ORIGINAL PAPER

INFINITE EXACT SOLUTIONS FOR THE WAVE EQUATION OF NONLINEAR ELASTIC ROD

PENG GUO[1,*], XIANG WU[2,3], XIAO-WEI SUN[1,3], LIANG-BI WANG[3,4]

Manuscript received: 20.10.2025; Accepted paper: 18.04.2026;
Published online: 30.06.2026.

Abstract. *The generalized auxiliary equation method (GAEM) was employed to solve the wave equation of a nonlinear elastic rod (NER), yielding several exact infinite solutions. Numerical simulations were conducted to illustrate the graphs of some of these solutions. The findings contribute to the ongoing investigation into the existence of solitary waves in nonlinear elastic rods. It is demonstrated that the GAEM is a straightforward, efficient, and versatile approach, applicable to a wide range of other nonlinear evolution equations.*

Keywords: *generalized auxiliary equation method; nonlinear elastic rod; infinite exact solution.*

Mathematics Subject Classification: *35C10; 74B20; 74J30.*

1. INTRODUCTION

The rod is one of the most commonly used structural components, forming part of many structures such as blood vessels in the human body and bridges, all of which can be modelled as rods. Therefore, investigating the propagation characteristics of nonlinear waves in rods is significant. The propagation of linear waves in elastic rods has already been utilised in the study of the dynamic mechanical properties of materials and in non-destructive testing. Nonlinear waves, particularly solitary waves, can still be detected after propagating over long distances, making them more precise for non-destructive testing. This is particularly relevant to long-distance pipelines used to transport oil and gas [1-6].

In [1], Zhang and Zhuang established the wave equation for a nonlinear elastic rod (NER) incorporating transverse inertia effects for arbitrarily shaped cross-sections. The wave equation of NER is given by

$$\frac{\partial^2 u}{\partial t^2} - c_0^2[1 + na_n(\frac{\partial u}{\partial x})^{n-1}]\frac{\partial^2 u}{\partial x^2} - \frac{\nu^2 J_\rho}{s}\frac{\partial^4 u}{\partial t^2 \partial x^2} = 0, \tag{1}$$

where s represents the cross-sectional area of the rod, J_ρ is the polar moment of inertia of the rod's cross-section, $c_0^2 = E/\rho$ denotes the square of the longitudinal wave speed in linear elasticity, E is the Young's modulus, ρ is the density of the rod, ν is Poisson's ratio, and both a_n and n are material constants. In [1], Zhang and Zhuang set the material constant

[1] Lanzhou Jiaotong University, School of Mathematics and Physics, 730070 Lanzhou, China.
[2] Tianjin University, School of Mechanical Engineering, 300350 Tianjin, China.
[3] Lanzhou Jiaotong University, School of Mechatronic Engineering, 730070 Lanzhou, China.
E-mail: wuxiang@mail.lzjtu.cn; sunxw@lzjtu.edu.cn.
[4] Lanzhou Jiaotong University, Key Laboratory of Railway Vehicle Thermal Engineering, Ministry of Education, 730070 Lanzhou, China. E-mail: lbiwang@yeah.net.
[*] *Corresponding author*: *guopenglzjtu@126.com*.

https://doi.org/10.46939/J.Sci.Arts-26.2-a04

$n=2$. They transformed Equation (1) into the KdV equation and obtained soliton solutions using inverse scattering method, demonstrating the existence of solitons in NER when $n=2$. Many scholars have studied the soliton problem in nonlinear elastic rods. In [7], Hu et al. set the material constant $n=2$ and conducted numerical simulations of Equation (1) using the multi-symplectic method, discussing the effects of nonlinear and geometric dispersion effects on soliton propagation. In [8], Duan and Zhao set the material constant $n=3$ and transformed Equation (1) into the nonlinear Schrödinger equation using the reduced perturbation method, obtaining envelope soliton solutions. In [9], Lü et al. set the material constant $n \geq 2$ and transformed Equation (1) into the nonlinear Schrödinger equation using the reduced perturbation method. The above studies demonstrate the existence of solitons and envelope solitons in nonlinear elastic rods when the material constant $n \geq 2$. However, it is important to acknowledge that the solutions derived via the reduced perturbation method remain approximate solutions of Equation (1). In [10], Kabir et al. employed three different methods to derive exact solutions of Equation (1) with the material constant $n=2$. In [11], Çelik et al. utilised the Lie group analysis method alongside the F-expansion method to obtain exact solutions of Equation (1) with the material constant $n=2$. In [12] and [13], Li et al. studied exact solutions of Equation (1) using the planar dynamical systems method for the material constants $n=2,\ 3,\ 5$, and odd numbers greater than 5.

Despite the aforementioned research on solving Equation (1), obtaining solutions (especially exact solutions) remains challenging when the material constant κ in Equation (1) takes different values, including fractional values. It is important to note that, owing to the efforts of scholars over the past few decades, numerous methods have been developed for obtaining exact solutions to nonlinear evolution equations [14-25]. Among these methods, the generalized auxiliary equation method (GAEM) is considered a relatively straightforward approach [23-25]. In this paper, we apply the GAEM to solve Equation (1).

The paper is organised as follows: Section 2 introduces the GAEM. In Section 3, the infinite exact solutions of Equation (1) are derived. Finally, Section 4 presents the conclusion and includes numerical simulations.

2. AN INTRODUCTION TO GAEM

The GAEM was first introduced by Li and Wang in [23] and later extended by Sirendaoreji in [25]. We now outline the main steps of the GAEM. Consider a given nonlinear evolution equation

$$N(w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial t^2}, \frac{\partial^2 w}{\partial x^2}, \cdots) = 0. \tag{2}$$

We seek exact solutions of Equation (2) by taking

$$w = w(\xi),\ \ \xi = kx - ct, \tag{3}$$

where k and c are undetermined constants. Under the transformation Equation (3), Equation (2) becomes an ordinary differential equation

$$P(w, \frac{\partial w}{\partial \xi}, \frac{\partial^2 w}{\partial \xi^2}, \frac{\partial^3 w}{\partial \xi^3}, \cdots) = 0. \tag{4}$$

We assume that Equation (4) has the exact solutions in the form

$$w(\xi) = \mu F^{m}(\xi), \tag{5}$$

where μ and m are parameters to be determined. $F(\xi)$ satisfies a generalized ordinary differential equation, which includes arbitrary powers of the dependent variable

$$\left[\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}\right]^2 = AF^{2}(\xi) + BF^{2+p}(\xi) + CF^{2+2p}(\xi), \tag{6}$$

where A, B, C, and p are constants to be determined.

Equation (6) admits the following solutions [25]

$$F(\xi) = \left(\frac{1}{\cosh\ p\sqrt{A}\xi - \frac{B}{2A}}\right)^{\frac{1}{p}},\ C = \frac{B^2}{4A} - A, \tag{7}$$

$$F(\xi) = \left(\frac{1}{\sinh(p\sqrt{A}\xi) - \frac{B}{2A}}\right)^{\frac{1}{p}},\ C = \frac{B^2}{4A} + A, \tag{8}$$

$$F(\xi) = \left[\sqrt{\frac{A}{C}}\left(\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}p\sqrt{A}\xi)\right)\right]^{\frac{1}{p}},\ B = -2\sqrt{AC}, \tag{9}$$

$$F(\xi) = \left[\sqrt{\frac{A}{C}}\left(\frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{2}p\sqrt{A}\xi)\right)\right]^{\frac{1}{p}},\ B = -2\sqrt{AC}, \tag{10}$$

$$F(\xi) = \left[\sqrt{\frac{A}{C}}\left(\frac{1}{2} + \frac{1}{2}\coth(\frac{1}{2}p\sqrt{A}\xi)\right)\right]^{\frac{1}{p}},\ B = -2\sqrt{AC}, \tag{11}$$

$$F(\xi) = \left[\sqrt{\frac{A}{C}}\left(\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}p\sqrt{A}\xi)\right)\right]^{\frac{1}{p}},\ B = -2\sqrt{AC}, \tag{12}$$

$$F(\xi) = \left(\frac{1}{(\frac{1}{2}p\xi)^2 - C}\right)^{\frac{1}{p}},\ A = 0,\ B = 1, \tag{13}$$

$$F(\xi)=\left(\frac{1}{\sigma+\sin(p\xi)}\right)^{\frac{1}{p}},\ A=-1,\ B=2\sigma,\ C=1-\sigma^2, \tag{14}$$

$$F(\xi)=\left(\frac{1}{\sigma-\sin(p\xi)}\right)^{\frac{1}{p}},\ A=-1,\ B=2\sigma,\ C=1-\sigma^2, \tag{15}$$

$$F(\xi)=\left(\frac{1}{\sigma+\cos(p\xi)}\right)^{\frac{1}{p}},\ A=-1,\ B=2\sigma,\ C=1-\sigma^2, \tag{16}$$

$$F(\xi)=\left(\frac{1}{\sigma-\cos(p\xi)}\right)^{\frac{1}{p}},\ A=-1,\ B=2\sigma,\ C=1-\sigma^2, \tag{17}$$

where σ is an undetermined constant. Substituting Equation (5) and Equation (6) into Equation (4) results in an algebraic system of equations in terms of powers of $F(\xi)$, which will determine the parameters μ, m, A,B,C, p, and σ. By employing Equation (5) and the solutions from Equation (6), we can derive multiple exact solutions for Equation (2).

3. INFINITE EXACT SOLUTIONS OF EQUATION (1)

Substituting Equation (3) into Equation (1) yields

$$\frac{\partial^2 u}{\partial\xi^2}-\frac{na_n c_0^2k^3}{c^2-c_0^2k^2}\left(\frac{\partial u}{\partial\xi}\right)^{n-1}\frac{\partial^2 u}{\partial\xi^2}-\frac{v^2J_\rho c^2k^2}{s(c^2-c_0^2k^2)}\frac{\partial^4 u}{\partial\xi^4}=0. \tag{18}$$

Integrating Equation (18) with respect to ξ once and setting the integrating constant to zero, we have

$$w-\alpha w^n-\beta\frac{\partial^2 w}{\partial\xi^2}=0, \tag{19}$$

where

$$w=\frac{\partial u}{\partial\xi}, \tag{20}$$

$$\alpha=\frac{a_n c_0^2k^3}{c^2-c_0^2k^2}, \tag{21}$$

$$\beta=\frac{v^2J_\rho c^2k^2}{s(c^2-c_0^2k^2)}. \tag{22}$$

Substituting Equation (5) and Equation (6) into Equation (19), we obtain

$$\begin{aligned}2\mu F^{m}(\xi)-2Am^{2}\beta\mu F^{m}(\xi)-2Bm^{2}\beta\mu F^{m+p}(\xi)-Bmp\beta\mu F^{m+p}(\xi)\\ -2Cm^{2}\beta\mu F^{m+2p}(\xi)-2Cmp\beta\mu F^{m+2p}(\xi)-2\alpha(\mu F^{m}(\xi))^{n}=0.\end{aligned}\tag{23}$$

Balancing $F^{m+2p}(\xi)$ with $F^{mn}(\xi)$ gives

$$m+2p=mn \quad\Rightarrow\quad p=\frac{m(n-1)}{2}.\tag{24}$$

Substituting Equation (24) into Equation (23), we obtain

$$\begin{aligned}(2\mu-2Am^{2}\beta\mu)F^{m}(\xi)-(2Bm^{2}\beta\mu+Bmp\beta\mu)F^{m+p}(\xi)\\ -(2Cm^{2}\beta\mu+2Cmp\beta\mu+2\alpha\mu^{n})F^{m+2p}(\xi)=0.\end{aligned}\tag{25}$$

All the coefficients of the different powers of $F(\xi)$ in Equation (25) must vanish. This implies that the following equations are established

$$\begin{cases}2\mu-2Am^{2}\beta\mu=0,\\ 2Bm^{2}\beta\mu+Bmp\beta\mu=0,\\ 2Cm^{2}\beta\mu+2Cmp\beta\mu+2\alpha\mu^{n}=0.\end{cases}\tag{26}$$

Solving Equation (26) leads to the following solutions

$$\mu\neq 0,\ m\neq 0,\ n\neq -1,\ A=\frac{1}{m^{2}\beta},\ B=0,\ C=-\frac{2\alpha\mu^{n-1}}{m^{2}\beta(n+1)},\ p=\frac{m(n-1)}{2},\ \sigma=0.\tag{27}$$

Substituting Equation (27), Equation (7), Equation (8) and Equations (14)-(17) into Equation (5) we have the corresponding solution as

$$w(\xi)=\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\cosh\left(\frac{n-1}{2}\sqrt{\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}},\tag{28}$$

$$w(\xi)=\sqrt[n-1]{-\frac{n+1}{2\alpha}}\left(\frac{1}{\sinh\left(\frac{n-1}{2}\sqrt{\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}},\tag{29}$$

$$w(\xi)=\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\sin\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}, \tag{30}$$

$$w(\xi)=\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(-\frac{1}{\sin\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}, \tag{31}$$

$$w(\xi)=\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\cos\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}, \tag{32}$$

$$w(\xi)=\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(-\frac{1}{\cos\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}. \tag{33}$$

In accordance with Equation (20), integrating Equations (28)-(33) with respect to ξ once, we have

$$u=\int\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\cosh\left(\frac{n-1}{2}\sqrt{\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi, \tag{34}$$

$$u=\int\sqrt[n-1]{-\frac{n+1}{2\alpha}}\left(\frac{1}{\sinh\left(\frac{n-1}{2}\sqrt{\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi, \tag{35}$$

$$u=\int\sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\sin\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi, \tag{36}$$

$$u=\int \sqrt[n-1]{\frac{n+1}{2\alpha}}\left(-\frac{1}{\sin\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi, \tag{37}$$

$$u=\int \sqrt[n-1]{\frac{n+1}{2\alpha}}\left(\frac{1}{\cos\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi, \tag{38}$$

$$u=\int \sqrt[n-1]{\frac{n+1}{2\alpha}}\left(-\frac{1}{\cos\left(\frac{n-1}{2}\sqrt{-\frac{1}{\beta}}\xi\right)}\right)^{\frac{2}{n-1}}\mathrm{d}\xi. \tag{39}$$

In a mathematical sense, since n can take infinitely many values excluding 1, Equations (34)-(39) represent infinitely many exact solutions of Equation (1). Specifically, Equation (34) and Equation (35) denote solitary wave solutions of Equation (1), whereas Equations (36)-(39) correspond to trigonometric periodic wave solutions of Equation (1). A comparison between the exact solutions obtained in this paper and those reported by Kabir et al. in [10] and Çelik et al. in [11] demonstrates that the solutions presented herein take novel and entirely distinct mathematical forms.

With the infinite exact solutions described above, once the material constant n is determined, we can obtain exact solutions of Equation (1) through integration. We can provide a few examples to demonstrate these solutions.

For $n=\frac{1}{3}$, Equations (34)-(39) yield

$$u=\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{\frac{3\beta}{2}}\left[9\sinh\left(\frac{kx-ct}{3\sqrt{\beta}}\right)+\sinh\left(\frac{kx-ct}{\sqrt{\beta}}\right)\right]+\delta, \tag{40}$$

$$u=\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{-\frac{3\beta}{2}}\left[-9\cosh\left(\frac{kx-ct}{3\sqrt{\beta}}\right)+\cosh\left(\frac{kx-ct}{\sqrt{\beta}}\right)\right]+\delta, \tag{41}$$

$$u=-\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{-\frac{3\beta}{2}}\left[-9\cos\left(\frac{kx-ct}{3\sqrt{-\beta}}\right)+\cos\left(\frac{kx-ct}{\sqrt{-\beta}}\right)\right]+\delta, \tag{42}$$

$$u=\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{-\frac{3\beta}{2}}\left[-9\cos\left(\frac{kx-ct}{3\sqrt{-\beta}}\right)+\cos\left(\frac{kx-ct}{\sqrt{-\beta}}\right)\right]+\delta, \tag{43}$$

$$u=\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{-\frac{3\beta}{2}}[9\sin(\frac{kx-ct}{3\sqrt{-\beta}})+\sin(\frac{kx-ct}{\sqrt{-\beta}})]+\delta, \tag{44}$$

$$u=-\frac{3\alpha^{\frac{3}{2}}}{8}\sqrt{-\frac{3\beta}{2}}[9\sin(\frac{kx-ct}{3\sqrt{-\beta}})+\sin(\frac{kx-ct}{\sqrt{-\beta}})]+\delta, \tag{45}$$

where δ represents the integration constant.

For $n=\frac{1}{2}$, Eqs. (34)-(39) yield

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)+\sqrt{\beta}[8\sinh(\frac{kx-ct}{2\sqrt{\beta}})+\sinh(\frac{kx-ct}{\sqrt{\beta}})]\}+\delta, \tag{46}$$

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)+\sqrt{\beta}[-8\sinh(\frac{kx-ct}{2\sqrt{\beta}})+\sinh(\frac{kx-ct}{\sqrt{\beta}})]\}+\delta, \tag{47}$$

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)-\sqrt{-\beta}[8\sin(\frac{kx-ct}{2\sqrt{-\beta}})-\sin(\frac{kx-ct}{\sqrt{-\beta}})]\}+\delta, \tag{48}$$

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)-\sqrt{-\beta}[8\sin(\frac{kx-ct}{2\sqrt{-\beta}})-\sin(\frac{kx-ct}{\sqrt{-\beta}})]\}+\delta, \tag{49}$$

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)+\sqrt{-\beta}[8\sin(\frac{kx-ct}{2\sqrt{-\beta}})+\sin(\frac{kx-ct}{\sqrt{-\beta}})]\}+\delta, \tag{50}$$

$$u=\frac{2\alpha^2}{9}\{3(kx-ct)+\sqrt{-\beta}[8\sin(\frac{kx-ct}{2\sqrt{-\beta}})+\sin(\frac{kx-ct}{\sqrt{-\beta}})]\}+\delta, \tag{51}$$

where δ represents the integration constant.

For $n=\frac{4}{3}$, Eqs. (34)-(39) yield

$$u=\frac{343\sqrt{\beta}}{540\alpha^3}\tanh(\frac{kx-ct}{6\sqrt{\beta}})\{8+[5+2\cosh(\frac{kx-ct}{3\sqrt{\beta}})]\mathrm{sech}^4(\frac{kx-ct}{6\sqrt{\beta}})\}+\delta, \tag{52}$$

$$u=\frac{343\sqrt{\beta}}{540\alpha^3}\coth(\frac{kx-ct}{6\sqrt{\beta}})\{8+[5-2\cosh(\frac{kx-ct}{3\sqrt{\beta}})]\mathrm{csch}^4(\frac{kx-ct}{6\sqrt{\beta}})\}+\delta, \tag{53}$$

$$u=\frac{343\sqrt{-\beta}}{540\alpha^3}\cot(\frac{kx-ct}{6\sqrt{-\beta}})\csc^4(\frac{kx-ct}{6\sqrt{-\beta}})[8-6\cos(\frac{kx-ct}{3\sqrt{-\beta}})+\cos(\frac{2kx-2ct}{3\sqrt{-\beta}})]+\delta, \tag{54}$$

$$u=\frac{343\sqrt{-\beta}}{540\alpha^3}\cot(\frac{kx-ct}{6\sqrt{-\beta}})\csc^4(\frac{kx-ct}{6\sqrt{-\beta}})[8-6\cos(\frac{kx-ct}{3\sqrt{-\beta}})+\cos(\frac{2kx-2ct}{3\sqrt{-\beta}})]+\delta, \tag{55}$$

$$u=\frac{343\sqrt{-\beta}}{540\alpha^3}\tan(\frac{kx-ct}{6\sqrt{-\beta}})[8+4\sec^2(\frac{kx-ct}{6\sqrt{-\beta}})+3\sec^4(\frac{kx-ct}{6\sqrt{-\beta}})]+\delta, \tag{56}$$

$$u=\frac{343\sqrt{-\beta}}{540\alpha^3}\tan(\frac{kx-ct}{6\sqrt{-\beta}})[8+4\sec^2(\frac{kx-ct}{6\sqrt{-\beta}})+3\sec^4(\frac{kx-ct}{6\sqrt{-\beta}})]+\delta, \tag{57}$$

where δ represents the integration constant.

For $n=3$, Eqs. (34)-(39) yield

$$u=2\sqrt{\frac{2\beta}{\alpha}}\arctan(\tanh(\frac{kx-ct}{2\sqrt{\beta}}))+\delta, \tag{58}$$

$$u=\sqrt{-\frac{2\beta}{\alpha}}\ln(\tanh(\frac{kx-ct}{2\sqrt{\beta}}))+\delta, \tag{59}$$

$$u=\sqrt{-\frac{2\beta}{\alpha}}\ln(\tan(\frac{kx-ct}{2\sqrt{-\beta}}))+\delta, \tag{60}$$

$$u=-\sqrt{-\frac{2\beta}{\alpha}}\ln(\tan(\frac{kx-ct}{2\sqrt{-\beta}}))+\delta, \tag{61}$$

$$u=2\sqrt{-\frac{2\beta}{\alpha}}\operatorname{arctanh}(\tan(\frac{kx-ct}{2\sqrt{-\beta}}))+\delta, \tag{62}$$

$$u=-2\sqrt{-\frac{2\beta}{\alpha}}\operatorname{arctanh}(\tan(\frac{kx-ct}{2\sqrt{-\beta}}))+\delta, \tag{63}$$

where δ represents the integration constant.

Since the material constant n can take various values (including fractional values), infinite exact solutions can be obtained, which are not enumerated here individually.

4. NUMERICAL SIMULATIONS

The wave equation of NER was solved using the GAEM, yielding infinite solitary wave solutions and infinite trigonometric periodic wave solutions. Based on these obtained infinite exact solutions, numerical simulations can also be conveniently performed. Figures 1 and 2 show the solitary wave solutions of Equation (1) for specific values of the parameters given in Equation (40) and Equation (52). As can be observed from the figures, these are two types of kink-shaped solitary waves. Figures 3 and 4 show the trigonometric periodic wave solutions of Equation (1) for specific values of the parameters given in Equation (43) and Equation (48). As can be observed from the figures, these are two periodic wave forms of different shapes. Figures 5 and 6 illustrate the variation of u with parameters α and β in Equation (52). It can be observed from Figure 5 that the amplitude of u decreases as α increases. Similarly, Figure 6 shows that the amplitude of u increases with the growth of β. For other exact solutions obtained, numerical simulations can be conducted using the same method, which will not be listed here individually.

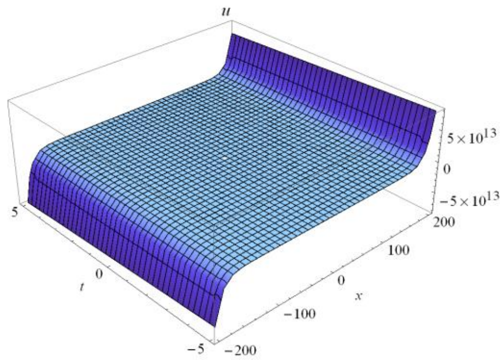

Figure 1. 3D plot of Equation (40) for $c=0.6$, $k=3$ $\alpha=1500$, $\beta=1000$, $\delta=2$.

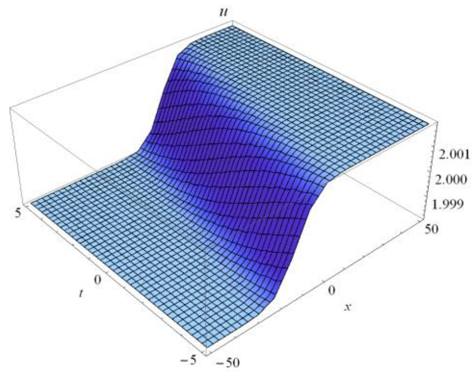

Figure 2. 3D plot of Equation (52) for $c=10$, $k=4$ $\alpha=30$, $\beta=100$, $\delta=2$.

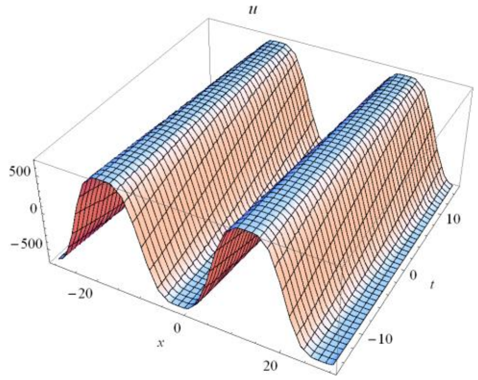

Figure 3. 3D plot of Equation (43) for $c=0.3$, $k=2$ $\alpha=15$, $\beta=-10$, $\delta=2$.

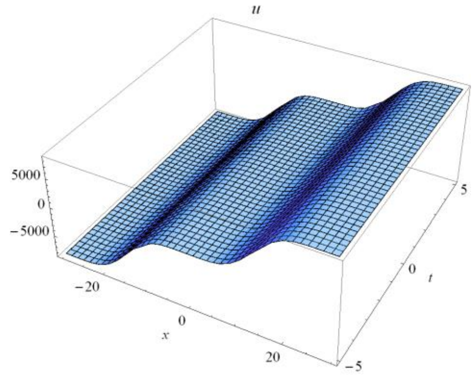

Figure 4. 3D plot of Equation (48) for $c=0.3$, $k=2$ $\alpha=15$, $\beta=-20$, $\delta=2$.

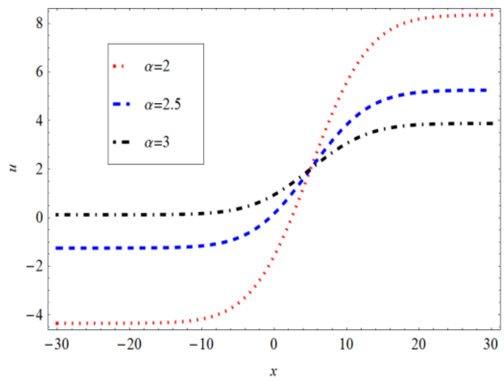

Figure 5. 2D plot of Equation (52) for $c=10$, $k=4$ $t=2$, $\beta=100$, $\delta=2$.

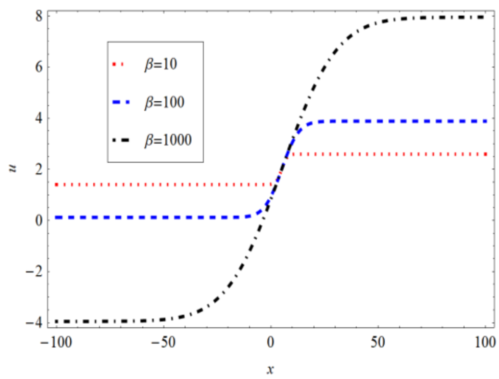

Figure 6. 2D plot of Equation (52) for $c=10$, $k=4$ $t=2$, $\alpha=3$, $\delta=2$.

5. CONCLUSION

The infinite exact solutions obtained in this paper demonstrate the existence of solitary waves and trigonometric periodic waves in NER for the material constant n not equal to 1, including fractional values of n. The infinite exact solutions presented in this work are of great significance for the dynamic analysis and optimal design of various slender rod structures, such as high-rise building columns, bridge cables, offshore drilling risers and submarine pipelines. These findings provide a foundation for further research on the relevant problem. The solution procedure for the wave equation of NER shows that the GAEM is a concise and effective method that can also be extended to other nonlinear evolution equations.

REFERENCES

[1] Zhang, S. Y., Zhuang, W., The Strain Solitary Waves in a Nonlinear Elastic Rod, *Acta Mechanica Sinica*, **20**, 58, 1988. https://doi.org/10.1007/BF02486784

[2] Zhuang, W., Yang, G. T., The Propagation of Solitary Waves in a Nonlinear Elastic Rod, *Applied Mathematics and Mechanics*, **7**, 615, 1986. https://doi.org/10.1007/BF01895973

[3] Zhang, S. Y., Liu, Z. F., Three Kinds of Nonlinear-Dispersive Waves in Finite Deformation Elastic Rods, *Applied Mathematics and Mechanics*, **29**, 909, 2008. https://doi.org/10.1007/s10483-008-0709-2

[4] Samsonov, A. M., Evolution of a Soliton in a Nonlinearly Elastic Rod of Variable Cross-Section, *Soviet Physics Doklady*, **29**, 586,1985.

[5] Samsonov, A. M., Sokurinskaya, E. V., Solitary Longitudinal Waves in an Inhomogeneous Nonlinear Elastic Rod, *Journal of Applied Mathematics & Mechanics*, **51**, 376,1988. https://doi.org/10.1016/0021-8928(87)90116-X

[6] Dai, H. H., Huo, Y., Solitary Waves in an Inhomogeneous Rod Composed of a General Hyperelastic Material, *Wave Motion*, **35**, 55, 2002. https://doi.org/10.1016/S0165-2125(01)00083-X

[7] Hu, W. P, Han, A. H., Deng, Z. C., Numerical Simulation on the Longitudinal Wave in Nonlinear Elastic Rod, *Chinese Journal of Computational Mechanics*, **27**, 8, 2010.

[8] Duan, W. S., Zhao, J. B., Solitary Waves in a Quadratic Nonlinear Elastic Rod, *Chaos Solitons and Fractals*, **11**, 1265, 2000. https://doi.org/10.1016/S0960-0779(99)00014-4

[9] Lü, K. P., Guo, P., Zhang, L., Yi, J. Q., Duan, W. S., Perturbation Analysis for Wave Equation of Nonlinear Elastic Rod, *Applied Mathematics and Mechanics*, **27**, 1233, 2006. https://doi.org/10.1007/s10483-006-0910-z

[10] Kabir, M. M., Exact Traveling Wave Solutions for Nonlinear Elastic Rod Equation, *Journal of King Saud University - Science*, **31**, 390, 2019. https://doi.org/10.1016/j.jksus.2017.08.010

[11] Çelik, N., Seadawy, A. R., Sağlam, Özkan, Y., Yaşar, E., A Model of Solitary Waves in a Nonlinear Elastic Circular Rod: Abundant Different Type Exact Solutions and Conservation Laws, *Chaos Solitons and Fractals*, **143**, 110486, 2021. https://doi.org/10.1016/j.chaos.2020.110486

[12] Li, J. B., Zhang, Y., Exact Travelling Wave Solutions in a Nonlinear Elastic Rod Equation, *Applied Mathematics and Computation*, **202**, 504, 2008. https://doi.org/10.1016/j.amc.2008.02.027

[13] Li, J. B., He, T. L., Exact Travelling Wave Solutions and Bifurcations in a Nonlinear Elastic Rod Equation, *Acta Mathematicae Applicatae Sinica*, **26**, 283, 2010. https://doi.org/10.1007/s10255-008-8139-1

[14] Hirota, R., Exact Solution of the Korteweg-De Vries Equation for Multiple Collisions of Solitons, *Physical Review Letters*, **27**, 1192, 1971. https://doi.org/10.1103/PhysRevLett.27.1192
[15] Parkes, E. J., Duffy, B. R., An Automated Tanh-Function Method for Finding Solitary Wave Solutions to Non-Linear Evolution Equations, *Computer Physics Communications*, **98**, 288, 1996. https://doi.org/10.1016/0010-4655(96)00104-X
[16] Yan, C. T., A Simple Transformation for Nonlinear Waves, *Physics Letters A*, **224**, 77, 1996. https://doi.org/10.1016/S0375-9601(96)00770-0
[17] Sakthivel, R., Chun, C., Lee, J., New Travelling Wave Solutions of Burgers Equation with Finite Transport Memory, *Zeitschrift Für Naturforschung A*, **65**, 633, 2010. https://doi.org/10.1515/zna-2010-8-903
[18] Abdou, M. A., The Extended F-Expansion Method and Its Application for a Class of Nonlinear Evolution Equations, *Chaos Solitons and Fractals*, **31**, 95, 2007. https://doi.org/10.1016/j.chaos.2005.09.030
[19] Sakthivel, R., Chun, C., New Soliton Solutions of Chaffee-Infante Equations Using the Exp-Function Method, *Zeitschrift Für Naturforschung A*, **65**, 197, 2010. https://doi.org/10.1515/ZNA-2010-0307
[20] Zayed, E. M. E., Gepreel, K. A., The (G'/G)-Expansion Method for Finding Travelling Wave Solutions of Nonlinear Partial Differential Equations in Mathematical Physics, *Journal of Mathematical Physics*, **50**, 013502, 2009. https://doi.org/10.1063/1.3033750
[21] Wang, M. L., Exact Solutions for a Compound KdV-Burgers Equation, *Physics Letters A*, **213**, 279, 1996. https://doi.org/10.1016/0375-9601(96)00103-X
[22] Fan, E. G., Extended Tanh-Function Method and its Applications to Nonlinear Equations, *Physics Letters A*, **277**, 212, 2000. https://doi.org/10.1016/S0375-9601(00)00725-8
[23] Li, X. Z., Wang, M. L., A sub-ODE Method for Finding Exact Solutions of a Generalized Kdv-Mkdv Equation with Higher-Order Nonlinear Terms, *Physics Letters A*, **361**, 115, 2007. https://doi.org/10.1016/j.physleta.2006.09.022
[24] Sabi'u, J., Rezazadeh, H., Cimpoiasu, R., Constantinescu, R., Travelling Wave Solutions of the Generalized Rosenau-Kawahara-Rlw Equation via the Sine-Cosine Method and a Generalized Auxiliary Equation Method, *International Journal of Nonlinear Sciences and Numerical Simulation*, **23**, 539, 2022. https://doi.org/10.1515/ijnsns-2019-0206
[25] Sirendaoreji, N., *Travelling Wave Solutions for Nonlinear Wave Equations—Theory and Applications of the Auxiliary Equation Method*, Science Press, Beijing, China, 2019.