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EXPLORING PRE-LORENTZ BICOMPLEX SEQUENCE SPACES: STRUCTURE, PROPERTIES, AND APPLICATIONS

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Abstract. *In this article, the authors introduced pre-Lorentz sequence spaces with bicomplex terms and studied their properties. They encountered difficulty when they considered applying the known properties of pre-Lorentz function spaces to sequence spaces with bicomplex terms. They managed to overcome these difficulties by using idempotent representations of bicomplex numbers. After comprehensively examining the distribution and rearrangement functions in the space of sequences with bicomplex terms in their first study, presented in the references, it was time to examine pre-Lorentz spaces, and this study emerged.*

Keywords: *$\mathbb{D}$-Distribution function; $\mathbb{D}$-Decreasing rearrangement function; Hyperbolic-valued norm; Bicomplex pre-Lorentz sequence space.*

1. INTRODUCTION

Bicomplex numbers were mentioned for the first time in [1]. A comprehensive review of the bicomplex spaces and the relevant context was given in [2]. Alpay et al. [3] developed bicomplex versions of functional analysis with complex scalars, and this was an important step for subsequent works on the theory of functions of bicomplex variables. Bicomplex numbers have new applications in areas such as neural networks [4], smart radio access networks [5], electromagnetic wave propagation [6], integral transformations and fractional calculus [7]. Thus, researchers working on bicomplex analysis reveal the importance of these numbers in real-world problems. Other recent notable applications can be found in [8-12].

We will now present a basic overview of bicomplex numbers. We also refer to books [2,3, 13] for more comprehensive information.

The elements of the set represented by $\mathbb{BC} = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\}$ is called bicomplex numbers, where $\mathbb{C}$ is the set of complex numbers with the imaginary unit i, and also where i and $i \neq j$ are commutative imaginary units, i.e., $ij = ji = k, i^2 = j^2 = -1$ and

$$k^2 = (ij)^2 = (ij)(ij) = i(ji)j = i(ij)j = (ii)(jj) = i^2j^2 = 1.$$

Any two elements $z = z_1 + jz_2$ and $w = w_1 + jw_2$ in the set $\mathbb{BC}$ can be added and multiplied as follows:

$$z + w = (z_1 + w_1) + j(z_2 + w_2) \text{ and } z.w = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1).$$

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According to the two operations described above, the set $\mathbb{BC}$ is a commutative and unitary ring with $1_{\mathbb{BC}} = 1 + j.0$. In addition, when $z_2 = 0$ in $z = z_1 + jz_2$, that is, $z = z_1$, the set of these numbers is represented by $\mathbb{C}(i)$. If the coefficients z_1 and z_2 are real numbers, that is, $z = x + jy$ with $x, y \in \mathbb{R}$, then the set of those numbers is represented by $\mathbb{C}(j)$. $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields.

The set of hyperbolic numbers is described by

$$\mathbb{D} = \{x + ky : x, y \in \mathbb{R}, k = ij\},$$

where k is a hyperbolic imaginary unit, i.e., $k^2 = 1$. In the studies conducted in the current literature, hyperbolic numbers are sometimes called duplex, double, or bireal numbers. The following subsets $\mathbb{D}^+$ and $\mathbb{D}^+\backslash\{0\}$ of $\mathbb{D}$ are called non-negative and positive hyperbolic numbers, respectively:

$$\mathbb{D}^+ = \{x + ky : x^2 - y^2 \geq 0, x \geq 0\},$$

$$\mathbb{D}^+\backslash\{0\} = \{x + ky : x^2 - y^2 \geq 0, x > 0\}.$$

Similarly, non-positive and negative hyperbolic numbers are defined as follows:

$$\mathbb{D}^- = \{x + ky : x^2 - y^2 \geq 0, x \leq 0\} \text{ and } \mathbb{D}^-\backslash\{0\} = \{x + ky : x^2 - y^2 \geq 0, x < 0\}.$$

Given $u, v \in \mathbb{D}^+$. If $u - v \in \mathbb{D}^+$, then we write $u \succcurlyeq v$ or $v \preccurlyeq u$, and say that u is $\mathbb{D}$-greater than or $\mathbb{D}$-equal to v, or that v is $\mathbb{D}$-less than or $\mathbb{D}$-equal to u. If $u = u_1e_1 + u_2e_2$ and $v = v_1e_1 + v_2e_2$ with real numbers u_1, u_2, v_1 and v_2, we can write $v \preccurlyeq u \Leftrightarrow v_1 \leq u_1$ and $v_2 \leq u_2$ ($v \prec u \Leftrightarrow v_1 < u_1$ and $v_2 < u_2$). If u is a (strictly) positive hyperbolic number, then it is inversible, and its inverse is also positive. Additionally, if $u \succ 0$ and $u \prec v$, then $v^{-1} \succ 0$ and $v^{-1} \prec u^{-1}$ [14].

Consider the bicomplex numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. It can be easily seen that $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$. There are also equations $(e_1)^n = e_1$, $(e_2)^n = e_2$ with $n \in \mathbb{N}$. For any bicomplex number $u = u_1 + ju_2 \in \mathbb{BC}$, we have

$$\begin{aligned} u &= \frac{u_1 + iu_2 + u_1 - iu_2}{2} + j\frac{u_2 + iu_1 + u_2 - iu_1}{2} \\ &= \frac{u_1 - iu_2}{2} + \frac{u_1 + iu_2}{2} + j\left(i\frac{u_1 - iu_2}{2} - i\frac{u_1 + iu_2}{2}\right) \\ &= \frac{u_1 - iu_2}{2}(1 + ij) + \frac{u_1 + iu_2}{2}(1 + ij) \\ &= (u_1 - iu_2)e_1 + (u_1 + iu_2)e_2 = \delta_1e_1 + \delta_2e_2 \end{aligned}$$

with $\delta_1 = (u_1 - iu_2)$ and $\delta_2 = (u_1 + iu_2)$ in $\mathbb{C}(i)$. This equality is named as $\mathbb{C}(i)$ idempotent representation of the bicomplex number u.

Similarly, along with the coefficients in $\mathbb{C}(j)$, there is also a representation of the bicomplex number u with respect to e_1 and e_2.

As a result, any bicomplex number has an idempotent representation with its coefficients in any of $\mathbb{C}(i)$ or $\mathbb{C}(j)$, that is, $u = \delta_1e_1 + \delta_2e_2 = \rho_1e_1 + \rho_2e_2$ where $\delta_1, \delta_2 \in \mathbb{C}(i)$ and $\rho_1, \rho_2 \in \mathbb{C}(j)$.

If a function $|.|_k$ from $\mathbb{BC}$ to $\mathbb{D}^+$ is defined by $|z|_k = |z_1|e_1 + |z_2|e_2$ for each $z = z_1e_1 + z_2e_2 \in \mathbb{BC}$ and provides the following properties, then it is called a $\mathbb{D}$-Norm or a hyperbolic-valued norm:
N1) Since $|z_1| \geq 0$ and $|z_2| \geq 0$ for a $z = (z_1e_1 + z_2e_2) \in \mathbb{BC}$, $|z|_k = |z_1|e_1 + |z_2|e_2 \succcurlyeq 0e_1 + 0e_2 = 0$.
N2) $|z|_k = |z_1|e_1 + |z_2|e_2 = 0 = 0e_1 + 0e_2$ if and only if $|z_1| = 0$ ve $|z_2| = 0$, and so $z = 0e_1 + 0e_2 = 0$.
N3) $|\lambda z|_k = (|\lambda_1|e_1 + |\lambda_2|e_2)(|z_1|e_1 + |z_2|e_2) = |\lambda|_k|z|_k$ for $\lambda \in \mathbb{D}$.
N4) $|z + w|_k \preccurlyeq |z|_k + |w|_k$ for $z = z_1e_1 + z_2e_2, w = w_1e_1 + w_2e_2 \in \mathbb{BC}$.

We will now establish the basic theory of pre-Lorentz spaces. Duyar and Işık [6] laid the foundations of Lorentz sequence spaces. For more information on classical Lorentz sequence spaces, one can be examined [8,15].

Let $\omega_{\mathbb{BC}}$ be denoted the set of all sequences with bicomplex terms, and G be the power set of $\mathbb{N}$, namely $G = 2^{\mathbb{N}}$, and μ be the counting measure on G. Also, let $(G, \mathcal{G}, \mu)$ be a measure space and $M(G, \mathcal{G})$ be the set of all $\mathcal{G}$-measurable complex-valued functions on G. The distribution function $D_g(\lambda)$ of a function g in $M(G, \mathcal{G})$ is given in [4,9] by

$$D_g(\lambda) = \mu\{x \in G: |g(x)| > \lambda \geq 0\}.$$

We now define the $\mathbb{D}$-distribution function, using a sequence with bicomplex terms instead of the measurable function and the non-negative hyperbolic numbers instead of the non-negative real numbers in the definition of the distribution function, and the counting measure instead of the measure.

The following two definitions, lemma and theorem, are known [9].

Definition 1.1. Let $z = \big(z(n)\big)$ be an arbitrary sequence in $\omega_{\mathbb{BC}}$ with $z(n) = z_1(n)e_1 + z_2(n)e_2$ for all $n \in \mathbb{N}$ and an arbitrary number $\lambda_1e_1 + \lambda_2e_2 = \lambda \in \mathbb{D}^+$ be given. The $\mathbb{D}$-distribution function D_z of z is defined by

$$D_z(\lambda) = D_{z_1}(\lambda_1)e_1 + D_{z_2}(\lambda_2)e_2.$$

Definition 1.2. A function h on $\mathbb{D}^+$ into itself is called a $\mathbb{D}$-decreasing function, if there is a $\mathbb{D}$-inequality $h(\beta) \preccurlyeq h(\alpha)$, whenever $\alpha \prec \beta$.

Lemma 1.1. Let λ be an element in $\mathbb{D}$ with $\lambda = \lambda_1e_1 + \lambda_2e_2$ and δ be another element in $\mathbb{D}$ with $\delta = \delta_1e_1 + \delta_2e_2, \delta_1 \neq 0$ and $\delta_2 \neq 0$. There exists the equality

$$\lambda/|\delta|_k = (\lambda_1/|\delta_1|)e_1 + (\lambda_2/|\delta_2|)e_2.$$

Theorem 1.1. Let z=$\big(z(n)\big)$ and $w = \big(w(n)\big)$ be two sequences in $\omega_{\mathbb{BC}}$ and λ, δ and c be the elements of $\mathbb{D}^+$. Then, the following features are satisfied:
(a) $\mathbb{D}$-distribution function is $\mathbb{D}$-decreasing.
(b) If $|z(n)|_k \preccurlyeq |w(n)|_k$ for all $n \geq 1$, then $D_w(\lambda) \preccurlyeq D_z(\lambda)$.
(c) $D_{cz}(\lambda) = D_z(\lambda/|c|_k)$ for all $c = c_1e_1 + c_2e_2 \in \mathbb{D}^+$ with $c_1 \neq 0, c_2 \neq 0$.
(d) $D_{z+w}(\lambda + \delta) \preccurlyeq D_z(\lambda) + D_w(\delta)$.
(e) $D_{z+w}(\lambda.\delta) \preccurlyeq D_z(\lambda) + D_w(\delta)$.

Definition 1.3. If there is a $\rho \in \mathbb{D}$ such that $g \preccurlyeq \rho(\rho \preccurlyeq g)$ for all $g \in G$, then it is said that a subset $G \subset \mathbb{D}$ is a $\mathbb{D}$-bounded from above(below). This number $\rho \in \mathbb{D}$ is called a $\mathbb{D}$-upper($\mathbb{D}$-lower) boundary of G. If $G \subset \mathbb{D}$ is a $\mathbb{D}$-bounded set from above, then we describe its $\mathbb{D}$-supremum, showed by $\mathbb{D} - \sup G$, the smallest upper bound of G, and its $\mathbb{D}$-infimum, showed by $\mathbb{D} - \inf G$, largest lower bound of G. Given a set $G \subset \mathbb{D}$, let the set G_1 and G_2 be defined by

$$G_1 = \{g_1 : g_1e_1 + g_2e_2 \in G\} \text{ and } G_2 = \{g_2 : g_1e_1 + g_2e_2 \in G\}.$$

If G is a $\mathbb{D}$-bounded set from above(below), then the $\mathbb{D} - \sup G(\mathbb{D} - \inf G)$ can be computed by the formula

$$\mathbb{D} - \sup G = \sup G_1 e_1 + \sup G_2 e_2 \ (\mathbb{D} - \inf G = \inf G_1 e_1 + \inf G_2 e_2).$$

If G and H are two $\mathbb{D}$-bounded set from below, then so is $G + H$ and

$$\mathbb{D} - inf(G + H) = \mathbb{D} - infG + \mathbb{D} - infH.$$

If two subsets $G \subset \mathbb{D}^+$ and $H \subset \mathbb{D}^+$ are $\mathbb{D}$-bounded from below, then so is $G.H$ and

$$\mathbb{D} - \inf(G.H) = (\mathbb{D} - \inf G).(D - \inf H).$$

For the $\mathbb{D}$-bounded subsets from above of $\mathbb{D}$, the last two equations are still true when $\mathbb{D} - \sup$ is written instead of $\mathbb{D}$-inf [8,14].

Definition 1.4. The $\mathbb{D}$-decreasing rearrangement of a sequence $z = \big(z(n)\big)$ in $\omega_{\mathbb{BC}}$ is a function z^*, defined by

$$z^*(t) = \mathbb{D} - inf\{\lambda \in D^+ : D_z(\lambda) \preccurlyeq \ t\}$$

from $\mathbb{D}^+$ into itself. It is known that z^* is the left inverse of D_z. Moreover, If $z(n) = z_1(n)e_1 + z_2(n)e_2, \lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $t = t_1e_1 + t_2e_2$, then

$$z^*(t) = {z_1}^*(t_1)e_1 + {z_2}^*(t_2)e_2$$

[9].

Definition 1.5. The pre-Lorentz sequence space $\ell_{(p,q)}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is the set of all complex sequences $\|u\|_{(p,q)} < \infty$, where

$$\|u\|_{(p,q)} = \begin{cases} \left(\sum_{n=1}^{\infty} n^{(q/p)-1} \, [u^*(n)]^q\right)^{1/q}, 1 < p \leq \infty, 1 \leq q < \infty \\ sup_{n\geq 1} n^{1/p} u^*(n) \, , 1 < p \leq \infty, q = \infty. \end{cases}$$

The pre-Lorentz sequence space $\ell_{(p,q)}, 1 < p \leq \infty$, $1 \leq q \leq \infty$ is a linear space and $\|.\|_{(p,q)}$ is a quasinorm. Also, $\ell_{(p,q)}, 1 < p \leq \infty$, $1 \leq q \leq \infty$ is complete with respect to the quasi-norm $\|.\|_{(p,q)}$ and $\ell_{(p,q)}, 1 \leq q \leq p \leq \infty$, $1 \leq q \leq \infty$ is a complete normed linear space with respect to $\|.\|_{(p,q)}$ [14].

Theorem 1.2. Let $z = \big(z(n)\big)$ be a sequence in $\omega_{\mathbb{BC}}$ and $\lambda = \lambda_1 e_1 + \lambda_2 e_2$ and $t = t_1 e_1 + t_2 e_2$ be two elements in $\mathbb{D}^+$. The $\mathbb{D}$-decreasing rearrangement function z^* of z has the following properties [9]:

(a) z^* is $\mathbb{D}$-decreasing.

(b) $z^*(t) \succ \lambda$ iff $D_z(\lambda) \succ t$.

(c) $(\kappa z)^*(t) = |\kappa|_k z^*(t)$ with $\kappa \in \mathbb{D}^+$.

(d) Let $\big(z(n)\big)$ and $\big(w(n)\big)$ be two sequences in $\omega_{\mathbb{BC}}$. If $|z(n)|_k \prec |w(n)|_k$ for every $n = 1,2,...$, then $z^*\,(t) \preccurlyeq w^*\,(t)$.

(e) $[|z|_k^p]^*(n) = (z^*)^p(n)$ with $p \geq 1$.

Lemma 1.2. Let $(G, \mathcal{G}, \mu)$ is a measure space, further g and h be two measurable functions. Then, the inequalities

$$(g+h)^*(\alpha+\beta) \leq g^*(\alpha) + h^*(\beta)$$

and

$$(g.h)^*(\alpha+\beta) \leq g^*(\alpha).h^*(\beta),$$

are true for all $\alpha, \beta \geq 0$ [9].

Theorem 1.3. Let $(N, \mathcal{G}, \mu)$ be a measure space, where μ is the counting measure, and $z = \big(z(n)\big), w = \big(w(n)\big)$ be two sequences in $\omega_{\mathbb{BC}}$. Then, the $\mathbb{D}$-inequalities

$$(z+w)^*(\alpha+\beta) \preccurlyeq z^*(\alpha) + w^*(\beta)$$

and

$$(z.w)^*(\alpha+\beta) \preccurlyeq z^*(\alpha).w^*(\beta)$$

occur [9].

2. MAIN RESULTS

As it is known, classical pre-Lorentz sequence spaces are defined for sequences with complex terms. We use the set of sequences with bicomplex terms in $\mathbb{D}$-pre-Lorentz space. To define this space, we give firstly the definition the $\mathbb{D}$- quasinorm and its some properties.

We secondly define the $\mathbb{D}$-pre-Lorentz space of sequences with bicomplex terms using the $\mathbb{D}$-quasinorm and examine its some properties.

We finally construct the $\mathbb{D}$-norm using the $\mathbb{D}$-rearrangement functions, and create $\mathbb{D}$-pre-Lorentz sequence space consisting of the sequences with bicomplex terms, and examine some characteristic properties of this space.

Definition 2.1. A function $\|.\|$ from $\omega_{\mathbb{BC}}$ into $\mathbb{D}^+$ is called a $\mathbb{D}$-quasinorm, if it satisfies the following properties:

(QN_1) $\|z\| = 0 \Leftrightarrow z = 0$ for $z \in \omega_{\mathbb{BC}}$.

(QN_2) $\|\lambda z\| = |\lambda|_k \|z\|$ for $\lambda \in \mathbb{BC}$ and $z \in \omega_{\mathbb{BC}}$.

(QN_3) $\|z+w\| \preccurlyeq \rho(\|z\|+\|w\|)$ for a fixed $\rho \in \mathbb{R}^{+}$ and every $z, w \in \omega_{\mathbb{BC}}$ ($\mathbb{D}$-triangle inequality).

Lemma 2.1. Let $a = (a_r)$ be a sequence with complex terms and $1 \leq p < q \leq \infty$. Then, there exists the inequality

$$\sum_{r=1}^{\infty} r^{(q/p)-1}[a^*(r/2)]^q \leq 2^{q/p}\sum_{r=1}^{\infty} r^{(q/p-1)}[a^*(r)]^q.$$

Proof: Since function $\psi: [0,\infty) \to R, \psi(t) = t^q$ is a convex function, we can write

$$((x+y)/2)^q \leq (1/2)(x^q + y^q)$$

for all $x, y \geq 0$. If $p < q$, then $(q/p) - 1 > 0$, and hence $s^{(q/p)-1} < (s+1)^{(q/p)-1}$ for all $s \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\sum_{r=1}^{\infty} r^{(q/p)-1}[a^*(r/2)]^q &= 1^{(q/p)-1}[a^*(1/2)]^q \\ &+2^{(q/p)-1}[a^*(2/2)]^q + 3^{(q/p)-1}[a^*(3/2)]^q + \ldots \\ &< 2^{(q/p)-1}[a^*(1)]^q + 2^{(q/p)-1}[a^*(1)]^q + 4^{(q/p)-1}[a^*(2)]^q \\ &+4^{(q/p)-1}[a^*(2)]^q + 6^{(q/p)-1}[a^*(3)]^q + \ldots \\ &= 2\left(2^{(q/p)-1}[a^*(1)]^q + 4^{(q/p)-1}[a^*(2)]^q + 6^{(q/p)-1}[a^*(3)]^q + \ldots\right) \\ &= 2^{q/p}\sum_{r=1}^{\infty} r^{(q/p)-1}[a^*(r)]^q.\end{aligned}$$

Definition 2.2. Let us think of the set of all sequences in $\omega_{\mathbb{BC}}$ with idempotent representation $z = (z(s)) = (z_1(s)e_1 + z_2(s)e_2)_{s\geq 1}$ such that $\|z\|_{(p,q)}^{\mathbb{BC}} \prec \infty$, whenever

$$\|z\|_{(p,q)}^{\mathbb{BC}} = \begin{cases} \left(\sum_{s=1}^{\infty} s^{(q/p)-1}[z^*(s)]^q\right)^{1/q} ; \ 1 < p \leq \infty, 1 \leq q < \infty \\ \mathbb{D} - sup_{s\geq 1} s^{\frac{1}{p}} z^*(s) \ ; \ 1 < p \leq \infty, q = \infty. \end{cases}$$

This set is denoted by $\ell_{(p,q)}^{\mathbb{BC}}$.

Theorem 2.1. $\ell_{(p,q)}^{\mathbb{BC}}$ is a complex linear space under vector summation and scalar product operations defined as follows:

$$+: \ell_{(p,q)}^{\mathbb{BC}} \times \ell_{(p,q)}^{\mathbb{BC}} \to \ell_{(p,q)}^{\mathbb{BC}}, +(z,w) = z + w = (z_1(s) + w_1(s))e_1 + (z_2(s) + w_2(s))e_2,$$

$$\cdot: \mathbb{C} \times \ell_{(p,q)}^{\mathbb{BC}} \to \ell_{(p,q)}^{\mathbb{BC}}, \cdot(t,z) = t \cdot z = tz_1(s)e_1 + tz_2(s)e_2.$$

Proof: If $z = (z(s)) \in \omega_{\mathbb{BC}}$ and $t \in \mathbb{C}$, then we have

$$\|t\cdot z\|_{(p,q)}^{\mathbb{BC}} = \begin{cases} \left(\sum_{r=1}^{\infty} s^{(q/p)-1}[(t\cdot z)^*(s)]^q\right)^{1/q} ; \ 1 \leq p \leq \infty, 1 \leq q < \infty \\ \mathbb{D} - sup_{s\geq 1} s^{1/p}(t\cdot z)^*(s) \ ; \ 1 \leq p \leq \infty, q = \infty. \end{cases}$$

$$= \begin{cases} |t|.\left(\sum_{r=1}^{\infty} s^{(q/p)-1}\,[z^*(s)]^q\right)^{1/q} \;;\; 1 < p \le \infty, 1 \le q < \infty \\ |t|.\,\mathbb{D} - sup_{s\ge1} s^{1/p} z^*(s) \;;\; 1 < p \le \infty, q = \infty. \end{cases}$$
$$= |t|\|z\|_{(p,q)}^{\mathbb{BC}}.$$

Let $1 < \mathrm{p} \le \infty$ and $1 \le \mathrm{q} < \infty$. If $\mathrm{z} = (\mathrm{z}(\mathrm{s})) \in \mathrm{w}_{\mathbb{BC}}$, $\mathrm{w} = (\mathrm{w}(\mathrm{s})) \in \omega_{\mathbb{BC}}$ and $1 < \mathrm{p} < \mathrm{q} < \infty$, then, using Lemma 2.1, we have

$$\|z+w\|_{(p,q)}^{\mathbb{BC}} = \left(\sum_{s=1}^{\infty} s^{\left(\frac{q}{p}\right)-1}\,[(z+w)^*(s)]^q\right)^{\frac{1}{q}}$$
$$\preccurlyeq \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\,[z^*(s/2)+w^*(s/2)]^q\right)^{1/q}$$
$$= \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left[\left(z_1{}^*(s/2)+w_1{}^*(s/2)\right)e_1 + \left(z_2{}^*(s/2)+w_2{}^*(s/2)\right)e_2\right]^q\right)^{1/q}$$
$$= \left(\sum_{s=1}^{\infty} s^{(q/p)-1} 2^q \left[\left(\frac{z_1{}^*(s/2)+w_1{}^*(s/2)}{2}\right)e_1 + \left(\frac{z_2{}^*(s/2)+w_2{}^*(s/2)}{2}\right)e_2\right]^q\right)^{1/q} \preccurlyeq \left(\sum_{s=1}^{\infty} s^{(q/p)-1} 2^{q-1}\left[\left(\left(z_1{}^*(s/2)\right)^q + \left(w_1{}^*(s/2)\right)^q\right)e_1 + \left(\left(z_2{}^*(s/2)\right)^q + \left(w_2{}^*(s/2)\right)^q\right)e_2\right]\right)^{\frac{1}{q}}$$
$$= 2^{1-1/q}\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(\left(\left(z_1^*(s/2)\right)^q + \left(w_1^*(s/2)\right)^q\right)e_1 + \left(\left(z_2^*(s/2)\right)^q + \left(w_2^*(s/2)\right)^q\right)e_2\right)\right)^{1/q}.$$
$$\preccurlyeq 2^{1-1/q}\left(2^{q/p}\sum_{s=1}^{\infty} s^{(q/p)-1}\left(\left(\left(z_1^*(s)\right)^q + \left(w_1^*(s)\right)^q\right)e_1 + \left(\left(z_2^*(s)\right)^q + \left(w_2^*(s)\right)^q\right)e_2\right)\right)^{1/q}$$
$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(\left(\left(z_1^*(s)\right)^q + \left(w_1^*(s)\right)^q\right)e_1 + \left(\left(z_2^*(s)\right)^q + \left(w_2^*(s)\right)^q\right)e_2\right)\right)^{1/q}$$
$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(\left(z_1^*(s)\right)^q + \left(w_1^*(s)\right)^q\right)\right)e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(\left(z_2^*(s)\right)^q + \left(z_2^*(s)\right)^q\right)\right)e_2\right)^{1/q}$$
$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_1^*(s)\right)^q + \sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_1^*(s)\right)^q\right)^{1/q} e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_2^*(s)\right)^q + \sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_2^*(s)\right)^q\right)^{1/q} e_2\right\}$$
$$\preccurlyeq 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_1^*(s)\right)^q\right)^{1/q} + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_1^*(s)\right)^q\right)^{1/q}\right)e_1 + \left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_2^*(s)\right)^q\right)^{1/q} + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_2^*(s)\right)^q\right)^{1/q}\right)e_2\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_1^*(s)\right)^q\right)^{1/q} e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_2^*(s)\right)^q\right)^{1/q} e_2\right) + \left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_1^*(s)\right)^q\right)^{1/q} e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_2^*(s)\right)^q\right)^{1/q} e_2\right)\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_1^*(s)\right)^q\right) e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(z_2^*(s)\right)^q\right) e_2\right)^{1/q} + \left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_1^*(s)\right)^q\right) e_1 + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left(w_2^*(s)\right)^q\right) e_2\right)^{1/q}\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left[\left(z_1^*(s)\right)^q e_1 + \left(z_2^*(s)\right)^q e_2\right]\right)\right)^{1/q} + \left(\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left[\left(w_1^*(s)\right)^q e_1 + \left(w_2^*(s)\right)^q e_2\right]\right)\right)^{1/q}\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left[z_1^*(s)e_1+z_2^*(s)e_2\right]^q\right)^{1/q} + \left(\sum_{s=1}^{\infty} s^{(q/p)-1}\left[w_1^*(s)e_1+w_2^*(s)e_2\right]^q\right)^{1/q}\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left\{\left(\sum_{s=1}^{\infty} s^{\frac{q}{p}-1}\left[z^*(s)\right]^q\right)^{\frac{1}{q}} + \left(\sum_{s=1}^{\infty} s^{\frac{q}{p}-1}\left[w^*(s)\right]^q\right)^{1/q}\right\}$$

$$= 2^{1-\frac{1}{q}+\frac{1}{p}}\left(\|z\|_{(p,q)}^{\mathbb{BC}} + \|w\|_{(p,q)}^{\mathbb{BC}}\right).$$

If $q = \infty$, then, using Theorem 1.3, we have

$$\|z+w\|_{(p,\infty)}^{\mathbb{BC}} = \mathbb{D} - \sup_{n\geq 1}\left(n^{1/p}(z+w)^*(n)\right) \preccurlyeq \mathbb{D} - \sup_{n\geq 1}\left(n^{1/p}\left(z^*(n/2) + w^*(n/2)\right)\right)$$
$$= \sup_{n\geq 1}\left(n^{1/p}\left(z_1^*(n/2) + w_1^*(n/2)\right)\right) e_1 + \sup_{n\geq 1}\left(n^{1/p}\left(z_2^*(n/2) + w_2^*(n/2)\right)\right) e_2$$
$$\preccurlyeq 2^{1/p}\left\{\sup_{m\geq 1}\left(m^{1/p}\left(z_1^*(m) + w_1^*(m)\right)\right) e_1 + \sup_{m\geq 1}\left(m^{1/p}\left(z_2^*(m) + w_2^*(m)\right)\right) e_2\right\}$$
$$= 2^{1/p}\left\{\left(\sup_{m\geq 1} m^{1/p} z_1^*(m)\right) e_1 + \left(\sup_{m\geq 1} m^{1/p} z_2^*(m)\right) e_2 + \left(\sup_{m\geq 1} m^{1/p} w_1^*(m)\right) e_1 + \left(\sup_{m\geq 1} m^{1/p} w_2^*(m)\right) e_2\right\}$$
$$= 2^{1/p}\left\{\mathbb{D} - \sup_{m\geq 1}\left(m^{1/p} z^*(m)\right) + \mathbb{D} - \sup_{m\geq 1}\left(m^{1/p} w^*(m)\right)\right\}$$
$$= 2^{1/p}\left\{\|z\|_{(p,\infty)}^{\mathbb{BC}} + \|w\|_{(p,\infty)}^{\mathbb{BC}}\right\}.$$

Now, let $1 < q < p < \infty$. Since $(q/p) - 1 < 0$ in this case, we can write

$$2^{(q/p)-1} < 1^{(q/p)-1}, 3^{(q/p)-1} < 2^{(q/p)-1}, 4^{(q/p)-1} < 2^{(q/p)-1}, 5^{(q/p)-1} < 3^{(q/p)-1}, 6^{(q/p)-1} < 3^{(q/p)-1} \ldots$$

and so, we have

$$\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_i(n/2)\big)^q = 1^{(q/p)-1}\big(z_i^*(1/2)\big)^q + 2^{(q/p)-1}\big(z_i^*(2/2)\big)^q$$
$$+3^{(q/p)-1}\big(z_i^*(3/2)\big)^q + 4^{(q/p)-1}\big(z_i^*(4/2)\big)^q$$
$$+5^{(q/p)-1}\big(z_i^*(5/2)\big)^q + 6^{(q/p)-1}\big(z_i^*(6/2)\big)^q + \cdots$$
$$< 1^{(q/p)-1}\big(z_i^*(1)\big)^q + 1^{(q/p)-1}\big(z_i^*(1)\big)^q + 2^{(q/p)-1}\big(z_i^*(2)\big)^q$$
$$+2^{(q/p)-1}\big(z_i^*(2)\big)^q + 3^{(q/p)-1}\big(z_i^*(3)\big)^q + 3^{(q/p)-1}\big(z_i^*(3)\big)^q + \cdots$$
$$= 2\{1^{(q/p)-1}\big(z_i^*(1)\big)^q + 2^{(q/p)-1}\big(z_i^*(2)\big)^q + 3^{(q/p)-1}\big(z_i^*(3)\big)^q \cdots\},$$

namely,

$$\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_i^*(n/2)\big)^q < 2\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_i^*(n)\big)^q$$

and similarly

$$\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_i^*(n/2)\big)^q < 2\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_i^*(n)\big)^q$$

for i $= 1,2$. Thus, we obtain

$$\|z+w\|_{(p,q)}^{\mathbb{BC}} = \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[(z+w)^*(n)]^q\right)^{1/q}$$
$$\preccurlyeq \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z^*(n/2)+w^*(n/2)]^q\right)^{1/q}$$
$$= \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big[\big(z_1^*(n/2)+w_1^*(n/2)\big)e_1 + \big(z_2^*(n/2)+w_2^*(n/2)\big)e_2\Big]^q\right)^{1/q}$$
$$= \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big[\big(z_1^*(n/2)+w_1^*(n/2)\big)^q e_1 + \big(z_2^*(n/2)+w_2^*(n/2)\big)^q e_2\Big]\right)^{1/q}$$
$$\preccurlyeq \left(\sum_{n=1}^{\infty} n^{(q/p)-1}2^{q-1}\Big[\Big(\big(z_1^*(n/2)\big)^q + \big(w_1^*(n/2)\big)^q\Big)e_1\right.$$
$$\left. + \Big(\big(z_2^*(n/]2)\big)^q + \big(w_2^*(n/2)\big)^q\Big)e_2\Big]\right)^{1/q}$$
$$= 2^{1-\frac{1}{q}}\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\left(\Big(\big(z_1^*(n/2)\big)^q + \big(w_1^*(n/2)\big)^q\Big)e_1\right.\right.$$
$$\left.\left. + \Big(\big(z_2^*(n/2)\big)^q + \big(w_2^*(n/2)\big)^q\Big)e_2\right)\right)^{\frac{1}{q}}$$
$$\preccurlyeq 2^{1-\frac{1}{q}}\left(2\sum_{n=1}^{\infty} n^{(q/p)-1}\left(\Big(\big(z_1^*(n)\big)^q + \big(w_1^*(n)\big)^q\Big)e_1 + \Big(\big(z_2^*(n)\big)^q + \big(w_2^*(n)\big)^q\Big)e_2\right)\right)^{1/q}$$
$$= 2\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\left(\Big(\big(z_1^*(n)\big)^q + \big(w_1^*(n)\big)^q\Big)e_1 + \Big(\big(z_2^*(n)\big)^q + \big(w_2^*(n)\big)^q\Big)e_2\right)\right)^{1/q}$$
$$= 2\left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big(\big(z_1^*(n)\big)^q + \big(w_1^*(n)\big)^q\Big)\right)e_1\right.$$
$$\left. + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big(\big(z_2^*(n)\big)^q + \big(w_2^*(n)\big)^q\Big)\right)e_2\right)^{1/q}$$

$$= 2\left\{\left(\sum_{n=!}^{\infty} n^{(q/p)-1}\big(z_1^*(n)\big)^q + \sum_{n=!}^{\infty} n^{(q/p)-1}\big(w_1^*(n)\big)^q\right)^{1/q} e_1 + \left(\sum_{n=!}^{\infty} n^{(q/p)-1}\big(z_2^*(n)\big)^q + \sum_{n=!}^{\infty} n^{(q/p)-1}\big(w_2^*(n)\big)^q\right)^{1/q} e_2\right\}$$

$$\preccurlyeq 2\left\{\left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_1^*(n)\big)^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_1^*(n)\big)^q\right)^{1/q}\right) e_1 + \left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_2^*(n)\big)^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_2^*(n)\big)^q\right)^{1/q}\right) e_2\right\}$$

$$= 2\left\{\left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_1^*(n)\big)^q\right)^{1/q} e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_2^*(n)\big)^q\right)^{1/q} e_2\right) + \left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_1^*(n)\big)^q\right)^{1/q} e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_2^*(n)\big)^q\right)^{1/q} e_2\right)\right\}$$

$$= 2\left\{\left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_1^*(n)\big)^q\right) e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z_2^*(n)\big)^q\right) e_2\right)^{\frac{1}{q}} + \left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_1^*(n)\big)^q\right) e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(w_2^*(n)\big)^q\right) e_2\right)^{\frac{1}{q}}\right\}$$

$$= 2\left\{\left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big(\big(z_1^*(n)\big)^q e_1 + \big(z_2^*(n)\big)^q e_2\Big)\right)\right)^{1/q} + \left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\Big(\big(w_1^*(n)\big)^q e_1 + \big(w_2^*(n)\big)^q e_2\Big)\right)\right)^{1/q}\right\}$$

$$= 2\left\{\left(\sum_{n=1}^{\infty} n^{(q/p)-1}(z_1^*(n)e_1 + z_2^*(n)e_2)^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}(w_1^*(n)e_1 + w_2^*(n)e_2)^q\right)^{1/q}\right\}$$

$$= 2\left\{\left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z^*(n)\big)^q\right)^{1/q} + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\big(z^*(n)\big)^q\right)^{1/q}\right\}$$

$$= 2\big(\|z\|_{(p,q)}^{\mathbb{BC}} + \|w\|_{(p,q)}^{\mathbb{BC}}\big).$$

This shows that $\ell_{(p,q)}^{\mathbb{BC}}$ is closed under vector summation and scalar product. Other properties of being a linear space for the set $\ell_{(p,q)}^{\mathbb{BC}}$ can be easily denoted.

Proposition 2.2. The transformation $\|.\|_{(p,q)}^{\mathbb{BC}}$ is a $\mathbb{D}^+$-valued $\mathbb{D}$-quasinorm on the space $\ell_{(p,q)}^{\mathbb{BC}}$.

Proof: Firstly, let $1 \le p \le \infty$ and $1 \le q < \infty$. If $z(s) = z_1(s)e_1 + z_2(s)e_2$ is a bicomplex-valued sequence, then $z^*(s) = z_1^*(s)e_1 + z_2^*(s)e_2$ [6]. Hence, we have

$$\|z\|_{(p,q)}^{\mathbb{BC}} = \left(\sum_{s=1}^{\infty} s^{\frac{q}{p}-1}\,[z^*(s)]^q\right)^{1/q} = 0$$

$$\Leftrightarrow \left(\sum_{r=1}^{\infty} s^{\frac{q}{p}-1}\,[z_1^*(s)]^q\right)^{1/q} e_1 + \left(\sum_{r=1}^{\infty} s^{\frac{q}{p}-1}\,[z_2^*(s)]^q\right)^{1/q} e_2 = 0e_1 + 0e_2$$

$$\Leftrightarrow \left(\sum_{s=1}^{\infty} s^{\frac{q}{p}-1}\,[z_1^*(s)]^q\right)^{1/q} = 0 \; and \; \left(\sum_{s=1}^{\infty} s^{\frac{q}{p}-1}\,[z_2^*(s)]^q\right)^{1/q} = 0$$
$$\Leftrightarrow z_1^*(s) = 0 \; and \; z_2^*(s) = 0, s = 1,2,\ldots \Leftrightarrow D_{z_1}(\lambda) = 0 \; and \; D_{z_1}(\lambda) = 0, \lambda \geq 0$$
$$\Leftrightarrow z_1(n) = 0 \; and \; z_2(n) = 0, n = 1,2,\ldots \Leftrightarrow z = 0.$$

Other two properties are immediately seen from the proof of Proposition 2.1. As a result, $\|.\|_{(p,q)}^{\mathbb{BC}}$ is a quasinorm for $1 \leq p, q \leq \infty$.

Corollary 2.1. If an element in the form $z = z_1 e_1 + z_2 e_2$ with $z_1 = z_2$ in $\ell_{(p,q)}^{\mathbb{BC}}$ is taken into account, then the classical pre-Lorentz space is obtained.

Definition 2.3. Let $z = (z(r))$ be a sequence in $\omega_{\mathbb{BC}}$. If, given any $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2 \in \mathbb{D}^+$ with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $N \in \mathbb{N}$ such that $|z(r) - z(s)|_k \prec \varepsilon$ for all $r, s > \mathbb{N}$, then z is called a $\mathbb{D}$-Cauchy sequence.

Lemma 2.2. A sequence $z = \big(z(r)\big) = (z_1(r)e_1 + z_2(r)e_2)$ is a $\mathbb{D}$-Cauchy sequence in $\omega_{\mathbb{BC}}$ if and only if $\big(z_1(r)\big)$ and $(z_2(r))$ are the Cauchy sequences in $\mathbb{C}(i)$.

Proof: If $z = \big(z(r)\big) = (z_1(r)e_1 + z_2(r)e_2)$ is a $\mathbb{D}$-Cauchy sequence in $\omega_{\mathbb{BC}}$, then, given any $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$\begin{aligned} |z(r) - z(s)|_k &= |(z_1(r)e_1 + z_2(r)e_2) - (z_1(s)e_1 + z_2(s)e_2)|_k \\ &= |\big(z_1(r) - z_1(s)\big)e_1 + \big(z_2(r) - z_2(s)\big)e_2|_k \\ &= |z_1(r) - z_1(s)|e_1 + |z_2(r) - z_2(s)|e_2 \\ &\prec \; \varepsilon = \varepsilon e_1 + \varepsilon e_2 \end{aligned}$$

for all $r, s \geq N$ and hence we have $|z_1(r) - z_1(s)| < \varepsilon$ and $|z_2(r) - z_2(s)| < \varepsilon$ for all $r, s \geq N$, showing that $\big(z_1(r)\big)$ and $\big(z_2(r)\big)$ are Cauchy sequences in $\mathbb{C}(i)$.

Conversely, if $\big(z_1(r)\big)$ and $(z_2(r))$ are two Cauchy sequences in $\mathbb{C}(i)$, then, given any $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2 \succ 0$ with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists a $N \in \mathbb{N}$ such that

$$|z_1(r) - z_1(s)| < \varepsilon_1 \; and \; |z_2(r) - z_2(s)| < \varepsilon_2$$

for all $r, s \geq N$. Thus, we have

$$|z(r) - z(s)|_k = |(z_1(r)e_1 + z_2(r)e_2) - (z_1(s)e_1 + z_2(s)e_2)|_k$$

$$= |z_1(r) - z_1(s)|e_1 + |z_2(r) - z_2(s)|e_2 \prec \; \varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2$$

for all $r, s \geq N$ and this shows that $z = \big(z(r)\big)$ is a $\mathbb{D}$-Cauchy sequence in $w_{\mathbb{BC}}$.

Theorem 2.2. The space $\ell_{(p,q)}^{\mathbb{BC}}$, $1 < p \leq \infty, 1 \leq q \leq \infty$ is complete with respect to the $\mathbb{D}$-quasinorm $\|.\|_{(p,q)}^{\mathbb{BC}}$.

Proof: Let $1 \leq q < \infty$ and $(z(r)) = (z_r)$ be an arbitrary $\mathbb{D}$-Cauchy sequence in $\ell^{\mathbb{BC}}_{(p,q)}$ with $z_r = z_{r,1}e_1 + z_{r,2}e_2$ for all $r \in \mathbb{N}$. Then, given any $\varepsilon > 0$, there is at least one $N \in \mathbb{N}$ such that $\|z_r - z_s\|^{\mathbb{BC}}_{(p,q)} \prec \varepsilon$, whenever $r, s > N$. Thus, we have

$$\begin{aligned}\|z_r - z_s\|^{\mathbb{BC}}_{(p,q)} &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[(z_r - z_s)^*(n)\right]^q\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(\left(z_{r,1}e_1 + z_{r,2}e_2\right) - \left(z_{s,1}e_1 + z_{s,2}e_2\right)\right)^*(n)\right]^q\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(\left(z_{r,1} - z_{s,1}\right)e_1 + \left(z_{r,2}e_1 - z_{s,2}\right)e_2\right)^*(n)\right]^q\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left\{\left[\left(z_{r,1} - z_{s,1}\right)^*(n)\right]^q e_1 + \left[\left(z_{r,2}e_1 - z_{s,2}\right)^*(n)\right]^q e_2\right\}\right)^{1/q}\\ &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(z_{r,1} - z_{s,1}\right)^*(n)\right]^q\right)^{1/q} e_1\\ &\quad + \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(z_{r,2} - z_{s,2}\right)^*(n)\right]^q\right)^{1/q} e_2\\ &\prec \varepsilon = \varepsilon e_1 + \varepsilon e_2\end{aligned}$$

and thus

$$\left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(z_{r,1} - z_{s,1}\right)^*(n)\right]^q\right)^{1/q} < \varepsilon \; and \; \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\left(z_{r,2} - z_{s,2}\right)^*(n)\right]^q\right)^{1/q} < \varepsilon.$$

Therefore, $(z_{r,1})$ and $(z_{r,2})$ are Cauchy sequences according to the pre-norm defined in the well-known classic Lorentz sequence space $\ell_{(p,q)}$. On the other hand, since the space $\ell_{(p,q)}$ is complete according to this pre-norm $\|.\|_{(p,q)}$, the sequences $(z_{r,1})$ and $(z_{r,2})$ converge to two elements with z_1 and z_2, respectively, in $\ell_{(p,q)}$. Then, obviously $(z_r) = z_{r,1}e_1 + z_{r,2}e_2$ converges to $z = z_1e_1 + z_2e_2$ in $\ell^{\mathbb{BC}}_{(p,q)}$. All that remains is to show $z \in \ell^{\mathbb{BC}}_{(p,q)}$. Indeed, since

$$\begin{aligned}\|z\|^{\mathbb{BC}}_{(p,q)} &= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z^*(n)\right]^q\right)^{1/q} = \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z_1^*(n)e_1 + z_2^*(n)e_2\right]^q\right)^{1/q}\\ &\preccurlyeq \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z_1^*(n)\right]^q\right)^{\frac{1}{q}} e_1 + \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z_2^*(n)\right]^q\right)^{\frac{1}{q}} e_2\end{aligned}$$

and also

$$\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z_1^*(n)\right]^q < \infty \; and \; \sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[z_2^*(n)\right]^q < \infty,$$

we have $\|z\|^{\mathbb{BC}}_{(p,q)} \prec \infty$ and thus $z \in \ell^{\mathbb{BC}}_{(p,q)}$, proving that the space $\ell^{\mathbb{BC}}_{(p,q)}$ is complete for $1 \leq q < \infty$.

Now let us examine the case $q = \infty$. Let $(z(r)) = (z_r)$ be an arbitrary $\mathbb{D}$-Cauchy sequence in $\ell^{\mathbb{BC}}_{(p,\infty)}$ with $z_r = z_{r,1}e_1 + z_{r,2}e_2$ for all $r \in \mathbb{N}$. Then

$$\|z_r - z_s\|^{\mathbb{BC}}_{(p,\infty)} = \mathbb{D} - sup_{t \geq 1} t^{\frac{1}{p}}(z_r - z_s)^*(t)$$

$$= \mathbb{D} - sup_{t\geq 1} t^{1/p}\left[\left(z_{r,1} - z_{s,1}\right)e_1 + \left(z_{r,2} - z_{s,2}\right)e_2\right]^*(t)$$
$$= \mathbb{D} - sup_{t\geq 1} t^{1/p}\left[\left(z_{r,1} - z_{s,1}\right)^*(t)e_1 + \left(z_{r,2} - z_{s,2}\right)^*(t)e_2\right]$$
$$= \left\{sup_{t\geq 1} t^{1/p}\left(z_{r,1} - z_{s,1}\right)^*(t)\right\}e_1 + \left\{sup_{t\geq 1} t^{1/p}\left(z_{r,2} - z_{s,2}\right)^*(t)\right\}e_2$$
$$\prec \varepsilon = \varepsilon e_1 + \varepsilon e_2,$$

and so $\mathrm{sup}_{t\geq 1} t^{1/p}\left(z_{r,1} - z_{s,1}\right)^*(t) < \varepsilon$ and $\mathrm{sup}_{t\geq 1} t^{1/p}\left(z_{r,2} - z_{s,2}\right)^*(t) < \varepsilon$. This means that $\left(z_{r,1}\right)$ and $\left(z_{r,2}\right)$ are Cauchy sequences in the classic Lorentz sequence $\ell_{(p,\infty)}$. Nevertheless, since the space $\ell_{(p,\infty)}$ is complete, the sequences $\left(z_{r,1}\right)$ and $\left(z_{r,2}\right)$ converge to two elements of this space with z_1 and z_2, respectively. Then, obviously $(z_r) = \left(z_{r,1}e_1 + z_{r,2}e_2\right)$ converges to $z = z_1e_1 + z_2e_2$ in $\ell^{\mathbb{BC}}_{(p,\infty)}$.

Consequently, any Cauchy sequence is convergent with respect to the $\mathbb{D}$-quasinorm in the space $\ell^{\mathbb{BC}}_{(p,\infty)}$ for both cases $1 \leq q < \infty$ and $q = \infty$. This shows that $\ell^{\mathbb{BC}}_{(p,q)}$ with $1 < p \leq \infty, 1 \leq q \leq \infty$ is complete with respect to the $\mathbb{D}$ -quasinorm.

Definition 2.4. A measure space $(G, \mathcal{G}, \mu)$ is called σ-finite, if there is a sequence $\{I_r\}_{r=1}^{\infty}$ in G such that $G \in \mathcal{G}$ with $G = \cup_{r=1}^{\infty} I_r$ and $\mu(I_r) < \infty$ for all $r \in \mathbb{N}$.

Theorem 2.3. Let $z = \left(z(n)\right)$ and $w = \left(w(n)\right)$ be two sequences in $\omega_{\mathbb{BC}}$, and also $w \in \ell^{\mathbb{BC}}_{(p,q)}, 1 < p \leq \infty, 1 \leq q \leq \infty$, If $|z(r)|_k \preccurlyeq |w(r)|_k$ for all $r \in \mathbb{N}$, then $z \in \ell^{\mathbb{BC}}_{(p,q)}$ and $\|z\|^{\mathbb{BC}}_{(p,q)} \preccurlyeq \|w\|^{\mathbb{BC}}_{(p,q)}$.

Proof: Let $|z(r)|_k \preccurlyeq |w(r)|_k$ for all $r \in \mathbb{N}$. Then, we can write $z^*(r) \preccurlyeq w^*(r)$ for all $r \in \mathbb{N}$ [6]. Firstly, let $1 \leq q < \infty$. Since $(\lambda_1e_1 + \lambda_2e_2)^p = \lambda_1^pe_1 + \lambda_2^pe_2$ for all $p \in \mathbb{R}$ with $\lambda = \lambda_1e_1 + \lambda_2e_2 \in \mathbb{BC}$, we have

$$\|z\|^{\mathbb{BC}}_{(p,q)} = \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[z^*(r)]^q\right)^{1/q} = \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[z_1^*(r)e_1 + z_2^*(r)e_2]^q\right)^{1/q}$$
$$= \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[z_1^*(r)]^q\right)^{1/q} e_1 + \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[z_2^*(r)]^q\right)^{1/q} e_2$$
$$\preccurlyeq \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[w_1^*(r)]^q\right)^{1/q} e_1 + \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[w_2^*(r)]^q\right)^{1/q} e_2$$
$$= \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[w_1^*(r)e_1 + w_2^*(r)e_2]^q\right)^{1/q} = \left(\sum_{r=1}^{\infty} r^{(q/p)-1}[w^*(r)]^q\right)^{1/q}$$
$$= \|w\|^{\mathbb{BC}}_{(p,q)} \prec \infty e_1 + \infty e_2 = \infty.$$

Secondly, let $q = \infty$. Then we have

$$\|z\|^{\mathbb{BC}}_{(p,\infty)} = \mathbb{D} - sup_{r\geq 1} r^{1/p} z^*(r) = \mathbb{D} - sup_{r\geq 1} r^{1/p}[z_1^*(r)e_1 + z_2^*(r)e_2]$$
$$= \left\{\mathbb{D} - sup_{r\geq 1} r^{1/p} z_1^*(r)\right\}e_1 + \left\{\mathbb{D} - sup_{r\geq 1} r^{1/p} z_2^*(r)\right\}e_2$$
$$\preccurlyeq \left\{\mathbb{D} - sup_{r\geq 1} r^{1/p} w_1^*(r)\right\}e_1 + \left\{\mathbb{D} - sup_{r\geq 1} r^{1/p} w_2^*(r)\right\}e_2$$
$$= \mathbb{D} - sup_{r\geq 1} r^{\frac{1}{p}}[w_1^*(r)e_1 + w_2^*(r)e_2] = \mathbb{D} - sup_{r\geq 1} r^{\frac{1}{p}} w^*(r) = \|w\|^{\mathbb{BC}}_{(p,\infty)}.$$

Lemma 2.3.
(a) If $1 < p_1 < p_2 \leq \infty$ and $1 \leq q \leq \infty$, then $\ell_{(p_1,q)} \subseteq \ell_{(p_2,q)}$.

(b) If $1 < p \leq \infty$ and $1 \leq q_1 < q_2 < \infty$, then $\ell_{(p,q_1)} \subseteq \ell_{(p,q_2)}$ [14].

Theorem 2.4.
(a) If $1 < p_1 < p_2 \leq \infty$ and $1 \leq q \leq \infty$, then $\ell^{\mathbb{BC}}_{(p_1,q)} \subseteq \ell^{\mathbb{BC}}_{(p_2,q)}$.
(b) If $1 < p \leq \infty$ and $1 \leq q_1 < q_2 < \infty$, then $\ell^{\mathbb{BC}}_{(p,q_1)} \subseteq \ell^{\mathbb{BC}}_{(p,q_2)}$.

Proof: Note that these properties are known to be true for classical pre-Lorentz spaces $\ell_{(p_1,q)}$. For any $z = (z(n)) = (z_1(n)e_1 + z_2(n)e_2) \in \ell^{\mathbb{BC}}_{(p,q)}$, if $q < \infty$, then we have

$$\|z\|^{\mathbb{BC}}_{(p,q)} = \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z^*(n)]^q\right)^{1/q} = \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z_1^*(n)e_1 + z_2^*(n)e_2]^q\right)^{1/q}$$
$$= \left(\sum_{n=1}^{\infty} n^{(q/p)-1}\left[\left(z_1^*(n)\right)^q e_1 + \left(z_2^*(n)\right)^q e_2\right]\right)^{1/q}$$
$$= \left(\sum_{n=1}^{\infty}\left[n^{(q/p)-1}\left(z_1^*(n)\right)^q e_1 + n^{(q/p)-1}\left(z_2^*(n)\right)^q e_2\right]\right)^{1/q}$$
$$= \left(\left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z_1^*(n)]^q\right)e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z_2^*(n)]^q\right)e_2\right)^{1/q}$$
$$= \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z_1^*(n)]^q\right)^{1/q} e_1 + \left(\sum_{n=1}^{\infty} n^{(q/p)-1}[z_2^*(n)]^q\right)^{1/q} e_2$$
$$= \|z_1\|_{(p,q)}e_1 + \|z_2\|_{(p,q)}e_2.$$

Hence, we can clearly write $z = (z(n)) \in \ell^{\mathbb{BC}}_{(p,q)}$ if and only if $z_1 = \left(z_1(n)\right) \in \ell_{(p,q)}$ and $z_1 = \left(z_1(n)\right) \in \ell_{(p,q)}$. From Lemma 2.3, it can be seen immediately the inclusion (a) and (b).

Example 2.1. Calculate the $\mathbb{D}$-quasinorm of the $\mathbb{D}$-characteristic function $\chi^{\mathbb{D}}_H$. Solution:

Let $q < \infty$. If the equation $\left(\chi^{\mathbb{D}}_H\right)^*(t) = \chi_{[1,\,\mu(H))}(t_1)e_1 + \chi_{[1,\,\mu(H))}(t_2)e_2$ is used [2], we have

$$\left\|\chi^{\mathbb{D}}_H\right\|^{\mathbb{BC}}_{(p,q)} = \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}[(\chi^{\mathbb{D}}_H)^*(n)]^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}[(\chi^{\mathbb{D}}_H)^*(ne_1 + ne_2)]^q\right)^{\frac{1}{q}}$$
$$= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\chi_{[1,\,\mu(H))}(n)e_1 + \chi_{[1,\,\mu(H))}(n)e_2\right]^q\right)^{\frac{1}{q}}$$
$$= \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\chi_{[1,\,\mu(H))}(n)\right]^q\right)^{\frac{1}{q}} e_1 + \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1}\left[\chi_{[1,\,\mu(H))}(n)\right]^q\right)^{\frac{1}{q}} e_2$$
$$= \left(\sum_{n=1}^{\mu(H)} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} e_1 + \left(\sum_{n=1}^{\mu(H)} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}} e_2 = \left(\sum_{n=1}^{\mu(H)} n^{\frac{q}{p}-1}\right)^{\frac{1}{q}}.$$

Again, if $q = \infty$, then we have

$$\left\|\chi_H^{\mathbb{D}}\right\|_{(p,\infty)}^{\mathbb{BC}} = \mathbb{D} - \sup_{n\geq 1}\left(n^{\frac{1}{p}}(\chi_H^{\mathbb{D}})^*(n)\right)$$
$$= \mathbb{D} - \sup_{n\geq 1}\left(\left(n^{\frac{1}{p}}\chi_{[1,\,\mu(H))}(n)\right)e_1 + \left(n^{\frac{1}{p}}\chi_{[1,\,\mu(H))}(n)\right)e_2\right)$$
$$= \sup_{n\geq 1}\left(n^{\frac{1}{p}}\chi_{[1,\,\mu(H))}(n)\right)e_1 + \sup_{n\geq 1}\left(n^{\frac{1}{p}}\chi_{[1,\,\mu(H))}(n)\right)e_2$$
$$= \mu(H)^{\frac{1}{p}}e_1 + \mu(H)^{\frac{1}{p}}e_2 = \mu(H)^{\frac{1}{p}}.$$

As a result, the $\mathbb{D}$-quasi-norm of the $\mathbb{D}$-characteristic function of the set H can be written as

$$\left\|\chi_H^{\mathbb{D}}\right\|_{(p,q)}^{\mathbb{BC}} = \begin{cases} \left(\sum_{n=1}^{\mu(H)} n^{(q/p)-1}\right)^{1/q}, 1 < p \leq \infty \; and \; 1 < q < \infty \\ \mu(H)^{\frac{1}{p}}, 1 < p \leq \infty \wedge q = \infty. \end{cases}.$$

3. CONCLUSION

In this study, the foundations of pre-Lorentz sequence spaces with bicomplex terms were laid. Thus, the authors paved the way for the study of Lorentz sequence spaces containing bicomplex terms and continue their work in this direction. For example, the product operators between these spaces will be the subject of their next studies.

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