ORIGINAL PAPER

ON GAUSSIAN FUZZY LUCAS NUMBERS

TÜLAY YAĞMUR[1]

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Abstract. *In the present paper, we introduce a new class of fuzzy numbers, which we will call Gaussian fuzzy Lucas numbers. We establish some fundamental properties and identities for these newly defined numbers, including the recurrence relation, Binet's formula, generating function, summation formula, and Vajda's identity, as special cases, Catalan's identity, Cassini's identity, and d'Ocagne's identity. We also establish some relationships between the Gaussian fuzzy Fibonacci and Gaussian fuzzy Lucas numbers, by virtue of the fuzzy Fibonacci and fuzzy Lucas numbers.*

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1. INTRODUCTION

Special integer sequences have numerous applications in science, including mathematics, physics, engineering, and economics. The most well-known integer sequence is the Fibonacci sequence. The Fibonacci sequence starts with 0 and 1, and each number in the sequence is formed by adding the two numbers preceding it. The Fibonacci sequence includes the numbers $0, 1, 1, 2, 3, 5, 8, 13,$ and so on. The Lucas sequence, which is closely related to the Fibonacci sequence, is another well-known integer sequence. The Lucas sequence begins with 2 and 1, and is defined by a formula in which the sum of the two preceding numbers yields the subsequent number. The Lucas sequence includes the numbers $2, 1, 3, 4, 7, 11, 18, 29,$ and so on. The Fibonacci and Lucas sequences share recursive similarities, which result in their many similarities. For further information on the Fibonacci and Lucas sequences, see [1-3].

A Gaussian number is a complex number with integer coefficients that Gauss investigated in 1832 [4]. In 1963, Horadam introduced the concepts of complex Fibonacci and complex Lucas numbers in [5]. Then, many authors studied the Gaussian Fibonacci and Gaussian Lucas numbers. A few instances of these studies can be found in [6-8].

Fuzzy set theory was introduced by Zadeh in [9]. Fuzzy set theory has applications in a variety of fields, including artificial intelligence, robotics, logic, decision theory, and psychology, see [10-16], among others. The fuzzy set is specified by a membership function. The membership function, represented by μ_A for a given set A, associates an element of a set A with an element in the interval $[0, 1]$. The membership function μ_A describes a fuzzy set and can be used to determine an element's membership grade in relation to a set. For further information, we refer to [17-19]. In particular, Dubois and Prade [18] described a fuzzy number, which is a generalization of a real number, as a fuzzy subset of the real line. In fuzzy set theory, there are numerous types of fuzzy membership functions used. Triangular, trapezoidal, Gaussian, and generalized Bell-shaped membership functions are the most commonly used fuzzy membership functions. Gao et al. provided some arithmetic operations

[1] Aksaray University, Department of Mathematics, 68100 Aksaray, Turkey. E-mail: tulayyagmurr@gmail.com.

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on triangular fuzzy numbers using intervals called α-cut functions in [20]. The membership functions, or α-cut functions, are typically used directly when performing arithmetic operations on fuzzy numbers.

In recent years, we have seen some studies on fuzzy numbers related to special integer sequences. For instance, Irmak and Demirtaş integrated the key ideas of fuzzy numbers, Fibonacci numbers, and Lucas numbers, and introduced the fuzzy Fibonacci numbers and fuzzy Lucas numbers in [21]. Moreover, the same authors gave several identities involving these new types of fuzzy numbers. Then, Duman presented some additional identities for fuzzy Fibonacci numbers in [22]. Furthermore, Spreafico et al. introduced the fuzzy Leonardo numbers and gave several results for these numbers in [23]. More recently, Erduvan introduced the Gaussian fuzzy Fibonacci numbers and obtained some significant formulas for these numbers in [24].

In this paper, we aim to introduce the Gaussian fuzzy Lucas numbers and provide some properties of these novel fuzzy numbers. We also obtain some relationships between the Gaussian fuzzy Fibonacci and Gaussian fuzzy Lucas numbers.

2. PRELIMINARIES

This section introduces some basic concepts that will serve as a foundation for the next section. We will first define triangular fuzzy numbers with their arithmetic operations based on the α-cut approach, where $\alpha \in [0, 1]$.

The triangular fuzzy number is a fuzzy number, which is denoted by $A = (a_1,\ a_2,\ a_3)$, is represented by three points, two of which are left and right of the interval, the remaining is a peak point, such that a_1, a_2, and a_3 are real numbers, see [20, 21]. The triangular membership function with $A = (a_1,\ a_2,\ a_3)$ is given as

$$\mu_A(x) = \begin{cases} 0, x \leq a_1, \\ \dfrac{x-a_1}{a_2-a_1}, a_1 < x \leq a_2, \\ \dfrac{a_3-x}{a_3-a_2}, a_2 < x \leq a_3, \\ 0, x > a_3. \end{cases}$$

Let $A = (a_1,\ a_2,\ a_3)$ be a triangular fuzzy number. The triangular fuzzy number can be represented by α-cut operation, which is denoted by A^α. To convert a triangular fuzzy number to α-cut interval, we follow that

$$A^\alpha = [{a_1}^\alpha, {a_3}^\alpha] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$$

with $\alpha \in [0, 1]$, and ${a_1}^\alpha$ and ${a_3}^\alpha$ are real numbers.

Let $A^\alpha = [{a_1}^\alpha, {a_3}^\alpha]$ and $B^\alpha = [{b_1}^\alpha, {b_3}^\alpha]$ be two α-cut intervals. Then the arithmetic operations of the α-cut intervals are as follows [20, 21]:

$$A^\alpha \pm B^\alpha = [{a_1}^\alpha \pm {b_1}^\alpha,\ {a_3}^\alpha \pm {b_3}^\alpha]$$
$$A^\alpha B^\alpha = [\min\{{a_1}^\alpha {b_1}^\alpha, {a_1}^\alpha {b_3}^\alpha, {a_3}^\alpha {b_1}^\alpha,\ {a_3}^\alpha {b_3}^\alpha\}, \max\{{a_1}^\alpha {b_1}^\alpha, {a_1}^\alpha {b_3}^\alpha, {a_3}^\alpha {b_1}^\alpha, {a_3}^\alpha {b_3}^\alpha\}]$$
$$kA^\alpha = [\min\{\ k{a_1}^\alpha, k{a_3}^\alpha\}, \max\{k{a_1}^\alpha, k{a_3}^\alpha\}]$$

with $k \in \mathbb{R}$. It must be noted that, for positive real numbers a_1, a_2, a_3, b_1, b_2, and b_3 such that $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$, $A^\alpha B^\alpha = [{a_1}^\alpha {b_1}^\alpha, {a_3}^\alpha {b_3}^\alpha]$. Furthermore, for a positive real number k, $kA^\alpha = [k{a_1}^\alpha, k{a_3}^\alpha]$ (see [20]).

The fuzzy Fibonacci numbers are defined in [21] as follows: For $n \geq 2$ and $\alpha \in [0,1]$, the fuzzy Fibonacci numbers are

$$ {F_n}^\alpha = [{F_{n-1}}^\alpha, {F_{n+1}}^\alpha] = [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}] $$

with ${F_0}^\alpha = [1-\alpha, 1+\alpha]$ and ${F_1}^\alpha = [\alpha, 1]$. Equivalently, for $n \geq 2$, the fuzzy Fibonacci numbers are

$$ {F_{n+1}}^\alpha = {F_n}^\alpha + {F_{n-1}}^\alpha $$

with ${F_0}^\alpha = [1-\alpha, 1+\alpha]$ and ${F_1}^\alpha = [\alpha, 1]$. Here, F_n is the *n*-th Fibonacci number [3] defined by the relation

$$ F_0 = 0, F_1 = 1;\ F_n = F_{n-1} + F_{n-2}, n \geq 2. $$

In a similar manner, the fuzzy Lucas numbers are defined in [21] as follows: For $n \geq 2$ and $\alpha \in [0,1]$, the fuzzy Lucas numbers are

$$ {L_n}^\alpha = [{L_{n-1}}^\alpha, {L_{n+1}}^\alpha] = [L_{n-1} + \alpha L_{n-2}, L_{n+1} - \alpha L_{n-1}] \tag{1} $$

with ${L_0}^\alpha = [-1-3\alpha, 1+\alpha]$ and ${L_1}^\alpha = [2-\alpha, 3-2\alpha]$. Equivalently, for $n \geq 2$, the fuzzy Lucas numbers are

$$ {L_{n+1}}^\alpha = {L_n}^\alpha + {L_{n-1}}^\alpha $$

with ${L_0}^\alpha = [-1-3\alpha, 1+\alpha]$ and ${L_1}^\alpha = [2-\alpha, 3-2\alpha]$. Here, L_n is the *n*-th Lucas number [3] defined by the relation

$$ L_0 = 2, L_1 = 1;\ L_n = L_{n-1} + L_{n-2}, n \geq 2. $$

In [24], Erduvan defined the Gaussian fuzzy Fibonacci numbers as follows:

$$ G{F_n}^\alpha = {F_n}^\alpha + i{F_{n-1}}^\alpha, n \geq 0, \tag{2} $$

where ${F_n}^\alpha$ is the *n*-th fuzzy Fibonacci number. Here, GF_n is the *n*-th Gaussian Fibonacci number in [6].

In this study, we consider the Gaussian Lucas numbers in [6]. The *n*-th Gaussian Lucas number GL_n is defined by the relation

$$ GL_n = GL_{n-1} + GL_{n-2}, n \geq 2 $$

with $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$. Moreover, for $n \geq 0$,

$$ GL_n = L_n + iL_{n-1}, \tag{3} $$

where L_n is the *n*-th Lucas number. The Binet's formula of the Gaussian Lucas numbers is

$$ GL_n = \phi_1^n + \phi_2^n + i(\phi_1^{n-1} + \phi_2^{n-1}), \tag{4} $$

where $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$ (see [6]).

3. MAIN RESULTS

In this section, we first define the Gaussian fuzzy Lucas numbers, and then we derive some formulas and identities for these numbers.

Definition 1. For $n \geq 0$, the Gaussian fuzzy Lucas numbers are defined by

$$GL_n{}^{\alpha} = L_n{}^{\alpha} + iL_{n-1}{}^{\alpha}, \tag{5}$$

where $L_n{}^{\alpha}$ is the *n*-th fuzzy Lucas number.

From (5), it is evident that

$$GL_n{}^{\alpha} = GL_{n-1}{}^{\alpha} + GL_{n-2}{}^{\alpha}, n \geq 2, \tag{6}$$

which is the recurrence relation for the Gaussian fuzzy Lucas numbers.

Let $GL_n{}^{\alpha}$ and $GL_m{}^{\alpha}$ be two Gaussian fuzzy Lucas numbers. Then the conjugate of $GL_n{}^{\alpha}$, denoted by $\overline{GL_n{}^{\alpha}}$, is

$$\overline{GL_n{}^{\alpha}} = L_n{}^{\alpha} - iL_{n-1}{}^{\alpha}, \tag{7}$$

and the addition, subtraction, multiplication, and multiplication with scalar for Gaussian fuzzy Lucas numbers are given as

$$\boldsymbol{GL_n{}^{\alpha} \pm GL_m{}^{\alpha} = (L_n{}^{\alpha} \pm L_m{}^{\alpha}) + i(L_{n-1}{}^{\alpha} \pm L_{m-1}{}^{\alpha})},$$
$$GL_n{}^{\alpha} GL_m{}^{\alpha} \cong L_n{}^{\alpha} L_m{}^{\alpha} - L_{n-1}{}^{\alpha} L_{m-1}{}^{\alpha} + i(L_{n-1}{}^{\alpha} L_m{}^{\alpha} + L_n{}^{\alpha} L_{m-1}{}^{\alpha}),$$
$$k\, GL_n{}^{\alpha} = kL_n{}^{\alpha} + ikL_{n-1}{}^{\alpha}$$

with $k \in \mathbb{R}$, respectively.

Proposition 1. Let $GL_n{}^{\alpha}$ be the *n*-th Gaussian fuzzy Lucas number. Then
(i) $GL_n{}^{\alpha} + \overline{GL_n{}^{\alpha}} = 2L_n{}^{\alpha}$
(ii) $GL_n{}^{\alpha}\, \overline{GL_n{}^{\alpha}} = (L_n{}^{\alpha})^2 + (L_{n-1}{}^{\alpha})^2$
(iii) $(GL_n{}^{\alpha})^2 \cong 2L_n{}^{\alpha} GL_n{}^{\alpha} - GL_n{}^{\alpha}\overline{GL_n{}^{\alpha}}$

Proof: From (5) and (7), and considering the addition and multiplication operations, the proof is straightforward.

The following result gives the some relationships between the Gaussian fuzzy Fibonacci and Gaussian fuzzy Lucas numbers.

Theorem 1. Let $GF_n{}^{\alpha}$ and $GL_n{}^{\alpha}$ be the *n*-th Gaussian fuzzy Fibonacci and the *n*-th Gaussian fuzzy Lucas numbers, respectively. Then we have
(i) $GF_n{}^{\alpha} + GL_n{}^{\alpha} = 2GF_{n+1}{}^{\alpha}$
(ii) $GL_{n+1}{}^{\alpha} + GL_{n-1}{}^{\alpha} = 5GF_n{}^{\alpha}$
(iii) $GF_{n+1}{}^{\alpha} + GF_{n-1}{}^{\alpha} = GL_n{}^{\alpha}$

Proof: (i) From (2) and (5), we get

$$\begin{aligned} GF_n{}^{\alpha} + GL_n{}^{\alpha} &= (F_n{}^{\alpha} + iF_{n-1}{}^{\alpha}) + (L_n{}^{\alpha} + iL_{n-1}{}^{\alpha}) \\ &= (F_n{}^{\alpha} + L_n{}^{\alpha}) + i(F_{n-1}{}^{\alpha} + L_{n-1}{}^{\alpha}) \\ &= 2(F_{n+1}{}^{\alpha} + iF_n{}^{\alpha}) \\ &= 2GF_{n+1}{}^{\alpha}. \end{aligned}$$

Here, we use the relation $F_n{}^{\alpha} + L_n{}^{\alpha} = 2F_{n+1}{}^{\alpha}$ in ([21], Theorem 3.2-(d)).

(ii) Using (2) and (5), we find

$$\begin{aligned} {GL_{n+1}}^{\alpha} + {GL_{n-1}}^{\alpha} &= ({L_{n+1}}^{\alpha} + i{L_n}^{\alpha}) + ({L_{n-1}}^{\alpha} + i{L_{n-2}}^{\alpha}) \\ &= ({L_{n+1}}^{\alpha} + {L_{n-1}}^{\alpha}) + i({L_n}^{\alpha} + {L_{n-2}}^{\alpha}) \\ &= 5({F_n}^{\alpha} + i{F_{n-1}}^{\alpha}) \\ &= 5{GF_n}^{\alpha}. \end{aligned}$$

Here, we use the relation ${L_{n+1}}^{\alpha} + {L_{n-1}}^{\alpha} = 5{F_n}^{\alpha}$ in ([21], Theorem 3.2-(c)).
(iii) From (2) and (5), we get

$$\begin{aligned} {GF_{n+1}}^{\alpha} + {GF_{n-1}}^{\alpha} &= ({F_{n+1}}^{\alpha} + i{F_n}^{\alpha}) + ({F_{n-1}}^{\alpha} + i{F_{n-2}}^{\alpha}) \\ &= ({F_{n+1}}^{\alpha} + {F_{n-1}}^{\alpha}) + i({F_n}^{\alpha} + {F_{n-2}}^{\alpha}) \\ &= {L_n}^{\alpha} + i{L_{n-1}}^{\alpha} \\ &= {GL_n}^{\alpha}. \end{aligned}$$

Here, we use the relation ${F_{n+1}}^{\alpha} + {F_{n-1}}^{\alpha} = {L_n}^{\alpha}$ in ([21], Theorem 3.2-(g)).

Theorem 2. For $n \geq 0$, the Binet's formula of the Gaussian fuzzy Lucas numbers is

$$\begin{aligned} {GL_n}^{\alpha} = [\phi_1^*\phi_1^{n-2} + \phi_2^*\phi_2^{n-2} + \alpha(\phi_1^*\phi_1^{n-3} + \phi_2^*\phi_2^{n-3}), \phi_1^*\phi_1^{n} + \phi_2^*\phi_2^{n} - \\ \alpha(\phi_1^*\phi_1^{n-2} + \phi_2^*\phi_2^{n-2})], \end{aligned} \tag{8}$$

where $\phi_1^* = \phi_1 + i$ with $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2^* = \phi_2 + i$ with $\phi_2 = \frac{1-\sqrt{5}}{2}$.

Proof: Using (1), (3) and (5), we have

$$\begin{aligned} {GL_n}^{\alpha} &= {L_n}^{\alpha} + i{L_{n-1}}^{\alpha} \\ &= [L_{n-1} + \alpha L_{n-2}, L_{n+1} - \alpha L_{n-1}] + i[L_{n-2} + \alpha L_{n-3}, L_n - \alpha L_{n-2}] \\ &= [L_{n-1} + iL_{n-2} + \alpha(L_{n-2} + i\,L_{n-3}),\ L_{n+1} + iL_n - \alpha(L_{n-1} + i\,L_{n-2})] \\ &= [GL_{n-1} + \alpha GL_{n-2}, GL_{n+1} - \alpha GL_{n-1}]. \end{aligned}$$

By virtue of (4), we get

$$\begin{aligned} {GL_n}^{\alpha} &= [\phi_1^{n-1} + \phi_2^{n-1} + i(\phi_1^{n-2} + \phi_2^{n-2}) + \alpha\big(\phi_1^{n-2} + \phi_2^{n-2} + i(\phi_1^{n-3} + \phi_2^{n-3})\big), \\ &\quad \phi_1^{n+1} + \phi_2^{n+1} + i(\phi_1^{n} + \phi_2^{n}) - \alpha\big(\phi_1^{n-1} + \phi_2^{n-1} + i(\phi_1^{n-2} + \phi_2^{n-2})\big)] \\ &= [\phi_1^{n-2}(\phi_1 + i) + \phi_2^{n-2}(\phi_2 + i) + \alpha(\phi_1^{n-3}(\phi_1 + i) + \phi_2^{n-3}(\phi_2 + i)), \\ &\quad \phi_1^{n}(\phi_1 + i) + \phi_2^{n}(\phi_2 + i) - \alpha(\phi_1^{n-2}(\phi_1 + i) + \phi_2^{n-2}(\phi_2 + i))]. \end{aligned}$$

Taking $\phi_1^* = \phi_1 + i$ and $\phi_2^* = \phi_2 + i$, the desired result can be obtained.

Theorem 3. The generating function of the Gaussian fuzzy Lucas numbers is

$$g_L^{\alpha}(t) = \frac{[-1-3\alpha, 1+\alpha] + [3+2\alpha, 2-3\alpha]t + i([3+2\alpha, 2-3\alpha] + [-4+5\alpha, -1+4\alpha]t)}{1 - t - t^2}.$$

Proof: Let $g_L^{\alpha}(t)$ be the generating function of the Gaussian fuzzy Lucas numbers, which is a power series where the coefficients of the series consist of the Gaussian fuzzy Lucas numbers. Then we have

$$g_L^{\alpha}(t) = \sum_{n=0}^{\infty} {GL_n}^{\alpha} t^n, \tag{9}$$

$$-t\, g_L^{\alpha}(t) = -\sum_{n=0}^{\infty} {GL_n}^{\alpha} t^{n+1}, \tag{10}$$

$$-t^2\, g_L^{\alpha}(t) = -\sum_{n=0}^{\infty} {GL_n}^{\alpha} t^{n+2}. \tag{11}$$

From (6), (9), (10) and (11), we obtain

$$\begin{aligned}(1-t-t^2)g_L^{\alpha}(t) &= {GL_0}^{\alpha} + ({GL_1}^{\alpha} - {GL_0}^{\alpha})t + \sum_{n=2}^{\infty} ({GL_n}^{\alpha} - {GL_{n-1}}^{\alpha} - {GL_{n-2}}^{\alpha})\, t^n \\ &= {GL_0}^{\alpha} + {GL_{-1}}^{\alpha} t,\end{aligned}$$

and it follows that

$$g_L^{\alpha}(t) = \frac{{GL_0}^{\alpha} + {GL_{-1}}^{\alpha} t}{1-t-t^2}.$$

By virtue of (5), we find

$$g_L^{\alpha}(t) = \frac{{L_0}^{\alpha} + i\,{L_{-1}}^{\alpha} + ({L_{-1}}^{\alpha} + i\,{L_{-2}}^{\alpha})t}{1-t-t^2}$$

$$= \frac{[-1-3\alpha, 1+\alpha] + i[3+2\alpha, 2-3\alpha] + ([3+2\alpha, 2-3\alpha] + i[-4+5\alpha, -1+4\alpha])t}{1-t-t^2}$$

$$= \frac{[-1-3\alpha, 1+\alpha] + [3+2\alpha, 2-3\alpha]t + i([3+2\alpha, 2-3\alpha] + [-4+5\alpha, -1+4\alpha]t)}{1-t-t^2}.$$

Lemma 1. For $n \geq 0$, let ${L_n}^{\alpha}$ be the n-th fuzzy Lucas number. Then

$$\sum_{m=0}^{n} {L_m}^{\alpha} = {L_{n+2}}^{\alpha} - {L_1}^{\alpha}. \tag{12}$$

Proof: Since ${L_{m+2}}^{\alpha} = {L_{m+1}}^{\alpha} + {L_m}^{\alpha}$, we can write ${L_m}^{\alpha} = {L_{m+2}}^{\alpha} - {L_{m+1}}^{\alpha}$. From telescoping sum, the desired result can be obtained.

Theorem 4. The summation formula of the Gaussian fuzzy Lucas numbers is

$$\sum_{m=0}^{n} {GL_m}^{\alpha} = {GL_{n+2}}^{\alpha} - {GL_1}^{\alpha}.$$

Proof: By virtue of (5) and (12), we get

$$\begin{aligned}\sum_{m=0}^{n} {GL_m}^{\alpha} &= \sum_{m=0}^{n} ({L_m}^{\alpha} + i{L_{m-1}}^{\alpha}) \\ &= \sum_{m=0}^{n} {L_m}^{\alpha} + i\sum_{m=0}^{n} {L_{m-1}}^{\alpha} \\ &= ({L_{n+2}}^{\alpha} - {L_1}^{\alpha}) + i({L_{n+1}}^{\alpha} - {L_0}^{\alpha}) \\ &= ({L_{n+2}}^{\alpha} + i{L_{n+1}}^{\alpha}) - ({L_1}^{\alpha} + i{L_0}^{\alpha}) \\ &= {GL_{n+2}}^{\alpha} - {GL_1}^{\alpha}.\end{aligned}$$

Theorem 5 (Vajda's Identity). For any non-negative integers $m, n,$ and r, we have

$${GL_{n+m}}^{\alpha}{GL_{n+r}}^{\alpha} - {GL_n}^{\alpha}{GL_{n+m+r}}^{\alpha} \cong 5(-2+i)(-1)^n F_m F_r(({F_1}^{\alpha})^2 - {F_0}^{\alpha}{F_2}^{\alpha}),$$

where F_m is the m-th Fibonacci number.

Proof: From (8), we get

$$\begin{gathered}
{GL_{n+m}}^{\alpha}{GL_{n+r}}^{\alpha}-{GL_{n}}^{\alpha}{GL_{n+m+r}}^{\alpha}\\
\cong[\phi_1^*\phi_1^{n+m-2}+\phi_2^*\phi_2^{n+m-2}+\alpha(\phi_1^*\phi_1^{n+m-3}+\phi_2^*\phi_2^{n+m-3}),\\
\phi_1^*\phi_1^{n+m}+\phi_2^*\phi_2^{n+m}-\alpha(\phi_1^*\phi_1^{n+m-2}+\phi_2^*\phi_2^{n+m-2})]\\
\times[\phi_1^*\phi_1^{n+r-2}+\phi_2^*\phi_2^{n+r-2}+\alpha(\phi_1^*\phi_1^{n+r-3}+\phi_2^*\phi_2^{n+r-3}),\\
\phi_1^*\phi_1^{n+r}+\phi_2^*\phi_2^{n+r}-\alpha(\phi_1^*\phi_1^{n+r-2}+\phi_2^*\phi_2^{n+r-2})]\\
-[\phi_1^*\phi_1^{n-2}+\phi_2^*\phi_2^{n-2}+\alpha(\phi_1^*\phi_1^{n-3}+\phi_2^*\phi_2^{n-3}),\\
\phi_1^*\phi_1^{n}+\phi_2^*\phi_2^{n}-\alpha(\phi_1^*\phi_1^{n-2}+\phi_2^*\phi_2^{n-2})]\\
\times[\phi_1^*\phi_1^{n+m+r-2}+\phi_2^*\phi_2^{n+m+r-2}+\alpha(\phi_1^*\phi_1^{n+m+r-3}+\phi_2^*\phi_2^{n+r+m-3}),\\
\phi_1^*\phi_1^{n+m+r}+\phi_2^*\phi_2^{n+m+r}-\alpha(\phi_1^*\phi_1^{n+m+r-2}+\phi_2^*\phi_2^{n+m+r-2})]\,.
\end{gathered}$$

Upon algebraic simplification, we obtain

$$\begin{gathered}
{GL_{n+m}}^{\alpha}{GL_{n+r}}^{\alpha}-{GL_{n}}^{\alpha}{GL_{n+m+r}}^{\alpha}\\
\cong[\phi_1^*\phi_2^*(\phi_1\phi_2)^{n-1}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r)\\
+\alpha\phi_1^*\phi_2^*(\phi_1\phi_2)^{n-2}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r)(\phi_1+\phi_2)\\
+\alpha^2\phi_1^*\phi_2^*(\phi_1\phi_2)^{n-2}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r),\\
\phi_1^*\phi_2^*(\phi_1\phi_2)^{n+1}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r)-\alpha\phi_1^*\phi_2^*(\phi_1\phi_2)^{n-1}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r)(\phi_1^2\\
+\phi_2^2)\\
+\alpha^2\phi_1^*\phi_2^*(\phi_1\phi_2)^{n-1}(\phi_1^m-\phi_2^m)(\phi_1^r-\phi_2^r)].
\end{gathered}$$

On the other hand, by virtue of $\phi_1=\frac{1+\sqrt{5}}{2}$ and $\phi_2=\frac{1-\sqrt{5}}{2}$, we find the followings:

$$\phi_1^*\phi_2^*=-2+i,\phi_1\phi_2=-1,\phi_1+\phi_2=1,\text{ and }\phi_1^2+\phi_2^2=3. \tag{13}$$

From (13) and the Binet's formula $F_m=\frac{\phi_1^m-\phi_2^m}{\sqrt{5}}$ in [3], we get

$$\begin{gathered}
{GL_{n+m}}^{\alpha}{GL_{n+r}}^{\alpha}-{GL_{n}}^{\alpha}{GL_{n+m+r}}^{\alpha}\\
\cong[5(-2+i)(-1)^{n-1}F_mF_r+5\alpha(-2+i)(-1)^{n-2}F_mF_r+5\alpha^2(-2+i)(-1)^{n-2}F_mF_r,\\
5(-2+i)(-1)^{n+1}F_mF_r-15\alpha(-2+i)(-1)^{n-1}F_mF_r+5\alpha^2(-2+i)(-1)^{n-1}F_mF_r]\\
=5(-2+i)(-1)^nF_mF_r[-1+\alpha+\alpha^2,-1-3\alpha-\alpha^2]\\
=5(-2+i)(-1)^n\,F_mF_r(({F_1}^{\alpha})^2-{F_0}^{\alpha}{F_2}^{\alpha}).
\end{gathered}$$

The following results follow from Theorem 5.

Corollary 1 (Catalan's Identity). For $m\to -r$, we have

$${GL_{n-r}}^{\alpha}{GL_{n+r}}^{\alpha}-({GL_n}^{\alpha})^2\ \cong 5(-2+i)(-1)^{n+r}\,{F_r}^2({F_0}^{\alpha}{F_2}^{\alpha}-({F_1}^{\alpha})^2).$$

Here, we consider the relation $F_{-r}=(-1)^{r+1}F_r$ in [3].

Corollary 2 (Cassini's Identity). For $m\to -r$ and $r=1$, we have

$${GL_{n-1}}^{\alpha}{GL_{n+1}}^{\alpha}-({GL_n}^{\alpha})^2\ \cong 5(-2+i)(-1)^n\,(({F_1}^{\alpha})^2-{F_0}^{\alpha}{F_2}^{\alpha}).$$

Corollary 3 (d'Ocagne's Identity). For $r\to p-n$ and $m=1$, we have

$${GL_{n+1}}^{\alpha}{GL_p}^{\alpha}-{GL_n}^{\alpha}{GL_{p+1}}^{\alpha}\ \cong 5(-2+i)(-1)^n\,F_{p-n}(({F_1}^{\alpha})^2-{F_0}^{\alpha}{F_2}^{\alpha}).$$

4. CONCLUSION

In this paper, combining the Gaussian Lucas numbers and fuzzy numbers, the Gaussian fuzzy Lucas numbers are defined and studied. The Binet's formula, generating function, and summation formula of these numbers are presented. Moreover, using the Binet's formula, Vajda's identity, as special cases, Catalan's identity, Cassini's identity and d'Ocagne's identity involving these numbers are derived.

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