ORIGINAL PAPER

STRUCTURAL PROPERTIES AND SPECIAL TRANSFORMATIONS k-COPPER FIBONACCI AND k-COPPER LUCAS SEQUENCES

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Abstract. *In this study, we define the k-Copper Fibonacci and k-Copper Lucas sequences, and the first few terms of these sequences are given. Then we obtain properties such as special generating functions, Binet formulas, etc. Also, we present special relations between these sequences and the roots of their characteristic equations. We obtain the relationship between the ratio of the biggest to the smallest of two consecutive terms of these sequences. In addition, we give important relations among the positive and negative index terms of these sequences, and the sum of the squares of two consecutive terms is related to these sequences. We calculate the significant identities of these sequences. Besides, we apply Catalan transformations of these sequences and obtain the first few terms. We present some features of these Catalan transformations, such as generating functions. Furthermore, we apply the Hankel transform to the Catalan transform of these sequences.*

Keywords: *Copper Fibonacci numbers; Generating function; Binet formula; Fibonacci sequence; Cassini Identity.*

Mathematics Subject Classification: *05A15; 11B39; 11B83; 11C20.*

1. INTRODUCTION

In mathematics, sequences play a crucial role in clarifying how a problem arises under specific conditions. One of the most famous of these problems is the Rabbit Problem from Leonardo Fibonacci's renowned book Liber Abaci. Based on the outcomes of the rabbit problem, the Fibonacci sequence is defined, and the relationships between consecutive terms of this sequence are derived.

The most famous number sequences are the Fibonacci and Lucas sequences. These sequences have attracted the attention of scientists for centuries. Fibonacci sequences have been applied in various fields such as engineering [1], chemistry [2], biomathematics [3], phylotaxis [4], cryptology [5], economy [6], group theory [7], etc. Scientists have conducted numerous studies on the generalisation of the Fibonacci sequence. Examples of studies carried out are k-Edouard [8], Fermat [9], Narayana [10], k-Mersenne [11], Discatenated [12], k-Chebyshev [13], k-Jacobsthal [14], Olivier [15], k-Jacobsthal-Lucas [16], Pell [17], Padovan [18], Bronze Leonardo [19], k-Oresme [20], k-Leonardo [21], Gaussian Mersenne [22], Fibonacci link [23], Francois [24], Perrin [25], Balancing [26], Benoulli [27], Bronze Fibonacci [28], k-Cullen [29], Raoul [30], Bell [31], k-Blaise [32], Perfect Square [33], Poly-Bernoulli [34], k-Quasi-Morgan-Voyce [35], (Fabulous) Fibonacci [36], Copper Fibonacci polynomials [37] sequences, etc.

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For $n \in \mathbb{N}$, the Fibonacci numbers [39] F_n, Bronze Fibonacci numbers [28] BF_n, Lucas numbers [39] L_n, Bronze Lucas numbers [28] BL_n, Jacobsthal numbers [39] J_n, and Fermat numbers [39] E_n are defined by, respectively, the recurrence relations,

$$F_{n+2} = F_{n+1} + F_n,\ BF_{n+2} = 3BF_{n+1} + BF_n,\ L_{n+2} = L_{n+1} + L_n,$$
$$BL_{n+2} = 3BL_{n+1} + BL_n,\ J_{n+2} = J_{n+1} + 2J_n, \text{ and } E_{n+2} = 3E_{n+1} - 2E_n$$

with

$$F_0 = 0, F_1 = 1, BF_0 = 0, BF_1 = 1, L_0 = 2, L_1 = 1, BL_0 = 2, BL_1 = 3, J_0 = 0, J_1 = 1,$$

and

$$E_0 = 0, E_1 = 1$$

For F_n, BF_n, L_n, BL_n, J_n, and E_n the Binet formulas are

$$F_n = \frac{k_1^n - k_2^n}{k_1 - k_2},\ BF_n = \frac{m_1^n - m_2^n}{m_1 - m_2},\ L_n = k_1^n + k_2^n,$$
$$BL_n = m_1^n + m_2^n,\ J_n = \frac{j_1^n - j_2^n}{j_1 - j_2}, \text{ and } E_n = \frac{e_1^n - e_2^n}{e_1 - e_2},$$

where $k_1 = \frac{1+\sqrt{5}}{2} = 1{,}618033988749894 \ldots$, $k_2 = \frac{1-\sqrt{5}}{2}$, $m_1 = \frac{3+\sqrt{13}}{2} = 3.302775638 \ldots$, $m_2 = \frac{3-\sqrt{13}}{2}$, $j_1 = 2$, $j_2 = -1$, and $e_1 = 2$, $e_2 = 1$ are the roots of the characteristic equations $k^2 - k - 1 = 0$, $m^2 - 3m - 1 = 0$, $j^2 - j - 2 = 0$, and $e^2 - 3e - 2 = 0$. The k_1, and m_1 numbers here are the golden ratio and bronze ratio, respectively, from the metallic ratios.

Some of the identities between Fibonacci and Lucas sequences are as follows (for more identities, see [38] and [39]).

i. $F_{n-1}F_{n+1} - F_{n+2}F_{n-2} = 2(-1)^n$,
ii. $L_n = F_{n+1} + F_{n-1}$
iii. $F_{3n} = F_{2n}L_n - F_n(-1)^n$,
iv. $L_n^2 = L_{2n} + 2(-1)^n$,
v. $F_{n+1}L_n - L_{n+1}F_n = 2(-1)^n$,
vi. $L_nL_{n+1} - L_{2n+1} = (-1)^n$,
vii. $5F_n^2 = L_{2n} - 2(-1)^n$,
viii. $\frac{F_{n+1}}{F_n} - \frac{L_{n+1}}{L_n} = \frac{2(-1)^n}{F_{2n}}$.

The Cassini formulas for Fibonacci and Lucas sequences

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \text{ and } L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1}.$$

One of the interesting applications of the Cassini identity is its use in geometric paradoxes (see details in [21] pages 100-108). For $n \in \mathbb{N}$, the Copper Fibonacci numbers [40] CF_n and Copper Lucas numbers [40] CL_n are defined by, respectively, the recurrence relations,

$$CF_{n+2} = 4CF_{n+1} + CF_n \text{ and } CL_{n+2} = 4CL_{n+1} + CL_n$$

with $CF_0 = 0$, $CF_1 = 1$ and $CL_0 = 2$, $CL_1 = 4$.

For CF_n and CL_n the Binet formulas are

$$CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta} \text{ and } CL_n = \kappa^n + \theta^n$$

where $\kappa = 2 + \sqrt{5} = 4.23606797749979\ldots$, and $\theta = 2 - \sqrt{5}$ are the roots of the characteristic equation $r^2 - 4r - 1 = 0$. The number κ here is the copper ratio from metallic ratios.

In [40], Özkan and Akkuş defined the Copper Fibonacci CF_n and Copper Lucas CL_n numbers. In addition, they obtained the following properties of these sequences. For $m, n \in \mathbb{N}$,

i. $CF_{n+m}CF_{n-m} - CF_n^2 = (-1)^{n-m+1}CF_m^2$,
ii. $CL_{n+m}CL_{n-m} - CL_n^2 = 20(-1)^{n-m}CF_m^2$,
iii. $CF_{n+m+1} = CF_{n+1}CF_{m+1} + CF_nCF_m$,
iv. $CL_{n+m+1} = CL_{m+1}CF_{n+1} + CL_mCF_n$,
v. $\theta^{2n} = \frac{CF_{2n}}{2}(2\sqrt{5} - \kappa)\sqrt{5} - \frac{CL_{2n-1}}{4}$,
vi. $\kappa^{2n} = \frac{CF_{2n}}{2}\kappa\sqrt{5} - \frac{CL_{2n-1}}{4}$.

With the help of the recurrence relation of the Fibonacci sequence, k-Fibonacci sequences were presented, and these sequences had a significant place in algebra and number theory. k-Fibonacci and k-Lucas sequences have applications in different scientific fields like graph theory [41], cryptology [42], group theory [43], etc.

For $n \in \mathbb{N}$, the k-Fibonacci sequence [48] $f_{k,n}$ and k-Lucas sequence [48] $l_{k,n}$ are defined by, respectively, the recurrence relations,

$$f_{k,n+2} = kf_{k,n+1} + f_{k,n} \text{ and } l_{k,n+2} = kl_{k,n+1} + l_{k,n},$$

with the initial conditions $f_{k,0} = 0$, $f_{k,1} = 1$ and $l_{k,0} = 2$, $l_{k,1} = k$.

For $f_{k,n}$ and $l_{k,n}$, the Binet formulas are

$$f_{k,n} = \frac{s_1^n - s_2^n}{s_1 - s_2} \text{ and } l_{k,n} = s_1^n + s_2^n$$

where $s_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $s_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $s^2 - ks - 1 = 0$.

In [44], Godase and Dhakne studied the properties of k-Fibonacci and k-Lucas sequences. In addition, Falcon defined the convolutions k-Fibonacci sequences, and he found many properties of these sequences [45]. In another work, Falcon defined the convolutional k-Lucas sequence and introduced the Binet formula and generating functions of this sequence [46]. Moreover, Ramirez [47] made different applications of the convolutional k-Fibonacci sequence and made the sequence attract the attention of scientists. In [48], Falcon and Plaza studied repeated partial sums of k-Fibonacci sequences and they showed interesting properties of these sequences. Özkan and Akkuş [49], inspired by Dickson polynomials and the Fibonacci sequence, defined Dickson k-Fibonacci polynomials and they calculated special identities of this sequence. In [50], Godase applied k-Fibonacci and k-Lucas sequences to hyperbolic quaternions. Also, Polatlı et al. brought a different perspective to these quaternions by applying them in various ways [51].

As can be seen from the studies carried out so far, many generalizations of the Fibonacci and Lucas sequences have been studied. In this study, we present a new generalization inspired by the Copper Fibonacci numbers and k-Fibonacci sequences. We call these sequences the k-Copper Fibonacci and k-Copper Lucas sequences and denote them as $CF_{k,n}$, and $CL_{k,n}$, respectively.

2. k-COPPER FIBONACCI AND k-COPPER LUCAS SEQUENCES

In this chapter, we define the k-Copper Fibonacci and k-Copper Lucas sequences, and several initial terms of these sequences are presented in order to illustrate their structures and behaviors. Furthermore, various algebraic properties of these sequences are investigated in detail. Recurrence relations, generating functions, and relationships between the two sequences are also examined. In addition, some fundamental identities and special cases are derived, revealing important characteristics of the sequences.

Definition 2.1. *For $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$, the k-Copper Fibonacci $CF_{k,n}$, and k-Copper Lucas $CL_{k,n}$ are defined by, respectively,*

$$CF_{k,n+2} = 4CF_{k,n+1} + kCF_{k,n} \tag{2.1}$$

with $CF_{k,0} = 0$ and $CF_{k,1} = 1$,

$$CL_{k,n+2} = 4CL_{k,n+1} + kCL_{k,n} \tag{2.2}$$

with $CL_{k,0} = 2$ and $CL_{k,1} = 4$.

Note that when $k = 1$, the Copper Fibonacci and Copper Lucas sequences are obtained [see details in [40]. Below, calculations for some k values are performed and correlated with sequences in OEIS:

- For $k = 2$, $CF_{2,n} = \{0, 1, 4, 18, 80, 356, 1584, 7048, \dots\}$ OEIS index is A090017,
- For $k = 3$, $CF_{3,n} = \{0, 1, 4, 19, 88, 409, 1900, 8827, \dots\}$ OEIS index is A015530,
- For $k = 5$, $CF_{5,n} = \{0, 1, 4, 21, 104, 521, 2604, \dots\}$ OEIS index is A015530,
- For $k = 6$, $CF_{6,n} = \{0, 1, 4, 22, 112, 580, 2992, \dots\}$ OEIS index is A085939.

The characteristic equation of the k-Copper Fibonacci and k-Copper Lucas sequences is

$$v^2 - 4v - k = 0. \tag{2.3}$$

The roots of this equation

$$v_1 = 2 + \sqrt{k+4}, \text{ and } v_2 = 2 - \sqrt{k+4}.$$

Next, we give the relationships between these roots below:

$$v_1 + v_2 = 4,\ v_1 - v_2 = 2\sqrt{4+k},\ v_1 v_2 = -k, \text{ and } v_1^2 + v_2^2 = 16 + 2k.$$

Some terms of the $CF_{k,n}$ and $CL_{k,n}$ sequences are

$$0, 1, 4, k + 16, 8k + 64, k^2 + 48\text{k} + 256, 12k^2 + 256\text{k} + 1024, \dots$$

and

$$2, 4, 2k + 16, 12k + 64, 2k^2 + 64k + 256, 20k^2 + 320k + 1024, \dots,$$

respectively.

Next, the Binet formulas of the $CF_{k,n}$, and $CL_{k,n}$ are expressed.

Theorem 2.1. Let $n \in \mathbb{N}$. We have
i. $CF_{k,n} = \frac{v_1^n - v_2^n}{v_1 - v_2}$,
ii. $CL_{k,n} = v_1^n + v_2^n$.

Proof: i. *The following general form is used to find equality:*

$$CF_{k,n}(x) = s v_1^n + t v_2^n.$$

s and t values are obtained by solving the equation obtained by giving different values to the above equation. For $n = 0$, $CF_{k,0} = 0$ and for $n = 1$, $CF_{k,1} = 1$. So, $s = \frac{1}{v_1 - v_2}$ and $t = \frac{-1}{v_1 - v_2}$.

Thus, we obtain

$$CF_{k,n} = \frac{v_1^n - v_2^n}{v_1 - v_2}.$$

Other proofs are shown in a similar way. □

In the following theorems, we establish relationships among the roots of the characteristic equation of these sequences and the sequences themselves.

Theorem 2.2. *The relationship between two consecutive terms of $CF_{k,n}$ and $CL_{k,n}$ sequences are as follows:*

$$\lim_{n\to\infty} \frac{CF_{k,n+1}}{CF_{k,n}} = v_1 \text{ and } \lim_{n\to\infty} \frac{CL_{k,n+1}}{CL_{k,n}} = v_1.$$

Proof: Binet formulas are used for the proof. We obtain

$$\lim_{n\to\infty} \frac{CF_{k,n+1}}{CF_{k,n}} = \frac{\frac{v_1^{n+1} - v_2^{n+1}}{v_1 - v_2}}{\frac{v_1^n - v_2^n}{v_1 - v_2}} = \lim_{n\to\infty} \frac{v_1^{n+1}(1 - (\frac{v_2}{v_1})^{n+1})}{v_1^n (1 - \left(\frac{v_2}{v_1}\right)^n)}$$

Thus, we obtain

$$\lim_{n\to\infty} \frac{CF_{k,n+1}}{CF_{k,n}} = v_1.$$

Similarly, we have

$$\lim_{n\to\infty} \frac{CL_{k,n+1}}{CL_{k,n}} = v_1.$$

□

Theorem 2.3. *Let $m = v_1$ or $m = v_2$, $k \in \mathbb{R}^+$, and $n, s, r, p \in \mathbb{N}$.* We obtain
i. $m^n = mCF_{k,n} + kCF_{k,n-1}$,
ii. $k + 4m + m^{2(2^{n+1}+1)} = m^{2(2^n+1)} CL_{k,2^{n+1}}$.
iii. $m^{2n} = m^n CL_{k,n} - (-k)^n$,
iv. $m^{np} = \frac{m^n CF_{k,np}}{CF_{k,n}} - (-k)^n \frac{CF_{k,n(p-1)}}{CF_{k,n}}$,
v. $m^n = m^s CF_{k,n-s+1} + k m^{s-1} CF_{k,n-s}$.

Proof: i. For $m = v_1$, we have

$$mCF_{k,n}+kCF_{k,n-1}=v_1\left(\frac{v_1^n-v_2^n}{v_1-v_2}\right)+k\left(\frac{v_1^{n-1}-v_2^{n-1}}{v_1-v_2}\right)=\frac{v_1^{n-1}(v_1^2+k)-v_2^{n-1}(v_1v_2+k)}{v_1-v_2}=v_1^n.$$

For $m=v_2$, we have

$$mCF_{k,n}+kCF_{k,n-1}=v_2\left(\frac{v_1^n-v_2^n}{v_1-v_2}\right)+\left(\frac{v_1^{n-1}-v_2^{n-1}}{v_1-v_2}\right)=\frac{v_1^{n-1}(v_1v_2+k)-v_2^n(v_2+\frac{k}{v_2})}{v_1-v_2}=v_2^n.$$

Other proofs are shown in a similar way. □

Theorem 2.4. *Let $k\in\mathbb{R}^+$, and $n\in\mathbb{N}$.* We have

i. $v_1^{2n}=\frac{CF_{k,2n}}{2}v_1\sqrt{4+k}-k\frac{CL_{k,2n-1}}{4}$,
ii. $v_1^{2n+1}=-k\frac{CF_{k,2n}}{2}\sqrt{4+k}+v_1\frac{CL_{k,2n+1}}{4}$,
iii. $v_2^{2n+1}=-k\frac{CF_{k,2n}}{2}\sqrt{4+k}-v_2\frac{CL_{k,2n+1}}{4}$,
iv. $2\sqrt{4+k}CF_{k,n}+CL_{k,n}=2v_1^n$,
v. $2\sqrt{4+k}CF_{k,n}-CL_{k,n}=-2v_2^n$.

Proof: Binet formulas are used for proofs. We have

i. $$\begin{aligned}\frac{CF_{k,2n}}{2}v_1\sqrt{4+k}-k\frac{CL_{k,2n-1}}{4}&=\frac{v_1^{2n}-v_2^{2n}}{2(v_1-v_2)}v_1\sqrt{4+k}-k\frac{v_1^{2n-1}+v_2^{2n-1}}{4}\\&=\frac{v_1^{2n+1}-v_1v_2^{2n}-v_1^{2n-1}-v_2^{2n-1}}{4}\\&=\frac{v_1^{2n}\left(v_1-\frac{k}{v_1}\right)+v_2^{2n}(-v_1-\frac{k}{v_2})}{4}\\&=v_1^{2n}.\end{aligned}$$

Other proofs are shown in a similar way. □

In the next theorems, we obtain special relations between the $CF_{k,n}$, and $CL_{k,n}$ sequences.

Theorem 2.5. *Let $k\in\mathbb{R}^+$, and $n\in\mathbb{N}$. The following equations are satisfied*:

i. $CF_{k,n}=\frac{1}{2k+8}CL_{k,n+1}-\frac{1}{k+4}CL_{k,n}$,
ii. $CL_{k,n}=2CF_{k,n+1}-4CF_{k,n}$.

Proof: i. *The following relation is used for proofs*:

$$CF_{k,n}=a\times CL_{k,n+1}+b\times CL_{k,n}.$$

For these n values, we obtain:

$$CF_{k,0}=a\times CL_{k,1}+b\times CL_{k,0},$$
$$CF_{k,1}=a\times CL_{k,2}+b\times CL_{k,1}.$$

If the values are substituted, the following equation system is obtained.

$$0=4a+2b,$$
$$1=16\mathrm{a}+2ka+4b.$$

We find

$$a=\frac{1}{2k+8}\text{ and }b=-\frac{1}{k+4}.$$

Other proof is shown in a similar way. □

Theorem 2.6. *Let* $k \in \mathbb{R}^+$, $m, n \in \mathbb{N}$, *and* $n > m$. We have
i. $CF_{k,n+m+1} = CF_{k,n+1}CF_{k,m+1} + CF_{k,n}CF_{k,m}$,
ii. $CL_{k,n} = \frac{CL_{k,-n}}{2}(CL_{k,n}^2 - CL_{k,2n})$,
iii. $CL_{k,n+m+1} = CL_{k,m+1}CF_{k,n+1} + CL_{k,m}CF_{k,n}$,
iv. $CF_{k,n}^2 + CF_{k,n+1}^2 = CF_{k,2n+1} - 2(-k)^n(1-k)$.
v. $CF_{k,n} = CF_{k,-n}(\sqrt{4+k}CF_{k,n}^2 - \frac{1}{2}CL_{k,2n})$,
vi. $CL_{k,n}^2 + CL_{k,n+1}^2 = CL_{k,2n} + CL_{k,2n+2}$.

Proof: Binet formulas are used for the proofs. We obtain:

i. $$\begin{aligned} CF_{k,n+1}CF_{k,m+1} + CF_{k,n}CF_{k,m} &= \frac{v_1^{n+1}-v_2^{n+1}}{v_1-v_2}\frac{v_1^{m+1}-v_2^{m+1}}{v_1-v_2} + \frac{v_1^{n}-v_2^{n}}{v_1-v_2}\frac{v_1^{m}-v_2^{m}}{v_1-v_2} \\ &= \frac{v_1^{n+m+1}(v_1-v_2)-v_2^{n+m+1}(v_1-v_2)}{(v_1-v_2)^2} \\ &= \frac{v_1^{n+m+1}-v_2^{n+m+1}}{v_1-v_2} \\ &= CF_{k,n+m+1}. \end{aligned}$$

v. $CF_{k,-n} = \frac{v_1^{-n}-v_2^{-n}}{v_1-v_2} = -\frac{v_1^{n}-v_2^{n}}{(v_1-v_2)v_1^n v_2^n} = -\frac{CF_{k,n}}{v_1^n v_2^n}$. So, we get $v_1^n v_2^n = -\frac{CF_{k,-n}}{CF_{k,n}}$.

Then we find

$$CF_{k,n}^2 = \frac{v_1^{n}-v_2^{n}}{v_1-v_2}\frac{v_1^{n}-v_2^{n}}{v_1-v_2} = \frac{v_1^{2n}+v_2^{2n}-2v_1^n v_2^n}{(v_1-v_2)(v_1-v_2)} = \frac{1}{2\sqrt{4+k}}CL_{k,2n} - \frac{1}{\sqrt{4+k}}v_1^n v_2^n.$$

Thus, we obtain

$$CF_{k,n} = CF_{k,-n}(\sqrt{4+k}CF_{k,n}^2 - \frac{1}{2}CL_{k,2n}).$$

Other proofs are shown in a similar way. □

In the following theorem, we examine the relationships among $CF_{k,n}$, $CL_{k,n}$, F_n, and L_n.

Theorem 2.7. *Let* $k \in \mathbb{R}^+$, *and* $n \in \mathbb{N}$. We have
i. $CF_{k,n} = k^n CF_{k,-n}\left(\frac{F_{3n}-F_{2n}(F_{n+1}+F_{n-1})}{F_n}\right)$,
ii. $CL_{k,n} = k^n CL_{k,-n}(L_n L_{n+1} - L_{2n+1})$,
iii. $CL_{k,n}^2 - (4k+16)CF_{k,n}^2 = 4k^n(\frac{F_{2n}F_{n+1}}{2F_n} - \frac{F_{2n}L_{n+1}}{2L_n})$.

Proof: Binet formulas are used for the proofs. We obtain

i. $CF_{k,-n} = \frac{v_1^{-n}-v_2^{-n}}{v_1-v_2} = -\frac{v_1^{n}-v_2^{n}}{(v_1-v_2)v_1^n v_2^n} = -\frac{CF_{k,n}}{v_1^n v_2^n}$. Thus, we get

$$\begin{aligned} CF_{k,n} &= -CF_{k,-n}(-1)^n k^n \\ &= CF_{k,-n}\frac{F_{3n}-F_{2n}L_n}{F_n}k^n \\ &= k^n CF_{k,-n}\left(\frac{F_{3n}-F_{2n}(F_{n+1}+F_{n-1})}{F_n}\right). \end{aligned}$$

ii. $CL_{k,-n} = v_1^{-n} + v_2^{-n} = \frac{v_1^n+v_2^n}{v_1^n v_2^n} = \frac{CL_{k,n}}{(-k)^n}$. Thus, we have

$$\begin{aligned} CL_{k,n} &= CL_{k,-n}(-1)^n k^n \\ &= CL_{k,-n}(L_n L_{n+1} - L_{2n+1})k^n. \end{aligned}$$

iii.
$$\begin{aligned} CL_{k,n}^2 - (4k+16)CF_{k,n}^2 &= (v_1^n + v_2^n)^2 - (4k+16)(\frac{v_1^n - v_2^n}{v_1 - v_2})^2 \\ &= v_1^{2n} + v_2^{2n} + 2v_1^n v_2^n - v_1^{2n} - v_2^{2n} + 2v_1^n v_2^n \\ &= 4v_1^n v_2^n \\ &= 4(-k)^n \\ &= 4k^n(\frac{F_{2n}F_{n+1}}{2F_n} - \frac{F_{2n}L_{n+1}}{2L_n}). \end{aligned}$$ □

Next, we find the sum formulas of the $CF_{k,n}$, and $CL_{k,n}$ sequences. Also, we give binomial formulas of these sequences.

Theorem 2.8. *Let* $m, n, p, r \in \mathbb{N}$. We obtain

i. $$\sum_{m=0}^{n} \mathrm{CF}_{k,rm+p} = \begin{cases} \frac{-\mathrm{CF}_{k,nr+r+p} + (-k)^r \mathrm{CF}_{k,nr+p} + \mathrm{CF}_{k,r-p} + \mathrm{CF}_{k,p}}{1+(-k)^r - \mathrm{CL}_{k,r}}, & \text{if } p < r \\ \frac{-\mathrm{CF}_{k,nr+r+p} + (-k)^r \mathrm{CF}_{k,nr+p} + \mathrm{CF}_{k,p-r} + \mathrm{CF}_{k,p}}{1+(-k)^r - \mathrm{CL}_{k,r}}, & \text{othervise} \end{cases},$$

ii. $$\sum_{m=0}^{n} \mathrm{CL}_{k,rm+p} = \begin{cases} \frac{-\mathrm{CL}_{k,nr+r+p} + (-k)^r \mathrm{CL}_{k,nr+p} - \mathrm{CL}_{k,r-p} + \mathrm{CL}_{k,p}}{1+(-k)^r - \mathrm{CL}_{k,r}}, & \text{if } p < r \\ \frac{(-1)^r \mathrm{CL}_{k,nr+p} - \mathrm{CL}_{k,p-r} + \mathrm{CL}_{k,p} - \mathrm{CL}_{k,nr+r+p}}{1+(-k)^r - \mathrm{CL}_{k,r}}, & \text{othervise} \end{cases}.$$

Proof: i. Binet formulas are used for proofs. We have

$$\begin{aligned} \sum_{m=0}^{n} CF_{k,rm+p} &= \sum_{m=0}^{n} \frac{v_1^{rm+p} - v_2^{rm+p}}{v_1 - v_2} = \frac{v_1^p}{v_1 - v_2}\sum_{y=0}^{n}(v_1^r)^m - \frac{v_2^p}{v_1 - v_2}\sum_{m=0}^{n}(v_2^r)^m \\ &= \frac{v_1^p}{v_1 - v_2}\frac{v_1^{nr+r}-1}{v_1^r - 1} - \frac{v_2^p}{v_1 - v_2}\frac{v_2^{nr+r}-1}{v_2^r - 1} \\ &= \begin{cases} \frac{-CF_{k,nr+r+p} + (-k)^r CF_{k,nr+p} + CF_{k,r-p} + CF_{k,p}}{1+(-k)^r - CL_{k,r}}, & \text{if } p < r \\ \frac{-CF_{k,nr+r+p} + (-1)^r CF_{k,nr+p} + CF_{k,p-r} + CF_{k,p}}{1+(-k)^r - CL_{k,r}}, & \text{othervise} \end{cases}. \end{aligned}$$

Another proof is shown in a similar way. □

Corollary 2.1. The special sum formulas of $CF_{k,n}$ and $CL_{k,n}$ sequences are as follows:

i. $\sum_{m=0}^{n} CF_{k,m} = \frac{CF_{k,n+1} + kCF_{k,n} - 1}{k+3}$,

ii. $\sum_{m=0}^{n} CL_{k,m} = \frac{CL_{k,n+1} + kCL_{k,n} - 2}{k+3}$,

iii. $\sum_{m=0}^{n} CF_{k,2m} = \frac{-CF_{k,2n+2} + k^2 CF_{k,2n} + 4}{k^2 - 2k - 15}$,

iv. $\sum_{m=0}^{n} CL_{k,2m} = \frac{-CL_{k,2n+2} + k^2 CL_{k,2n} + CL_{k,2n+1} + 2k + 14}{k+3}$,

v. $\sum_{m=0}^{n} CF_{k,2m+1} = \frac{-CF_{k,2n+3} + k^2 CF_{k,2n+1} + 2}{k^2 - 2k - 15}$,

vi. $\sum_{m=0}^{n} CL_{k,2m+1} = \frac{-CL_{k,2n+3} + k^2 CL_{k,2n+1}}{k^2 - 2k - 15}$.

Proof: i. *In Theorem 2.8, if* $p = 0$ *and* $r = 1$, we obtain

$$\sum_{m=0}^{n} CF_{k,m} = \frac{CF_{k,n+1}+kCF_{k,n}-1}{k+3}. \quad \square$$

Next, we obtain special generating functions of the $CF_{k,n}$, and $CL_{k,n}$ sequences.

Theorem 2.9. *Let $r, n, p \in \mathbb{N}$, and $p > r$, we obtain*
i. $\sum_{n=0}^{\infty} CF_{k,rn+p} t^n = \frac{CF_{k,p}-t(-k)^r CF_{k,p-r}}{1-tCL_{k,r}+(-k)^r t^2}$,
ii. $\sum_{n=0}^{\infty} CL_{k,rn+p} t^n = \frac{CL_{k,p}-t(-k)^r CL_{k,p-r}}{1-tCL_{k,r}+(-k)^r t^2}$,
iii. $\sum_{n=0}^{\infty} \frac{CF_{k,rn}}{n!} t^n = \frac{e^{v_1^r t}-e^{v_2^r t}}{v_1-v_2}$,
iv. $\sum_{n=0}^{\infty} \frac{CL_{k,rn}}{n!} t^n = e^{v_1^r t} + e^{v_2^r t}$.

Proof: Binet formulas are used for the proofs. We obtain

i.
$$\begin{aligned}\sum_{i=0}^{\infty} CF_{k,rn+p} t^n &= \sum_{n=0}^{\infty} \frac{v_1^{rn+p}-v_2^{rn+p}}{v_1-v_2} t^n \\ &= \frac{v_1^p}{v_1-v_2}\sum_{n=0}^{\infty}(v_1^r t)^n - \frac{v_2^p}{v_1-v_2}\sum_{n=0}^{\infty}(v_2^r t)^n \\ &= \frac{v_1^p}{v_1-v_2}\frac{1}{1-v_1^r t} - \frac{v_2^p}{v_1-v_2}\frac{1}{1-v_2^r t} \\ &= \frac{CF_{k,p}-t(-k)^r CF_{k,p-r}}{1-tCL_{k,r}+(-k)^r t^2}.\end{aligned}$$

iv.
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{CL_{k,rn}}{n!} t^n &= \sum_{n=0}^{\infty} \frac{v_1^{rn}+v_2^{rn}}{n!} t^n \\ &= \sum_{n=0}^{\infty} \frac{(v_1^r t)^n}{n!} + \sum_{n=0}^{\infty} \frac{(v_2^r t)^n}{n!} \\ &= e^{v_1^r t} + e^{v_2^r t}.\end{aligned}$$

Other proofs are shown in a similar way. $\square$

Corollary 2.2. The generating functions of $CF_{k,n}$ and $CL_{k,n}$ sequences are as follows:
i. $f(t) = \sum_{n=0}^{\infty} CF_{k,n} t^n = \frac{t}{1-4t-kt^2}$,
ii. $l(t) = \sum_{n=0}^{\infty} CL_{k,n} t^n = \frac{2-4t}{1-4t-kt^2}$.

Proof: i. In Theorem 2.9, if $p = 0$ and $r = 1$, we obtain

$$f(t) = \sum_{n=0}^{\infty} CF_{k,n} t^n = \frac{t}{1-4t-kt^2}. \quad \square$$

In the following theorem, we calculate some identities of the $CF_{k,n}$, and $CL_{k,n}$ sequences.

Theorem 2.10. i. ***(Cassini Identity)*** *Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$*. We get

- $CF_{k,n+1}CF_{k,n-1} - CF_{k,n}^2 = (-k)^n$,
- $CL_{k,n+1}CL_{k,n-1} - CL_{k,n}^2 = (4k+16)(-k)^{n-1}$.

ii. *(Halton Identity)* Let $k \in \mathbb{R}^+$, $n, i, j \in \mathbb{N}$, and $i, j < n$. We get

- $CF_{k,n+i}CF_{k,n-i} - CF_{k,n+j}CF_{k,n-j} = \frac{1}{(4k+16)}[(-k)^{n-j}CL_{k,2j} - (-k)^{n-i}CL_{k,2i}]$
- $CL_{k,n+i}CL_{k,n-i} - CL_{k,n+j}CL_{k,n-j} = (-k)^{n-i}CL_{k,2i} - (-k)^{n-j}CL_{k,2j}$.

iii. *(d'ocagne Identity)* Let $k \in \mathbb{R}^+$, $n, i \in \mathbb{N}$, and $i < n$. We get

- $CF_{k,n+1}CF_{k,i} - CF_{k,n}CF_{k,i+1} = (-k)^{i+1}CF_{k,n-i}$,
- $CL_{k,n+1}CL_{k,i} - CL_{k,n}CL_{k,i+1} = (4k+16)(-k)^{i}CF_{k,n-i}$.

iv. *(Vajda Identity)* Let $k \in \mathbb{R}^+$, $n,i,j \in \mathbb{N}$, and $i,j < n$. We get

- $CF_{k,n+i}CF_{k,n+j} - CF_{k,n}CF_{k,n+i+j} = (-k)^{n}CF_{k,i}CF_{k,j}$,
- $CL_{k,n+i}CL_{k,n+j} - CL_{k,n}CL_{k,n+i+j} = (4k+16)(-k)^{n+1}CF_{k,i}CF_{k,j}$.

v. *(Catalan Identity)* Let $k \in \mathbb{R}^+$, $n,i \in \mathbb{N}$, and $i < n$. We get

- $CF_{k,n+i}CF_{k,n-i} - CF_{k,n}^2 = (-k)^{n-i+1}CF_{k,i}^2$,
- $CL_{k,n+i}CL_{k,n-i} - CL_{k,n}^2 = (4k+16)(-k)^{n-i}CF_{k,i}^2$.

vi. *(Gelin-Cesaro Identity)* Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. We get

- $CF_{k,n+2}CF_{k,n+1}CF_{k,n-1}CF_{k,n-2} - CF_{k,n}^4 = \frac{1}{(4k+16)}(\frac{1}{(-k)^{n-3}}CL_{k,2n+4} + \frac{1}{(-k)^{n-2}}CL_{k,2n+2}$
$+\frac{1}{(-k)^{2n-3}}CL_{k,6} + \frac{1}{(-k)^{n}}CL_{k,2n-2} + \frac{1}{(-k)^{n+1}}CL_{k,2n-4} + \frac{1}{(-k)^{n-3}}CL_{k,2n} + 16k^2 - 3)$,
- $CL_{k,n+2}CL_{k,n+1}CL_{k,n-1}CL_{k,n-2} - CL_{k,n}^4 = \frac{1}{(-k)^{n-2}}CL_{k,2n+4} + \frac{1}{(-k)^{n-1}}CL_{k,2n+2}$
$+\frac{1}{(-k)^{2n-3}}CL_{k,6} + \frac{1}{(-k)^{n+1}}CL_{k,2n-2} + \frac{1}{(-k)^{2n-1}}CL_{k,2} + \frac{1}{(-k)^{n+2}}CL_{k,2n-4} - \frac{1}{(-k)^{n-2}}CL_{k,2n} - k$.

vii. *(Padilla Identity)* Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. We get

- $CF_{k,n+2}^3 + CF_{k,n-1}^3 - 3CF_{k,n}CF_{k,n+1}CF_{k,n+2} = \frac{1}{(4k+16)}(CF_{k,3n+6} - 3CF_{k,3n+3} + CF_{k,3n-3}$
$+3CF_{k,2n+3} - 3CF_{k,n+2} + 3CF_{k,n+1})$,
- $CL_{k,n+2}^3 + CL_{k,n-1}^3 - 3CL_{k,n}CL_{k,n+1}CL_{k,n+2} = CL_{k,3n+6} + 3CL_{k,n+2} + CL_{k,3n-3}$
$+3CL_{k,n-1} + (4k+16)(-CF_{k,3n+3} - 3CF_{k,n-1} + 3CF_{k,n+1} + 3CF_{k,2n+3})$.

viii. *(Melham Identity)* Let $k \in \mathbb{R}^+$, $n \in \mathbb{N}$. We get

- $CF_{k,n+1}CF_{k,n+2}CF_{k,n+6} - CF_{k,n}^3 = \frac{1}{(4k+16)}(CF_{k,3n+9} - CF_{k,3n} - \frac{1}{(-k)^{n+1}}CF_{k,n+7}$
$-\frac{1}{(-k)^{n+2}}CF_{k,n+5} + \frac{3}{(-k)^{n}}CF_{k,n} - \frac{1}{(-k)^{n+6}}CF_{k,n-3})$,
- $CL_{k,n+1}CL_{k,n+2}CL_{k,n+6} - CL_{k,n}^3 = CL_{k,3n+9} - CL_{k,3n} + \frac{1}{(-k)^{n+1}}CL_{k,n+7}$
$+\frac{1}{(-k)^{n+2}}CL_{k,n+5} - \frac{3}{(-k)^{n}}CL_{k,n} + \frac{1}{(-k)^{n+6}}CL_{k,n-3}$.

Proof: Binet formulas are used for proofs. We have

i. $$CF_{k,n+1}CF_{k,n-1} - CF_{k,n}^2 = \frac{v_1^{n+1}-v_2^{n+1}}{v_1-v_2}\frac{v_1^{n-1}-v_2^{n-1}}{v_1-v_2} - \frac{v_1^{n}-v_2^{n}}{v_1-v_2}\frac{v_1^{n}-v_2^{n}}{v_1-v_2}$$
$$= \frac{v_1^{2n}-v_1^{n+1}v_2^{n-1}-v_2^{n+1}v_1^{n-1}+v_2^{2n}}{(v_1-v_2)^2} - \frac{v_1^{2n}-2v_1^{n}v_2^{n}+v_2^{2n}}{(v_1-v_2)^2}$$
$$= \frac{(v_1v_2)^n\frac{-v_1}{v_2}}{(v_1-v_2)^2} + \frac{(v_1v_2)^n\frac{-v_2}{v_1}}{(v_1-v_2)^2} + \frac{2(v_1v_2)^n}{(v_1-v_2)^2}$$
$$= (-k)^n.$$

ii. $$CL_{k,n+1}CL_{k,n-1} - CL_{k,n}^2 = (v_1^{n+1}+v_2^{n+1})(v_1^{n-1}+v_2^{n-1}) - (v_1^{n}+v_2^{n})(v_1^{n}+v_2^{n})$$
$$= v_1^{2n} + v_1^{n+1}v_2^{n-1} + v_1^{n-1}v_2^{n+1} + v_2^{2n} - v_1^{2n} - 2v_1^{n}v_2^{n} - v_2^{2n}$$
$$= (v_1v_2)^n(-\frac{v_1}{v_2} - \frac{v_2}{v_1} + 2)$$
$$= (4k+16)(-k)^{n-1}.$$

Other proofs are shown in a similar way. □

3. CATALAN TRANSFORMATION

In this section, we apply the Catalan transformation to these sequences and find their first few terms. In addition, we obtain some properties of the Catalan transformation of these sequences. Moreover, the Hankel transform is applied to the Catalan transformation of these sequences. Moreover, very interesting results are obtained with the terms of the Hankel transformation.

Catalan numbers are one of the most important sequences in combinatorics and discrete mathematics. They appear naturally in many counting problems involving recursive or hierarchical structures. The sequence was named after the Belgian mathematician Eugène Charles Catalan.

In [52], for $n \in \mathbb{N}^+$, the Catalan numbers and the generating function of the Catalan numbers are

$$C_n = \frac{C(2n,n)}{n+1}, \text{ and } c(t) = \frac{1-\sqrt{1-4t}}{2t}.$$

Using the definition of the Catalan transformation, the Catalan transformation of the k-Copper Fibonacci and k-Copper Lucas sequences are defined as follows:

$$CCF_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} CF_{k,i}, n \in \mathbb{N}^+ \text{ with } CCF_{k,0} = 0$$

and

$$CCL_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} CL_{k,i}, n \in \mathbb{N}^+ \text{ with } CCL_{k,0} = 0.$$

Next, the first five values of $CCF_{k,n}$, and $CCL_{k,n}$ are given below:

$$CCF_{k,0} = 0,\ CCF_{k,1} = 1,\ CCF_{k,2} = 5,\ CCF_{k,3} = k + 26,$$
$$CCF_{k,4} = 11k + 137,\ CCF_{k,5} = k^2 + 89k + 726,$$

and

$$CCL_{k,0} = 0,\ CCL_{k,1} = 4,\ CCL_{k,2} = 2k + 20,\ CCL_{k,3} = 16k + 104,$$
$$CCL_{k,4} = 2k^2 + 110k + 548,\ CCL_{k,5} = 28k^2 + 712k + 2904.$$

We can write $CCF_{k,n}$ and $CCL_{k,n}$ as the product of $nx1$ type $CF_{k,n}$ and $CL_{k,n}$'s of the lower triangular Catalan matrix C:

$$\begin{bmatrix} CCF_{k,1} \\ CCF_{k,2} \\ CCF_{k,3} \\ CCF_{k,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CF_{k,1} \\ CF_{k,2} \\ CF_{k,3} \\ CF_{k,4} \\ \vdots \end{bmatrix} \text{ and } \begin{bmatrix} CCL_{k,1} \\ CCL_{k,2} \\ CCL_{k,3} \\ CCL_{k,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} CL_{k,1} \\ CL_{k,2} \\ CL_{k,3} \\ CL_{k,4} \\ \vdots \end{bmatrix}.$$

So,

$$\begin{bmatrix} 1 \\ 5 \\ k+26 \\ 11k+37 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ k+16 \\ 8k+64 \\ \vdots \end{bmatrix}$$

and

$$\begin{bmatrix} 4 \\ 2k+20 \\ 16k+104 \\ 2k^2+110k+548 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 4 \\ 2k+16 \\ 12k+64 \\ 2k^2+64k+256 \\ \vdots \end{bmatrix}.$$

Next, we obtain the generating functions of the Catalan transformation of $CF_{k,n}$, and $CL_{k,n}$ sequences.

Theorem 3.1. *Let* $k \in \mathbb{R}^+$, $n \in \mathbb{N}$. We get

i. $F(t) = \sum_{n=0}^{\infty} CCF_{k,n} t^n = \frac{2-2\sqrt{1-4t}}{-4-k-k(1-4t)-(8+2k)\sqrt{1-4t}}$,

ii. $L(t) = \sum_{n=0}^{\infty} CCL_{k,n} t^n = \frac{2\sqrt{1-4t}}{-4-k-k(1-4t)-(8+2k)\sqrt{1-4t}}$.

Proof: **i.** *The following equations are used for the proof*:

$$F(t) = CCF_{k,n}(t) = CCF_{k,n}\big(t * c(t)\big).$$

Thus, we obtain

$$F(t) = \frac{\frac{1-\sqrt{1-4t}}{2}}{1-4\left(\frac{1-\sqrt{1-4t}}{2}\right)-k\left(\frac{1-\sqrt{1-4t}}{2t}\right)^2}$$
$$= \frac{2-2\sqrt{1-4t}}{-4-k-k(1-4t)-(8+2k)\sqrt{1-4t}}.$$

The other proof is shown in a similar way. □

In [53], for $n \in \mathbb{N}^+$, the Hankel transform H_n of the terms of this sequence was defined as follows:

$$H_\mathrm{n} = det \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & \vdots & a_n & \vdots \\ a_3 & \vdots & a_n & \vdots & \vdots \\ \vdots & a_n & \vdots & \vdots & \vdots \\ a_n & \cdots & \cdots & \cdots & a_{2n-1} \end{bmatrix}.$$

Here $\{a_1, a_2, a_3, \dots\}$ are the elements of the sequence. Next, we apply Hankel's work to the k-Copper Fibonacci $CCF_{k,n}$ and Catalan k-Copper Lucas $CCL_{k,n}$ sequences, respectively. We obtain

- $HCCF_{k,1} = det\left[CCF_{k,1}\right] = det[1] = 1 = CF_{k,1}$,
- $HCCF_{k,2} = det\begin{bmatrix} CCF_{k,1} & CCF_{k,2} \\ CCF_{k,2} & CCF_{k,3} \end{bmatrix} = det\begin{bmatrix} 1 & 5 \\ 5 & k+26 \end{bmatrix} = k+1 = kCF_{k,1} + CF_{k,1}$,
- $HCCF_{k,3} = det\begin{bmatrix} CCF_{k,1} & CCF_{k,2} & CCF_{k,3} \\ CCF_{k,2} & CCF_{k,3} & CCF_{k,4} \\ CCF_{k,3} & CCF_{k,4} & CCF_{k,5} \end{bmatrix}$
$= det\begin{bmatrix} 1 & 5 & k+26 \\ 5 & k+26 & 11k+137 \\ k+26 & 11k+137 & k^2+89k+726 \end{bmatrix}$
$= k^2+3k+1 = kCF_{k,2} + (1-\frac{13}{4}k)CF_{k,1}$,

and

- $HCCL_{k,1} = det\left[CCL_{k,1}\right] = det[4] = 4 = CL_{k,1}$,

- $HCCL_{k,2} = det\begin{bmatrix} CCL_{k,1} & CCL_{k,2} \\ CCL_{k,2} & CCL_{k,3} \end{bmatrix} = det\begin{bmatrix} 4 & 2k+20 \\ 2k+20 & 16k+104 \end{bmatrix}$
  $= -(2k-4)^2 = -(CL_{k,2} - 3CL_{k,1})^2,$
- $HCCL_{k,3} = det\begin{bmatrix} CCL_{k,1} & CCL_{k,2} & CCL_{k,3} \\ CCL_{k,2} & CCL_{k,3} & CCL_{k,4} \\ CCL_{k,3} & CCL_{k,4} & CCL_{k,5} \end{bmatrix} = -48k^2 - 192k + 64$
  $= -16kCL_{k,2} + 32CL_{k,1} + 32CL_{k,0}.$

4. CONCLUSIONS

In this study, the k-Copper Fibonacci and k-Copper Lucas sequences are defined. These sequences are generalized forms of the classical Fibonacci and Lucas sequences, and new sequences with different ratios are derived by using the parameter k. First, these sequences are defined, and the first few terms of both sequences are calculated and presented. This allows for a clearer understanding of the sequences' structure and differences. Subsequently, specific generating functions for these sequences are derived. These generating functions allow for closed expressions of the sequences, and their analytical properties are examined in detail. Furthermore, as with the classical Fibonacci and Lucas sequences, closed Binet-like formulas are derived for the k-Copper Fibonacci and k-Copper Lucas sequences. These Binet formulas are expressed in terms of the roots of the characteristic equations of the sequences. The characteristic equations of the sequences are established, and the specific relationships between the roots of these equations and the sequences are presented in detail. In particular, meaningful connections are established between the magnitudes of the absolute values of the characteristic roots and the limit behavior of the sequences. Furthermore, by examining the ratio between two consecutive terms of these sequences, it is shown that the ratio of the largest term to the smallest term converges to a certain value. This ratio is important for understanding the long-term growth rates of the sequences. In addition, symmetric relations between the positively and negatively indexed terms of the sequences are revealed, and these relations are proven analytically. Furthermore, the connections between these sequences and the sum of the squares of the consecutive terms are detailed, and various identities are derived in this context. These identities are found to provide in-depth information about the structural properties of the sequences. The study also applies Catalan transformations to the k-Copper Fibonacci and k-Copper Lucas sequences, and the first few terms of the new sequences resulting from these transformations are calculated and presented. The generating functions of the sequences resulting from these transformations are also obtained. Finally, the Hankel transformation is applied to these new sequences. The structures of the sequences obtained from the Hankel transformation are examined, and the effects of this transformation on the sequences in terms of symmetry and determinant relations are discussed. Thus, meaningful mathematical results have been obtained on both the original k-Copper Fibonacci and Lucas sequences and the derivative sequences obtained by applying Catalan and Hankel transformations to them. These results can shed light on the possible applications of such generalized sequences in number theory and combinatorics. Upon examining this work, several algebraic properties of new generalized sequences can also be investigated. In particular, the Parastichy Fibonacci [54], Ernst [55], and John [56] sequences may be defined and studied through their recurrence relations, generating functions, and matrix representations. Therefore, this study may provide a basis for future research on generalized recursive sequences.

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