of civil, mechanical, architectural, aerospace and marine engineering. Examples of shell structures in civil and architectural engineering include concrete shell roofs, concrete silos, cooling towers, and containment shells of nuclear power plants. Shell forms are used in piping systems and curved panels in mechanical engineering. In addition, aircraft, space vehicles, missiles, ships and submarines are also used extensively in aviation and marine engineering [6]. Curved surfaces are crucial for architects and engineers alike in the design of discrete, funicular (velaroidal) structures. It is noted that only three velaroid surfaces are known so far: sinusoidal velaroid, parabolic velaroid, and elliptical velaroid. In this study, specific velaroidal surface models have been created to contribute to the ongoing debate over the funicular shell form of such surfaces. We hope that these surfaces provide structural designers with guidance throughout the design process and offer insights into their internal workings. The obtained velaroidal surface families may provide useful geometric models for shell structures in structural engineering and architectural geometry.

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