ORIGINAL PAPER

A FIXED POINT APPROACH TO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS: CONVERGENCE AND DATA DEPENDENCE

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Manuscript received: 01.06.2025; Accepted paper: 17.12.2025;
Published online: 30.12.2025.

Abstract. *Fractional calculus, which involves derivatives and integrals of non-integer order, is widely used in modeling real-world problems in science and engineering. This study focuses on solving fractional integro-differential equations using Picard's three-step iteration algorithm. We apply Picard's three-step iteration algorithm to approximate solutions and show that it converges strongly. The method is shown to be not only efficient in approximating solutions but also robust with respect to perturbations in the initial data and operators, demonstrating its data dependence. A numerical example is provided to illustrate the theoretical results and to confirm the applicability and reliability of the proposed approach.*

Keywords: *Data dependency; fractional order differential equation; fixed point theory; Picard's three step iteration.*

Mathematics Subject Classification (2020): *26A33; 34A12; 47H10.*

1. INTRODUCTION

Fractional calculus is a mathematical field concerned with derivatives and integrals of arbitrary real or complex order. The increasing complexity of real-world models in science and engineering has led to a growing interest in fractional differential equations due to their ability to describe memory and hereditary properties. Differential equations involving non-integer derivatives have proven to be essential tools for accurately modeling a broad range of physical and engineering processes. Beyond its theoretical significance, fractional calculus has found practical applications across various disciplines, including physics, biology, engineering, and finance [1-11].

One of the most intriguing aspects of fractional calculus lies in its flexibility through different fractional operators. This flexibility enables the selection of operators that best capture the dynamics of specific real-world phenomena. The ongoing need to enhance model precision has led to the development of new types of fractional operators, thereby significantly improving the modeling capabilities of fractional differential equations [12-15].

Parallel to the growing importance of fractional calculus and related types of equations, studies involving various fractional operators and solution techniques are steadily increasing in the scientific literature. In particular, the analytical and numerical investigation of time-fractional equations such as Fornberg-Whitham, Newell-Whitehead-Segel, Cahn-Hilliard, and Jaulent-Miodek using methods like conformable derivatives, Caputo-Fabrizio

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https://doi.org/10.46939/J.Sci.Arts-25.4-a13

fractional derivatives, and natural transform decomposition demonstrates the dynamism of the field. These studies emphasize the modeling capability of fractional equations and the effectiveness of new solution algorithms developed for these equations [16-20].

Simultaneously, fixed point theory has emerged as an indispensable framework in nonlinear analysis, particularly for establishing the existence and uniqueness of solutions to differential, integral, and partial differential equations. Fixed point methods are especially valuable when dealing with fractional differential equations in initial and boundary value problems, where finding exact analytical solutions is often challenging. Consequently, there has been a surge in research efforts toward developing numerical and approximation methods for solving such equations [21-27].

Alongside the advancement of fixed point theory, a diverse range of contraction-type mappings has been introduced, including Lipschitzian mappings, classical contraction mappings, nonexpansive mappings, pseudo-contractions, semi-contractions, and weak contractions [28-32]. Numerous iteration methods have also been formulated to approximate fixed points, such as Mann iteration [33], Krasnosel'skii iteration [34], Kirk iteration [35], Ishikawa iteration [36], Noor iteration [37], S-iteration [38], three-step iteration [39], and the more recent Picard's three-step iteration [40]. Moreover, hybrid iteration methods that blend features of multiple classical methods have shown significant efficiency improvements. Examples include the Kirk-Noor, Kirk-Ishikawa, Kirk-Mann, Picard-Mann, Mann-Picard, and Kirk-MP iteration methods [41-44].

An additional aspect closely related to iterative processes is the concept of approximation operators. When iterative sequences are constructed using approximation operators instead of the original mappings, discrepancies between their fixed points may occur. The study of data dependence specifically addresses the nature and quantification of such discrepancies, ensuring that approximate solutions remain robust under small perturbations [45-54].

Fractional differential equations have become increasingly important in modeling systems, particularly in physics, biology, and engineering. Despite their broad applicability, solving such equations—especially in integro-differential form—poses significant challenges, as classical analytical and numerical techniques often fail to ensure stability or convergence. In response to these difficulties, iterative methods based on fixed point theory provide a flexible and effective framework. Among these, Picard's three-step iteration has shown improved convergence behavior and computational efficiency. Moreover, analyzing data dependence is crucial for assessing the robustness of solutions against small perturbations in the initial data or operators.

Although several iterative approaches have been proposed for solving fractional integro-differential equations, the convergence behavior and stability with respect to data perturbations have not been sufficiently clarified for multi-step Picard-type schemes. Addressing this gap is essential, since real-world models (particularly those involving memory-dependent processes) require numerical methods that are both fast-converging and robust. Motivated by these considerations, this study focuses on establishing strong convergence and data dependence results for the Picard's three-step iteration algorithm and demonstrating its practical reliability through a numerical example.

The general structure of the study is as follows: In Section 2, the basic concepts that are important for our analysis are presented. In Section 3, the following initial value problem for the integro-differential equation, where A is a constant and $f \in (C[0, A] \times \mathbb{R}, \mathbb{R})$, will be considered.

$$\begin{cases} \dfrac{d\varphi(t)}{dt} + I_{a^+}^{\alpha}\varphi(s) = f\big(s, \varphi(s)\big), \\ 0 < s < A \ , \ 0 < \alpha < 1, \\ \varphi(0) = \rho. \end{cases} \tag{1}$$

First, we approach the solution of initial value problem (1) using Picard's three-step algorithm introduced by Ali et al. [40]. We will also obtain the data dependence of the equations considered. A numerical example is provided to illustrate the obtained findings. Finally, our study concludes with a discussion of our findings and potential future research directions.

2. MATERIALS AND METHODS

In this section, we introduce essential definitions and preliminary results that form the foundation of our study.

Definition 2.1. ([1]) Let $\varphi(s) \in C([a,b])$, $\alpha \in (-\infty, \infty)$ and f be an integrable function on $[0,T]$. The Riemann–Liouville fractional integral of order α is defined as

$$I_{a^+}^{\alpha}\varphi(s) := \frac{1}{\Gamma(\alpha)} \int_a^s \frac{\varphi(r)}{(s-r)^{1-\alpha}} dr.$$

Similarly, for $\alpha \in (0,1)$ the Riemann-Liouville fractional derivative of order α can be expressed as

$$D_{a^+}^{\alpha}\varphi(s) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^s \frac{\varphi(r)}{(s-r)^{\alpha}} dr \ ,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s} ds \ , \quad (\alpha > 0).$$

Definition 2.2. ([55]) Denote by $(C[a,b], \mathbb{R})$ the Banach space of all continuous real-valued functions on $[a,b]$ equipped with the supremum norm

$$\|f\|_{\infty} = \sup_{s \in [a,b]} |f(s)| \, .$$

Definition 2.3. ([56]) Let (X,d) be a metric space and $T: X \to X$ be a mapping. T is called a Lipschitzian mapping, if there is a $L > 0$ real number such that $d(Tx,Ty) \leq Ld(x,y)$ for all $x, y \in X$. If in the definition, the constant L satisfies $L \in (0,1)$, then T is called a contraction mapping and L is referred to as the contraction ratio.

Theorem (Banach fixed point theorem) 2.1. ([28]) Let (X,d) be a complete metric space and $T: X \to X$ be a contraction mapping. Then, T has a unique fixed point $x^* \in X$. Furthermore, for any initial point $x_0 \in X$, the iterative sequence defined by $x_n = Tx_{n-1}$; $n = 0,1,2,...$ converges to x^*.

The reason we used Picard's three-step iteration algorithm in our study is that this algorithm has been proven to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, Picard-S, SP, S, CR, M*. Now let's express the definition of this algorithm [40].

Definition 2.5. Let X be a metric space or Banach space, $T: X \to X$ be a mapping. Starting from an initial guess $x_0 \in X$ and for $n \in \mathbb{Z}^+$ the Picard's three-step iteration algorithm is defined by

$$\begin{cases} \varphi_{n+1} = T\psi_n \\ \psi_n = T\phi_n \\ \phi_n = T\varphi_n \end{cases} \tag{2}$$

Theorem 2.2. ([23]) Let f be continuous function which is on the rectangle $B = \{(s,x): s \in [0,A]\} \subseteq \mathbb{R}^2$, and $|f(s,x)| \leq M$ for all $(s,x) \in B$. Also suppose that f satisfies a Lipschitz condition on B with respect to its second argument, i.e, there is a constant L such that for arbitrary $(s,x),(s,y) \in B$

$$|f(s,x) - f(s,y)| \leq L|x - y| \tag{3}$$

is valid. Moreover, let

$$h(\alpha, A, L) = \frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA$$

and suppose that

$$h(\alpha, A, L) < 1 \tag{4}$$

Then, the associated fractional integro-differential equation in (1) has a unique solution.

Definition 2.6. ([57]) An operator $\tilde{T}: X \to X$ is called an approximation operator of $T: X \to X$ if there exists a constant $\varepsilon > 0$ such that

$$\left\| T(x) - \tilde{T}(x) \right\| \leq \varepsilon, \ \ \forall x \in X.$$

Lemma 2.1. ([57]) Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative real sequence. Assume that there exists an integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the following inequality holds:

$$a_{n+1} \leq (1 - \mu_n)a_n + \mu_n \gamma_n,$$

where $\mu_n \in (0,1)$ satisfies $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$. Then, the sequence satisfies

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n \,.$$

3. RESULTS AND DISCUSSION

3.1. RESULTS

In this section, we present new findings regarding the existence, uniqueness, and data dependence of solutions for a class of fractional integro-differential equation (1) using the Picard's three-step iterative algorithm.

Theorem 3.1.1. Let $T: (C[0,A], \|\cdot\|_\infty) \to (C[0,A], \|\cdot\|_\infty)$ be a mapping. Then, the integro-differential equation defined in (1) admits a unique solution $x^* \in C[0,A]$, and the sequence $\{\varphi_n\}_{n=0}^{\infty}$ generated by the iterative algorithm described in (2) converges strongly to this unique solution.

Proof: By integrating both sides of the integro-differential equation (1), we obtain the following corresponding integral equation

$$\varphi(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} \varphi(r)\,dr\,dq + \int_0^s f\big(q, x(q)\big)dq + \rho. \tag{5}$$

It can be rewritten in the equivalent integral form given in (5), which is in the form $\varphi = T\varphi$, where $T: C[0,A] \longrightarrow C[0,A]$ is an operator defined by

$$T\varphi(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} \varphi(r)\,dr\,dq + \int_0^s f\big(q, \varphi(q)\big)dq + \rho, \tag{6}$$

where f is a continuous function on the rectangle B. Consider the sequence $\{\varphi_n\}_{n=0}^{\infty}$ generated by the iteration algorithm defined in (2) and constructed using the mapping $T: (C[0,A], \|\cdot\|_\infty) \to (C[0,A], \|\cdot\|_\infty)$. It will be shown that for $n \to \infty$ is $\varphi_n \to x^*$. Using equation (5) and the conditions of Theorem 2.2, we obtain the following inequality.

$$|\varphi_{n+1}(t) - x^*(t)| = |T\psi_n(t) - Tx^*(t)|$$

$$= \left| \begin{array}{l} -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} \psi_n(r)\,dr\,dq + \int_0^s f\big(q, \psi_n(q)\big)dq + \rho \\ +\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} x^*(r)\,dr\,dq - \int_0^s f\big(q, x^*(q)\big)dq - \rho \end{array} \right|$$

$$= \left| \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1}\left(\psi_n(r) - x^*(r)\right) dr dq \\ + \int_0^s \left(f\left(q, \psi_n(q)\right) - f\left(q, x^*(q)\right) \right) dq \end{array} \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q |q-r|^{\alpha-1} |\psi_n(r) - x^*(r)| dr dq$$

$$+ \int_0^s \left| f\left(q, \psi_n(q)\right) - f\left(q, x^*(q)\right) \right| dq$$

$$\leq \frac{1}{\Gamma(\alpha)} \|\psi_n - x^*\|_\infty \int_0^s \int_0^q |q-r|^{\alpha-1} dr dq + L \int_0^s |\psi_n(q) - x^*(q)| dq$$

$$\leq \left(\frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA \right) \|\psi_n - x^*\|_\infty.$$

Since $h(\alpha, A, L) = \frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA$, we have

$$\|\varphi_{n+1} - x^*\|_\infty \leq h \|\psi_n - x^*\|_\infty. \tag{7}$$

After performing the necessary calculations, the following inequalities are obtained.

$$|\psi_n(s) - x^*(s)| = |T\phi_n(s) - Tx^*(s)|$$

$$= \left| \begin{array}{l} -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} \phi_n(r) dr dq + \int_0^s f\left(q, \phi_n(q)\right) dq + \rho \\ + \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} x^*(r) dr dq - \int_0^s f\left(q, x^*(q)\right) dq - \rho \end{array} \right|$$

$$\leq \left| -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} \left(\phi_n(r) - x^*(r)\right) dr dq \right|$$

$$+ \left| \int_0^s f\left(q, \phi_n(q)\right) - f\left(q, x^*(q)\right) dq \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q |q-r|^{\alpha-1} |\phi_n(r) - x^*(r)| dr dq$$

$$+ \int_0^s \left| f\left(q, \phi_n(q)\right) - f\left(q, x^*(q)\right) \right| dq$$

$$\leq \frac{1}{\Gamma(\alpha)} \|\phi_n - x^*\|_\infty \int_0^s \int_0^q |q-r|^{\alpha-1} dr dq$$

$$+\int_0^s \left\|f\big(q,\phi_n(q)\big)-f\big(q,x^*(q)\big)\right\|_\infty dq$$
$$\leq \left(\frac{A^{\alpha+1}}{\Gamma(\alpha+2)}+LA\right)\|\phi_n-x^*\|_\infty$$
$$\|\psi_n-x^*\|_\infty \leq h\|\phi_n-x^*\|_\infty. \tag{8}$$

Similarly,

$$|\phi_n(s)-x^*(s)| = |T\varphi_n(s)-Tx^*(s)|$$
$$=\Bigg|-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}\varphi_n(r)drdq+\int_0^s f\big(q,\varphi_n(q)\big)dq+\rho$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}x^*(r)drdq-\int_0^s f\big(q,x^*(q)\big)dq-\rho\Bigg|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}\|\varphi_n-x^*\|_\infty drdq$$
$$+\int_0^s \left\|f\big(q,\varphi_n(q)\big)-f\big(q,x^*(q)\big)\right\|_\infty dq$$
$$\leq \left(\frac{A^{\alpha+1}}{\Gamma(\alpha+2)}+LA\right)\|\varphi_n-x^*\|_\infty,$$
$$\|\phi_n-x^*\|_\infty \leq h\|\varphi_n-x^*\|_\infty \tag{9}$$

is obtained. By substituting inequalities (9) and (8) into inequality (7), the following result is obtained.

$$\begin{gathered}\|\varphi_{n+1}-x^*\|_\infty \leq h^2\|\varphi_n-x^*\|_\infty\\ \|\varphi_n-x^*\|_\infty \leq h^2\|\varphi_{n-1}-x^*\|_\infty\\ \vdots\\ \|\varphi_1-x^*\|_\infty \leq h^2\|\varphi_0-x^*\|_\infty\end{gathered}$$

Applying induction to the last inequality, we obtain the following result.

$$\|\varphi_{n+1}-x^*\|_\infty \leq h^{2(n+1)}\|\varphi_0-x^*\|_\infty. \tag{10}$$

Since $h<1$ if the limit is taken as $n\to\infty$ in the inequality (10), we obtain

$$\lim_{n\to\infty}\|\varphi_n-x^*\|=0.$$

Hence, this contraction-type argument ensures that the iterative sequence generated by Picard's three-step method remains bounded and progressively closer to the fixed point.

Next, we investigate the data dependence of the solution to the integro-differential equation presented in initial value problem (1) by applying the iteration algorithm described in (2). Accordingly, we consider the following initial value problem associated with a second integro-differential equation

$$\begin{cases} \dfrac{du(s)}{ds} + I_{a^+}^{\alpha} u(s) = g\big(s, u(s)\big), \\ 0 < s < A \ , \ \ 0 < \alpha < 1, \\ u(0) = \tau. \end{cases} \tag{11}$$

By integrating both sides of integro-differential equation (11), we obtain integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} u(r) dr dq + \int_0^s g\big(q, u(q)\big) dq + \tau. \tag{12}$$

The equivalent integral form (12) can be expressed as $u = Su$, where $S: (C[0,A], \|\cdot\|_\infty) \to (C[0,A], \|\cdot\|_\infty)$ is a mapping defined by

$$Su(s) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-r)^{\alpha-1} u(r) dr dq + \int_0^s g\big(q, u(q)\big) dq + \tau, \tag{13}$$

where g is continuous function on the rectangle B.

If the iteration algorithm given in (2) is reformulated using the mapping T(6) and S(13), respectively, the following iteration algorithms can be obtained.

$$\begin{cases} \varphi_{n+1}(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} \psi_n(r) dr dq + \int_0^s f\big(q, \psi_n(q)\big) dq + \rho \\ \psi_n(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} \phi_n(r) dr dq + \int_0^s f\big(q, \phi_n(q)\big) dq + \rho \\ \phi_n(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} \varphi_n(r) dr dq + \int_0^s f\big(q, \varphi_n(q)\big) dq + \rho \end{cases} \tag{14}$$

$$\begin{cases} u_{n+1}(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} v_n(r) dr dq + \int_0^s g\big(q, v_n(q)\big) dq + \tau \\ v_n(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} w_n(r) dr dq + \int_0^s g\big(q, w_n(q)\big) dq + \tau \\ w_n(s) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^s \int_0^q (q-r)^{\alpha-1} u_n(r) dr dq + \int_0^s g\big(q, u_n(q)\big) dq + \tau \end{cases} \tag{15}$$

Theorem 3.1.2. Consider the sequence $\{\varphi_n\}_{n=0}^{\infty}$ obtained from (14) and the sequence $\{u_n\}_{n=0}^{\infty}$ obtained from (15) for each $n \in \mathbb{N}$. Let the solutions of the integral equations (5) and (12) be x^* and u^*, respectively, with the conditions of Theorem 3.1.1.

i) Let the constants ε_1 and ε_2 exist such that $\left\|f(q,\varphi(q)) - g(q,u(q))\right\|_\infty \leq \varepsilon_1$ for each $(s,r) \in [0,A]$ and $\|\rho-\tau\|_\infty \leq \varepsilon_2$.
ii) Let $K = \frac{A^{\alpha+1}}{\Gamma(\alpha+2)} < 1$.

If $\varphi_n \to x^*$ and $u_n \to u^*$ as $n \to \infty$, then the inequality

$$\|x^* - u^*\| \leq \frac{3(\varepsilon_1 A + \varepsilon_2)}{1-K}$$

is valid.

Proof: Under the hypotheses of Theorem 3.1.2, the following inequalities, namely (16), (17), and (18), are obtained.

$$\|\varphi_{n+1} - u_{n+1}\|_\infty = \|T\psi_n - Sv_n\|_\infty$$

$$= \left\| \begin{matrix} -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}\psi_n(r)drdq + \int_0^s f(q,\psi_n(q))dq + \rho \\ +\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}v_n(r)drdq - \int_0^s g(q,v_n(q))dq - \tau \end{matrix} \right\|_\infty$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}\|\psi_n - v_n\|_\infty drdq$$

$$+\int_0^s \left\|f(q,\psi_n(q)) - g(q,v_n(q))\right\|_\infty dq + \|\rho-\tau\|_\infty$$

$$\leq \frac{1}{\Gamma(\alpha)}\|\psi_n - v_n\|_\infty \int_0^s\int_0^q (q-r)^{\alpha-1}drdq + s\varepsilon_1 + \varepsilon_2$$

$$\leq \left(\frac{A^{\alpha+1}}{\Gamma(\alpha+2)}\right)\|\psi_n - v_n\|_\infty + A\varepsilon_1 + \varepsilon_2$$

$$\|\varphi_{n+1} - u_{n+1}\|_\infty \leq K\|\psi_n - v_n\|_\infty + A\varepsilon_1 + \varepsilon_2 \tag{16}$$

Similarly, inequalities (17) and (18) are obtained.

$$\|\psi_n - v_n\|_\infty = \|T\phi_n - Sw_n\|_\infty$$

$$= \left\| \begin{matrix} -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}\phi_n(r)drdq + \int_0^s f(q,\phi_n(q))dq + \rho \\ +\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-r)^{\alpha-1}w_n(r)drdq - \int_0^s g(q,w_n(q))dq - \tau \end{matrix} \right\|_\infty$$

$$\leq \frac{1}{\Gamma(\alpha)}\|\phi_n - w_n\|_\infty \int_0^s \int_0^q (q-r)^{\alpha-1} dr dq$$

$$+\left\|f\big(q,\phi_n(q)\big) - g\big(q,w_n(q)\big)\right\|_\infty \int_0^s dq + \|\rho - \tau\|_\infty$$

$$\leq K\|\phi_n - w_n\|_\infty + \varepsilon_1 A + \varepsilon_2$$

$$\|\psi_{n+1} - v_{n+1}\|_\infty \leq K\|\phi_n - w_n\|_\infty + \varepsilon_1 A + \varepsilon_2 \tag{17}$$

$$\|\phi_n - w_n\|_\infty = \|T\varphi_n - Su_n\|_\infty$$

$$= \left\| \begin{array}{l} -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^q (q-r)^{\alpha-1}\varphi_n(r) dr dq + \int_0^s f\big(q,\varphi_n(q)\big) dq + \rho \\ +\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^q (q-r)^{\alpha-1}u_n(r) dr dq - \int_0^s g\big(q,u_n(q)\big) dq - \tau \end{array} \right\|_\infty$$

$$\leq \frac{1}{\Gamma(\alpha)}\|\varphi_n - u_n\|_\infty \int_0^s \int_0^q (q-r)^{\alpha-1} dr dq$$

$$+\left\|f\big(q,\varphi_n(q)\big) - g\big(q,u_n(q)\big)\right\|_\infty \int_0^s dq + \|\rho - \tau\|_\infty$$

$$\leq K\|\varphi_n - u_n\|_\infty + \varepsilon_1 A + \varepsilon_2$$

$$\|\phi_n - w_n\|_\infty \leq K\|\varphi_n - u_n\|_\infty + \varepsilon_1 A + \varepsilon_2 \tag{18}$$

Thus, if the inequality (18) is written in inequality (17) and $K \leq 1$ is used, the following inequality is found.

$$\|\psi_n - v_n\|_\infty \leq \|\varphi_n - u_n\|_\infty + 2\varepsilon_1 A + 2\varepsilon_2 \tag{19}$$

If inequality (19) is substituted into inequality (16), inequality (20) is found.

$$\begin{aligned} \|\varphi_{n+1} - u_{n+1}\|_\infty &\leq K[\|\varphi_n - u_n\|_\infty + 2\varepsilon_1 A + 2\varepsilon_2] + A\varepsilon_1 + \varepsilon_2 \\ &\leq K\|\varphi_n - u_n\|_\infty + K2\varepsilon_1 A + K2\varepsilon_2 + A\varepsilon_1 + \varepsilon_2 \\ &\leq K\|\varphi_n - u_n\|_\infty + 3\varepsilon_1 A + 3\varepsilon_2 \end{aligned}$$

$$\|\varphi_{n+1} - u_{n+1}\|_\infty \leq [1-(1-K)]\|\varphi_n - u_n\|_\infty + (1-K)\frac{3\varepsilon_1 A + 3\varepsilon_2}{1-K} \tag{20}$$

From the last inequality, we get for each $n \in \mathbb{N}$.

$$\begin{aligned} a_n &= \|\varphi_n - u_n\|_\infty, \\ \mu_n &= (1-K) \in (0,1), \\ \gamma_n &= \frac{3(\varepsilon_1 A + \varepsilon_2)}{1-K} \geq 0. \end{aligned}$$

Thus, the inequality stated in (20) fulfills the requirements of Lemma 2.1. Hence,

$$0 \leq \limsup_{n\to\infty}\|\varphi_n - u_n\|_\infty \leq \limsup_{n\to\infty}\gamma_n = \limsup_{n\to\infty}\frac{3(\varepsilon_1 A + \varepsilon_2)}{1-K}$$

is obtained. Since $\varphi_n \to x^*$ and $u_n \to u^*$as $n \to \infty$, we found

$$\|x^* - u^*\|_\infty \leq \frac{3(\varepsilon_1 A + \varepsilon_2)}{1-K}.$$

Example 3.1.1. First, let us consider the following initial value problems

$$\begin{cases} \dfrac{d^2\varphi(s)}{ds^2} + I_{0^+}^{\frac{1}{2}}\varphi(s) = \dfrac{e^{-2s}+s}{5} + \dfrac{1}{3}\varphi(s) \\ 0 < s < 1, \\ \varphi(0) = 0. \end{cases} \tag{21}$$

and

$$\begin{cases} \dfrac{d^2\varphi(s)}{ds^2} + I_{0^+}^{\frac{1}{2}}\varphi(s) = \dfrac{e^{-s}+2s}{5} + \dfrac{1}{3}\varphi(s) \\ 0 < s < 1, \\ \varphi(0) = 0. \end{cases} \tag{22}$$

Iteration Steps and Existence of Solutions: To illustrate how the Picard three-step iteration algorithm is applied, consider the initial value problem (21). We denote the initial guess as $\varphi_0(s) = 0$ for all $\in [0,1]$. The operator T for this problem is defined as:

$$T: (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty),$$

$$T\varphi(s) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^s \int_0^q (q-r)^{\alpha-1}\varphi(r)\,dr\,dq + \int_0^s \left(\frac{e^{-2s}+s}{5} + \frac{1}{3}\varphi(t)\right)dq$$

Since, $L = \frac{1}{3}, \alpha = \frac{1}{2}$, $A = 1$

$$\begin{aligned} h(\alpha, A, L) &= \frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA \\ &= \frac{1^{\frac{1}{2}+1}}{\Gamma\left(\frac{1}{2}+2\right)} + \frac{1}{3}.\frac{1}{2} \\ &= \frac{1}{\Gamma\left(\frac{5}{2}\right)} + \frac{1}{6} \\ &= \frac{1}{1{,}329} + \frac{1}{6} \cong 0{,}918 < 1. \end{aligned}$$

Thus, from Theorem 2.2, the solution to the initial value problem (21) exists and is unique. Similarly, the initial value problem (22) can be expressed with the operator as follows:

$$S: (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty),$$

$$S\varphi(s) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^s \int_0^q (q-r)^{\alpha-1}\varphi(r)drdq + \int_0^s \left(\frac{e^{-s}+2s}{5} + \frac{1}{3}\varphi(s)\right)dq$$

In this case, $L = \frac{1}{3}, \alpha = \frac{1}{2}, A = 1$ and $h(\alpha, A, L) < 1$. Therefore, from Theorem 2.2, the initial value problem (22) has a solution and is unique.

Data Dependency:

$$\|f(s, \varphi_n(s)) - g(s, \varphi_n(s))\|_\infty = \left|\left(\frac{e^{-2s}+s}{5} + \frac{1}{3}\varphi(t)\right) - \left(\frac{e^{-s}+2s}{5} + \frac{1}{3}\varphi(t)\right)\right|$$

$$= \frac{1}{5}|e^{-2s} - e^{-s} - s| \leq 0{,}246 = \varepsilon_1$$

$$K = \frac{A^{\alpha+1}}{\Gamma\left(\frac{1}{2}+2\right)} = \frac{1}{1{,}329} = 0{,}752 < 1$$

$$\|\rho - \tau\|_\infty = 0 \leq \varepsilon_2 .$$

Thus, we obtain

$$\|x^* - u^*\|_\infty \leq \frac{3(\varepsilon_1 A + \varepsilon_2)}{1-K} \leq \frac{3.(0{,}246)}{1-0{,}752} \cong 2{,}97 .$$

To better demonstrate the practical performance of the Picard's three-step iteration method and to visualize the theoretical findings obtained above, we now present the numerical behavior of the iterative sequence generated for problems (21) and (22). In particular, the convergence pattern of the approximations and the decay of the iterative error are illustrated through graphical representations. These plots provide further insight into the efficiency, stability, and rapid convergence rate of the proposed algorithm.

The convergence pattern obtained from the iterative scheme is illustrated in Figure 1. Furthermore, the decay of the iterative error in logarithmic scale is presented in Figure 2, demonstrating the rapid convergence behavior of the method.

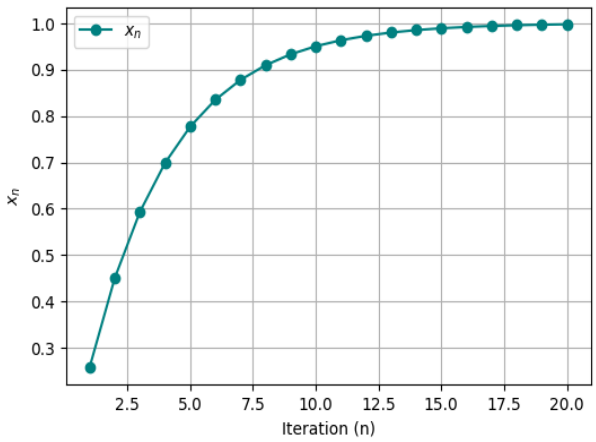

Figure 1. Convergence behavior of the iterative sequence $\{x_n\}$.

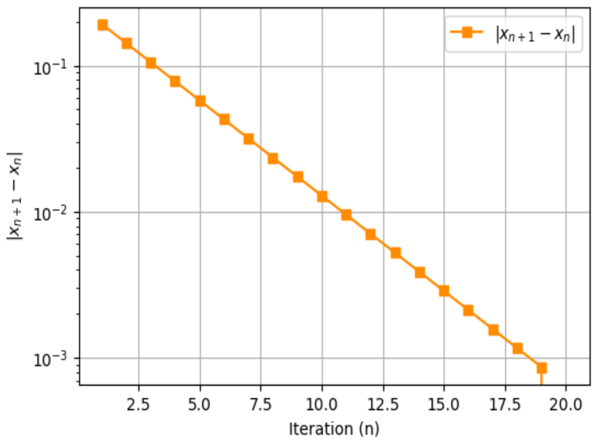

Figure 2. Error decay per iteration in logarithmic scale.

3.2. DISCUSSION

The theoretical analysis and numerical results presented in this study affirm the effectiveness of the Picard's three-step iteration method in solving fractional integro-differential equations. The main findings reveal that the method guarantees strong convergence to the unique solution under appropriate contractive conditions. Compared to classical iterative schemes, the three-step Picard method demonstrates superior convergence rates, aligning with previous results in the literature while offering an advantage in computational efficiency.

One of the central contributions of this study is the in-depth exploration of data dependence. The results confirm that the iterative approximations are not only stable but also robust against perturbations in the operator and initial conditions. This robustness is particularly important in real-world applications, where exact data may be unavailable or subject to measurement errors. The theoretical data dependence results (supported by rigorous inequalities) demonstrate that small deviations in input data result in proportionally small changes in the output, thereby ensuring the reliability of the method.

The numerical example provided illustrates how the theoretical framework applies in practice. Both the existence and uniqueness of solutions were verified via operator bounds and Lipschitz-type conditions. Furthermore, the estimation of the error bounds between two iterative sequences under data perturbation reinforces the practical relevance of the data dependence theory. The bounding inequalities and their eventual vanishing as the iteration count increases offer a quantitative measure of stability.

These results extend the applicability of fixed point methods (especially Picard-type schemes) to more general classes of fractional differential problems, particularly those involving integral components. The flexibility of the approach also suggests potential for adaptation to broader settings, such as equations with nonlinear kernels, memory-dependent operators, or multi-term fractional derivatives.

In summary, the discussion highlights the methodological strengths of the Picard's three-step iteration, especially its convergence behavior and its resilience to data changes. These features position it as a powerful and practical tool in the numerical analysis of fractional systems, with promising applications in physics, engineering, and biology where such equations frequently arise.

4. CONCLUSIONS

In this study, we have examined the approximate solutions of fractional integro-differential equations using Picard's three-step iteration algorithm, which is known for its faster convergence and strong stability compared to traditional iterative methods. The proposed approach proves to be both theoretically rigorous and computationally efficient. Furthermore, the study thoroughly investigates the concept of data dependence, which plays a critical role in ensuring the stability of the solution when subjected to perturbations in the operator or initial data. The derived results demonstrate that the Picard's three-step method yields robust approximations under small data changes, validating its reliability for practical applications. A numerical example is also provided to support the theoretical findings.

This work not only contributes to the existing literature by enhancing the applicability of the Picard's three-step method to fractional problems but also opens up new avenues for future research. Future research may focus on extending this approach to variable-order fractional models, systems involving delays, or partial differential equations with nonlocal boundary conditions, potentially enhancing its applicability to real-world engineering, physical, and biological problems.

Beyond the theoretical implications, the results of this study can be applied to several practical fields. In particular, fractional integro-differential equations commonly arise in control systems with memory, viscoelasticity, population dynamics, epidemiological models with hereditary effects, and signal processing. The strong convergence and data dependence properties established here suggest that the Picard's three-step method can be reliably utilized as a numerical tool in these applied settings.

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