ORIGINAL PAPER

STABILITY ANALYSIS AND WAVE PATTERN FORMATION TO FRACTIONAL ORDER NONLINEAR COMBINED KDV-MKDV EQUATION

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Abstract. *This research investigates the fractional order combined Korteweg-de Vries and modified Korteweg-de Vries equation by applying the Exp-function method. This method, with the help of travelling wave transformation, provides a systematic approach for deriving analytical solutions to nonlinear fractional partial differential equations. We derive a series of actual solutions, presenting the efficacy and simplicity of the exponential function method in handling the complexity of combined equations. The obtained solutions are analyzed and illustrated through various scenarios, demonstrating the potential of these solutions in understanding physical phenomena described by these equations. Our findings contribute to the broader understanding of nonlinear wave equations and offer a robust analytical tool for future studies in this domain.*

Keywords: *Fractional order combined KDV-mKDV equation; fractional travelling wave transform; Exp-function method; travelling wave solution.*

1. INTRODUCTION

Fractional partial differential equations (PDEs) extend the scope of classical differential equations by introducing fractional calculus into the modelling framework. Unlike traditional equations that use integer-order derivatives, fractional PDEs incorporate derivatives of arbitrary, non-integer orders, which allows them to capture more intricate dynamics of processes with memory and spatial heterogeneity. This fractional approach is particularly effective for representing systems with anomalous diffusion or complex boundary conditions, where traditional models might be inadequate. By employing fractional derivatives, fractional PDEs offer a way to describe phenomena where the influence of past states is distributed across time or space in a non-local manner, reflecting a broader range of behaviors such as slow relaxation or heterogeneous material properties. This capability makes fractional PDEs a powerful tool in diverse applications, from materials science to finance, where understanding and predicting irregular or multi-scale processes are crucial [1-5].

The Exp-function method is a direct technique used to solve complex equations by simplifying them into more manageable forms. It works by proposing that the solution to nonlinear fractional PDEs can be expressed as an Exp-function, which is a mathematical expression involving the powers of a constant base. By inserting this proposed solution into

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the original equation, researchers can transform it into an algebraic equation, which is often easier to solve [6-10]. Recently, various innovative methods have been introduced for tackling nonlinear wave equations, such as the tanh-function method, the F-expansion method, the Jacobian elliptic function method, and the variational iteration method. Nevertheless, these approaches have some constraints in their usage. In different papers this method has been utilized to investigate solitary, compact-like, and periodic solutions for different types of nonlinear wave equations [11-23]. In physical terms, this method is like finding a special kind of pattern or wave that fits perfectly within the system described by the equation. While this method is powerful for certain classes of nonlinear PDEs, other methods like the sine-cosine or tanh method might be more appropriate for specific equations, particularly where periodic or hyperbolic function solutions are more natural. For example, it can be used to model phenomena like waves in fluids or materials, where exact solutions can reveal important characteristics such as wave speed or shape. This method is valuable because it allows scientists and engineers to find precise solutions that describe how physical systems behave under various conditions, making it easier to understand and predict complex phenomena have been extensively studied for their ability to describe solitary wave propagation. However, certain complex wave phenomena necessitate a unified approach that incorporates the nonlinear characteristics of both equations [24-32].

The combined fractional Korteweg-de Vries and modified Korteweg-de Vries (KDV-mKDV) equation serves as a crucial framework for investigating water waves, allowing scientists to analyze and forecast various wave behaviors such as solitary waves, wave breaking, and turbulence, interactions between waves and structures, and tsunamis. Increasing attention has been drawn to this equation and its extensions, as demonstrated by a multitude of research efforts. In the section titled Rational Solutions with Free Multi-Parameters for the Combined Fractional KdV-mKdV Equation using this approach, we offer a thorough discussion of the insights obtained through the application of the Exp-function method, leading to the derivation of exact solutions. Expanding the use of this method allows us to discover multiple types of solutions [33-39]. To demonstrate its versatility, we provide examples featuring both rational and soliton solutions, employing polynomial and Exp-function, respectively. In particular, for the rational solutions to the combined fractional KdV-mKdV equation, free multi-parameters are incorporated. In physical terms, dispersion refers to the phenomenon where different wave frequencies travel at different speeds, leading to the spreading out of wave packets over time. This effect is captured by the KdV component of the equation, which is known for describing shallow water waves, such as tidal waves and solitons. Solitons are stable, solitary wave solutions that maintain their shape while traveling over long distances without dissipating. Moreover, we explicitly illustrate that the N-soliton solution can be represented as an Nth-order determinant. The result and discussion section further explores the dynamic characteristics of the exact solutions derived from the combined KdV-mKdV equation. In plasma physics, it helps model the movement of ion-acoustic waves in a plasma, where both wave steepening and dispersive effects are present. The equation is also used in optical fiber studies, where it models the propagation of intense light pulses, helping design systems that manage signal distortion [40-45].

This research aims to explore the combined fractional KDV-mKDV equation, which integrates these nonlinearities, providing a comprehensive model for analyzing intricate wave dynamics. We explore its derivation, mathematical properties, analytical solutions, and numerical methods for solving the equation. We also discuss various applications in physical contexts such as fluid dynamics, plasma physics, and optical fibers, highlighting the advantages of using the combined equation over the individual KDV-mKDV equations. The combined fractional KDV-mKDV equation is two of the most studied equations in this field, each describing different types of nonlinear wave behavior. The KDV equation, first presented by Diederik Korteweg and Gustav de Vries in 1895, describes the propagation of shallow water waves with long wavelengths and small amplitudes [46-50]. The mKDV equation, developed later, accounts for waves with a different type of nonlinearity. Many physical systems exhibit behaviors that cannot be fully captured by the fractional KDV-mKDV equation alone. This necessitated the development of the combined fractional KDV-mKDV equation, which integrates the nonlinearities from both equations to provide a more comprehensive model. This paper explores the formulation, solutions, and applications of the combined fractional KDV-mKDV equation. The combined KDV-mKDV equation is a nonlinear partial differential equation incorporating terms from the fractional (KDV) and mKDV equations to describe complex wave phenomena [51-54]. The modified Riemann-Liouville derivative of order α is used in this research which is defined as [55]:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \left(f(\xi) - f(0) \right) d\xi, \; 0 < \alpha < 1, \\ \left(f^n(t) \right)^{(\alpha-n)}, \;\; n \le \alpha < n+1, \; n \ge 1. \end{cases} \tag{1}$$

2. MATERIALS AND METHODS

2.1. METHOD DESCRIPTION

We will examine general nonlinear partial differential equations.

$$W(\psi, D_y{}^{\alpha}\psi, D_t{}^{\alpha}\psi, ...) = 0, \;\; 0 < \alpha \le 1 \tag{2}$$

where w represents a differential operator that involves nonlinear terms of , its partial derivatives, and possibly higher-order terms. The goal is to find exact solutions for which could be in the form of traveling waves, solitary waves, or other specific structures.

The first step involves applying an appropriate transformation to simplify the PDE. Commonly, this consists of converting the PDE into an ordinary differential equation (ODE) by assuming a specific type of solution. For instance, in the case of a traveling wave solution, we use the transformation [56-57]:

$$\psi(y,t) = \psi(\eta), \;\; \eta = k\left(\frac{y^{\alpha}}{\alpha} - \frac{rt^{\alpha}}{\alpha}\right), \tag{3}$$

where α is a parameter that typically represents the order of the fractional derivative, and is a constant. This transformation is designed to convert the original conformable fractional differential equation into a more manageable form, often an ordinary differential equation (ODE) or another simplified structure. Equation (1) rewrite again.

$$X(\psi, \psi', \psi'', \psi''') = 0, \tag{4}$$

where the primes represent the ordinary derivative concerning η.

We assume that the solution can be expressed in terms of exponential functions. Specifically, the solution is proposed in the form

$$\psi(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{5}$$

where c, d, p and q are positive integers, a_n and b_m are coefficients that to be dermined.

This corresponding construction shows a momentous and essential measure for determining the exact solution.

2.2. IMPLEMENTATION OF THE METHOD

The fractional KDV-mKDV equation is expressed as:

$$D_t^{\alpha}\psi + P\psi\, D_y^{\alpha}\psi + Q\psi^2 D_y^{\alpha}\psi + RD_y^{3\alpha}\psi = 0,\ 0<\alpha\le 1. \tag{6}$$

Here, $\psi(x,t)$ represents the wave amplitude, t is time, and y is the spatial coordinate. Each term in the equation has a specific physical interpretation, which is crucial for understanding the dynamics of the system it describes. D_t^{α} this term represents the time derivative of the wave amplitude ψ. It describes the temporal evolution of the wave. Essentially, it captures how the wave profile changes over time and $P\psi D_y^{\alpha}$ this is the nonlinear advection term similar to that in the fractional KDV equation. It represents the interaction between the wave amplitude and its gradient, leading to changes in wave shape and speed. The constant P determines the strength of this nonlinearity. The term $Q\psi^2 D_y^{\alpha}$ becomes significant for larger wave amplitudes, describing stronger nonlinear interactions that can lead to more complex wave behaviors. The term $RD_y^{3\alpha}$ spreads the wave energy over time and space, counter acting the nonlinear terms. This balance between nonlinearity and dispersion is essential for the formation and stability of solitons**.** The combined fractional KDV-mKDV equations can be analyzed and solved using the Exp-function method.

$$Rk^3\psi'' + \frac{1}{3}Qk\psi^3 + \frac{1}{2}Pk\psi^2 - kr\psi = 0. \tag{7}$$

By utilizing the proposed method on Eq. (5) and applying the fractional wave transformation as outlined in Eq. (2), we substitute Eq. (2) into Eq. (5). Upon performing integration, the following result is obtained.

We get $c=d=p=q=1$ with the help of homogeneous balance principal and then Eq. (4) reduce to

$$\psi(\eta)=\frac{a_{-1}\exp[-\eta]+a_0+a_1\exp[\eta]}{b_{-1}\exp[-\eta]+b_0+b_1\exp[\eta]}. \tag{8}$$

Substituting Eq. (7) into Eq. (6) and setting the coefficients to zero, we obtain a system of equations. These equations are then solved using software to produce the desired results.

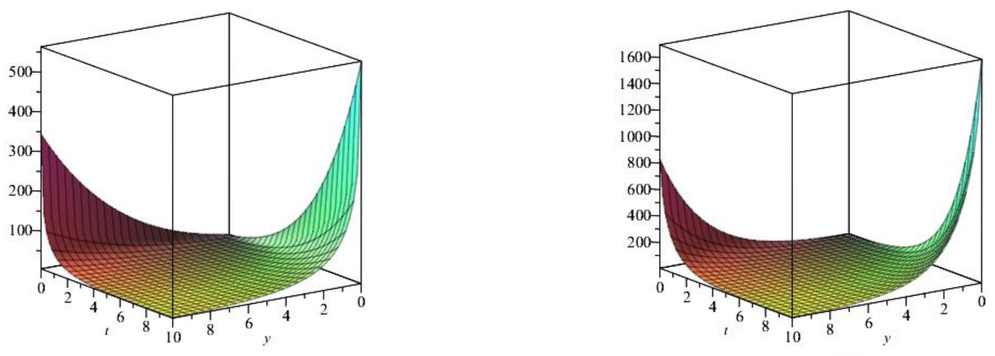
$$E_6 = 3a_1(\tfrac{2}{3}Qa_1{}^2 + Pa_1b_1 - 2rb^2{}_1) = 0,$$
$$E_5 = 3((Pb_0 + 2Qa_0)a_1{}^2 + 2(Pa_0 - b_0(Rk^2 + 2r))b_1a_1 + 2b_1{}^2a_0((Rk^2 - r)) = 0,$$
$$E_4 = 3((2Qa_1{}^2 + 2Pa_1b_1 + 8b_1{}^2(Rk^2 - \frac{r}{4}))a_{-1} + Pa_1{}^2b_{-1} + (2Qa_0{}^2 + 2Pa_0b_0 -$$
$$8b_1(Rk^2 + \frac{r}{2})b_{-1} + 2b_0{}^2(Rk^2 - r))a_1 + a_0(Pa_0 - 2b_0(Rk^2 + 2r))b_1 = 0,$$
$$E_3 = 3(((2Pb_0 + 4Qa_0)a_1 + 2b_1(Pa_0 + 3(Rk^2 - \frac{2r}{3})b_0))a_{-1}$$
$$+ 2(Pa_0 + 3(Rk^2 - \frac{2r}{3})b_0)b_{-1}a_1 + a_0(\frac{2Qa_0{}^2}{3} + Pa_0b_0 - 12(Rk^2 + \frac{r}{3})b_1b_{-1} - 2rb_0{}^2)) = 0, \quad (9)$$
$$E_2 = (2Pa_1b_{-1} + 2Qa^2{}_0 + 2Pa_0b_0 - 8b_1(Rk^2 + \frac{r}{2})b_{-1} + 2b_0{}^2(Rk^2 - r))a_{-1}$$
$$+ (8(Rk^2 - \frac{r}{4})b_{-1}a_1 + a_0(Pa_0 - 2b_0(Rk^2 + 2r)))b_{-1} = 0,$$
$$E_1 = 3((Pb_0 + 2Qa_0)a_{-1}{}^2 + 2(Pa_0 - b_0(Rk^2 + 2r)b_{-1}a_{-1} + 2b_{-1}{}^2a_0(Rk^2 - r)) = 0,$$
$$E_0 = 3a_{-1}(\frac{2}{3}Qa_{-1}{}^2 + Pa_{-1}b_{-1} - 2rb^2{}_{-1}) = 0.$$

Case 1:

$$P = 0,\ Q = 0,\ R = \frac{r}{k^2}, k = k, r = r, a_{-1} = a_{-1}, a_0 = 0, a_1 = a_1,$$
$$b_{-1} = 0, b_0 = b_0, b_1 = 0, \quad (10)$$

where a_{-1}, a_1 and b_0 are free parameters.

$$\psi_1(y,t) = \frac{a_{-1}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + a_1\exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}{b_0}. \quad (11)$$



(a) α=0.5 (b) α=0.75

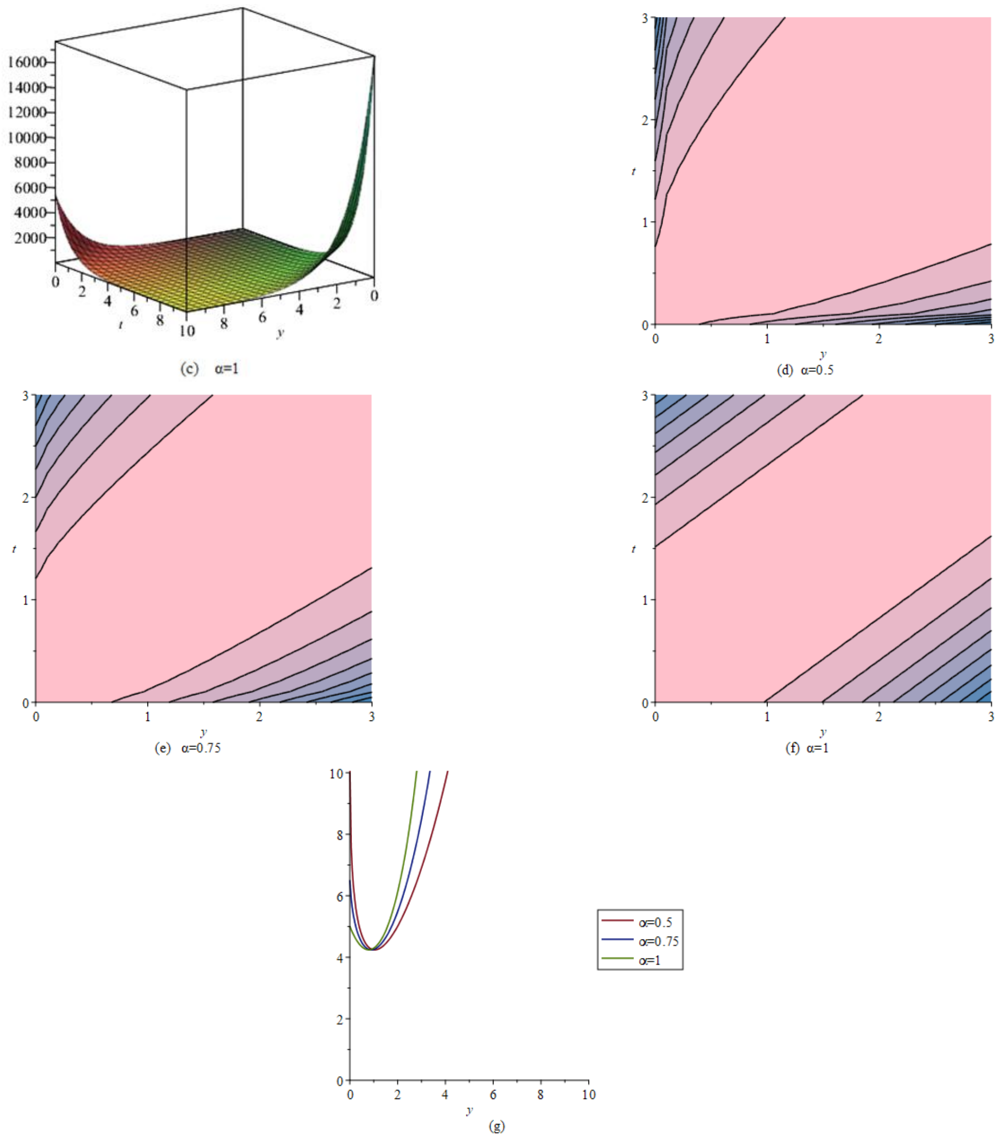


Figure 1. Plots (a) to (c) include 3D representation of $\psi_1(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] will $0 \le y \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 2:

$$P = 0,\ Q = \frac{3rb_1^2}{a_1^2},\ R = -\frac{2r}{k^2}, k = k, r = r, a_{-1} = \frac{a_0^2 b_1^2 - a_1^2 b_0^2}{4a_1 b_1^2}, a_0 = a_0, a_1 = a_1,$$
$$b_{-1} = \frac{a_0^2 b_1^2 - a_1^2 b_0^2}{4b_1 a_1^2}, b_0 = b_0, b_1 = b_1, \tag{12}$$

where a_0, a_1, b_0 and b_1 are free parameters.

$$\psi_2(y,t) = \frac{\dfrac{a_0^2 b_1^2 - a_1^2 b_0^2}{4a_1 b_1^2} \exp\left[-\dfrac{Ky^\alpha}{\alpha} + \dfrac{rkt^\alpha}{\alpha}\right] + a_0 + a_1 \exp\left[\dfrac{Ky^\alpha}{\alpha} - \dfrac{rkt^\alpha}{\alpha}\right]}{-\dfrac{a_0^2 b_1^2 - a_1^2 b_0^2}{4b_1 a_1^2} \exp\left[-\dfrac{Ky^\alpha}{\alpha} + \dfrac{rkt^\alpha}{\alpha}\right] + b_0 + b_1 \exp\left[\dfrac{Ky^\alpha}{\alpha} - \dfrac{rkt^\alpha}{\alpha}\right]}. \tag{13}$$

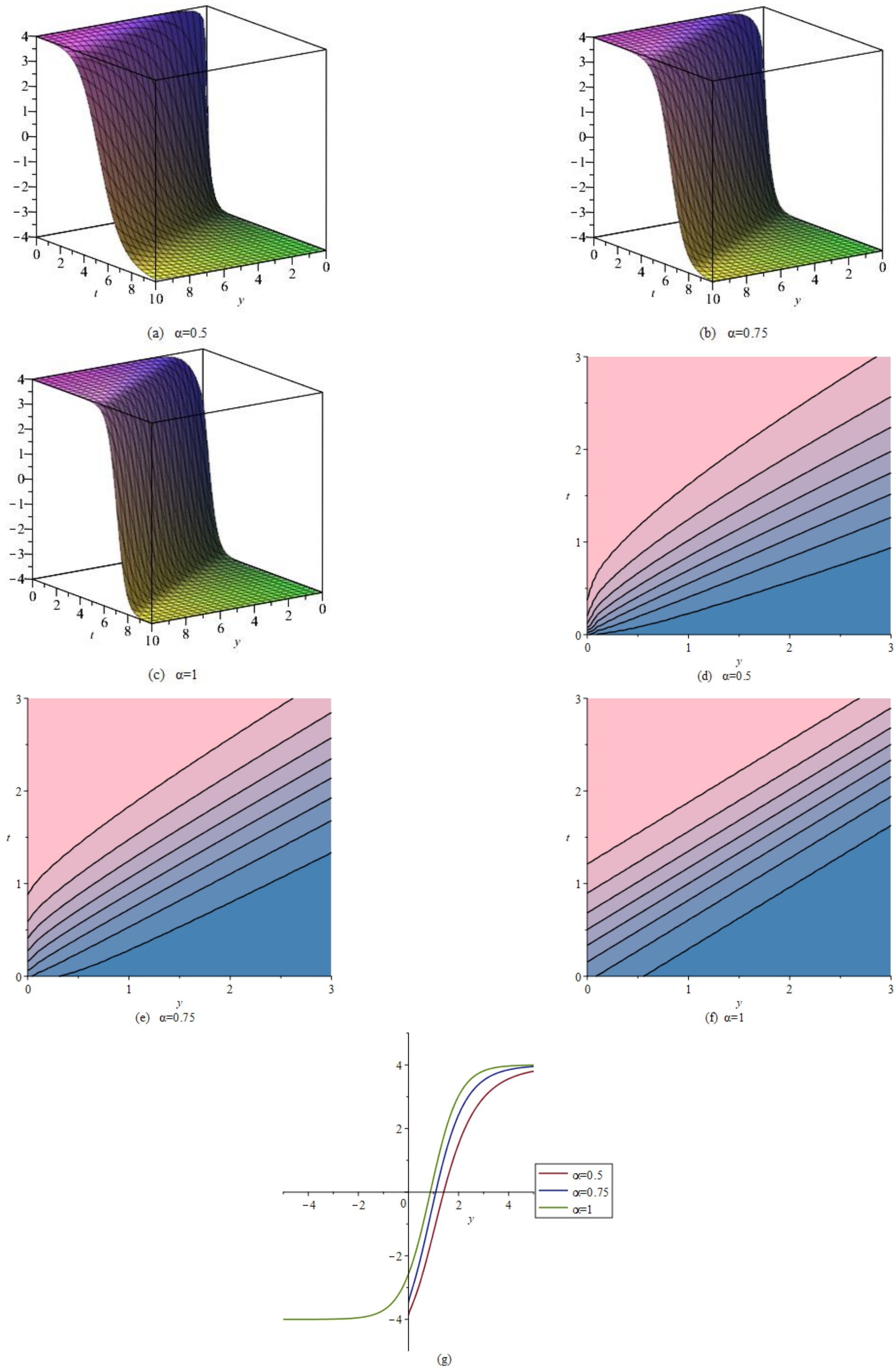


Figure 2. Plots (a) to (c) include 3D representation of $\psi_2(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] with $0 \le y \le 3, 0 \le t \le 3,$ provide a visual comparison using contour plots. (g) Presents the comparison as a 2D plot.

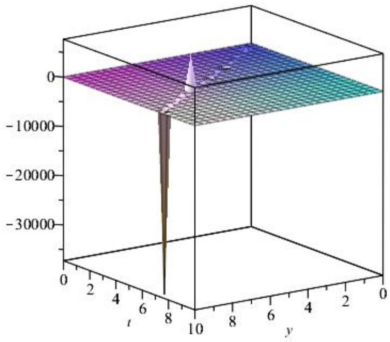
Case 3:

$$P=0,\ Q=\frac{3rb_1^2}{a_1^2},\ R=-\frac{r}{2k^2}, k=k, r=r, a_{-1}=a_{-1}, a_0=0, a_1=a_1, \tag{14}$$

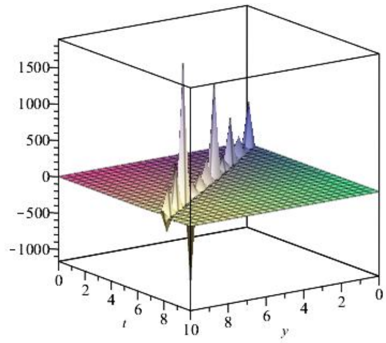
$$b_{-1}=\frac{a_{-1}b_1}{a_1}, b_0=0, b_1=b_1,$$

where a_{-1}, a_1 and b_1 are free parameters.

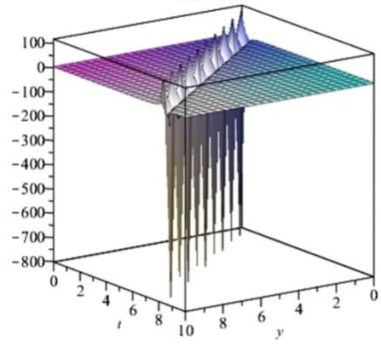
$$\psi_3(y,t)=\frac{a_{-1}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]+a_1\exp\left[\frac{Ky^{\alpha}}{\alpha}-\frac{rkt^{\alpha}}{\alpha}\right]}{-\frac{a_{-1}b_1}{a_1}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]+b_1\exp\left[\frac{Ky^{\alpha}}{\alpha}-\frac{rkt^{\alpha}}{\alpha}\right]}. \tag{15}$$



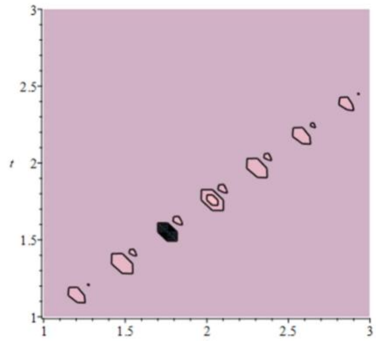
(a) α=0.5



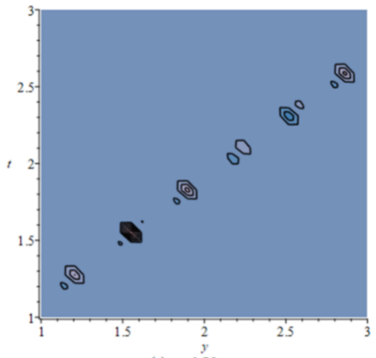
(b) α=0.75



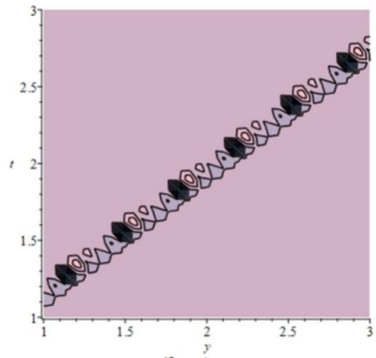
(c) α=1



(d) α=0.5



(e) α=0.75



(f) α=1

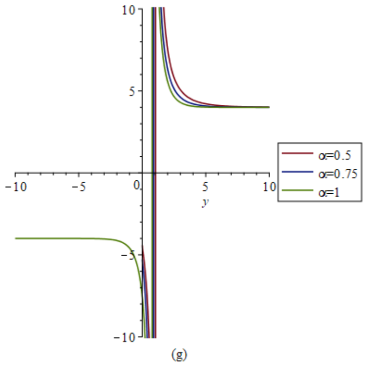


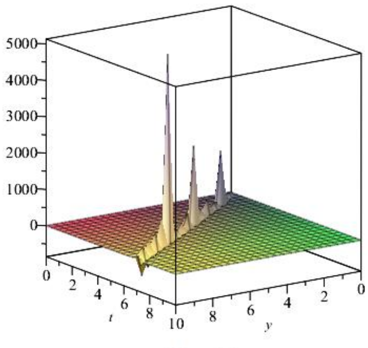
Figure 3. Plots (a) to (c) include 3D representation of $\psi_3(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] will $0 \le y \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

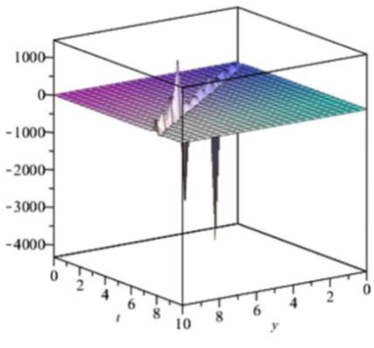
Case 4:

$$P=0,\ Q=\frac{3rb_0^2}{a_0^2},\ R=-\frac{2r}{k^2},k=k,r=r,a_{-1}=a_{-1},a_0=a_0,a_1=0,$$
$$b_{-1}=\frac{a_{-1}b_0}{a_0},b_0=b_0,b_1=0, \tag{16}$$

where a_{-1}, a_1 and b_1 are free parameters.

$$\psi_4(y,t)=\frac{a_{-1}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]+a_0}{-\frac{a_{-1}b_0}{a_0}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]+b_0}. \tag{17}$$





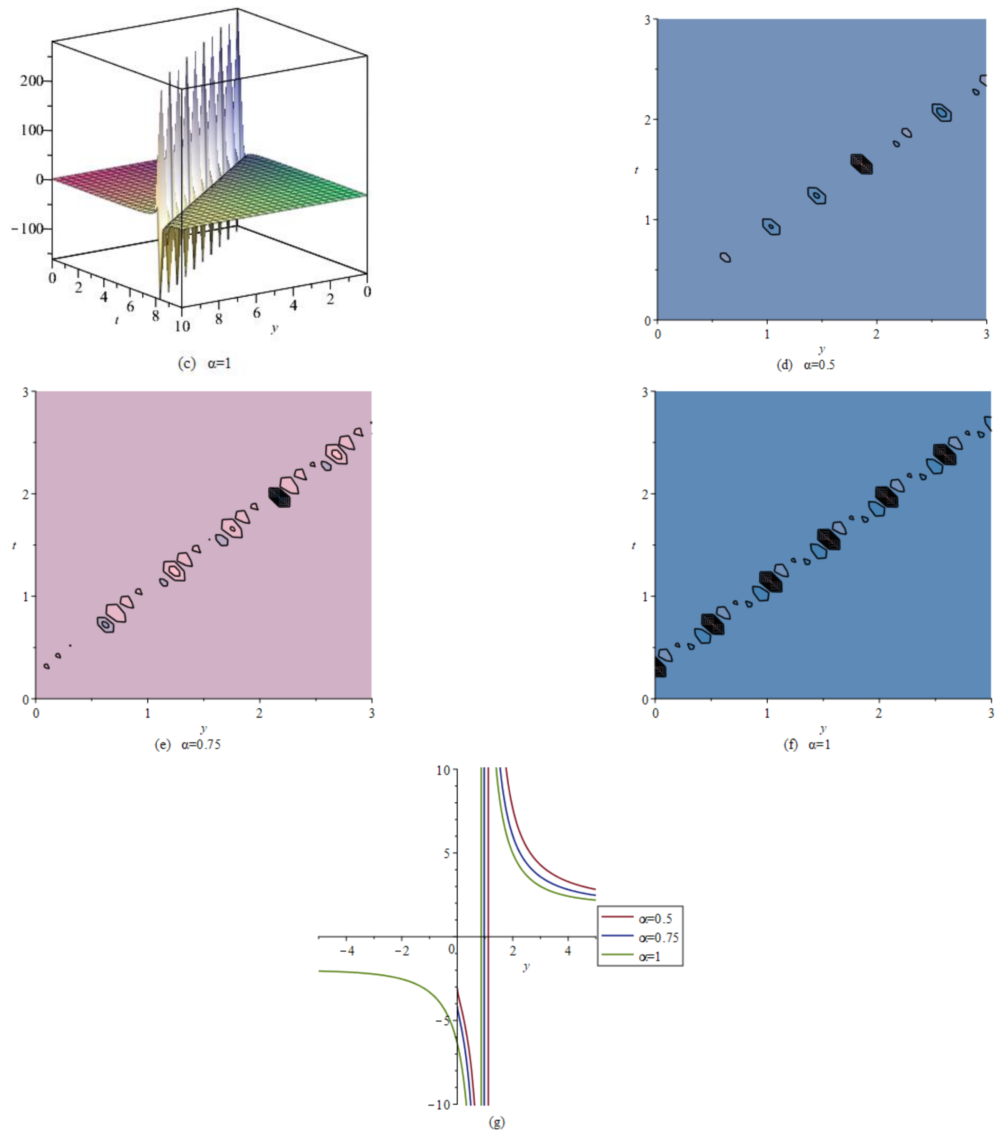


Figure 4. Plots (a) to (c) include 3D representation of $\psi_4(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] will $0 \le y \le 3, 0 \le t \le 3,$ provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 5:

$$P = P,\ Q = -\frac{3b_1(a_1P - 2rb_1)}{2a_1^2},\ R = R, k = k, r = r, a_{-1} = a_{-1}, a_0 = a_0, a_1 = a_1,$$
$$b_{-1} = \frac{a_{-1}b_1}{a_1}, b_0 = \frac{a_0 b_1}{a_1}, b_1 = b_1, \tag{18}$$

where a_{-1}, a_1 and b_1 are free parameters.

$$\psi_5(y,t) = \frac{a_{-1}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + a_0 + a_1\exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}{\frac{a_{-1}b_1}{a_1}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + \frac{a_0 b_1}{a_1} + b_1\exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}. \tag{19}$$

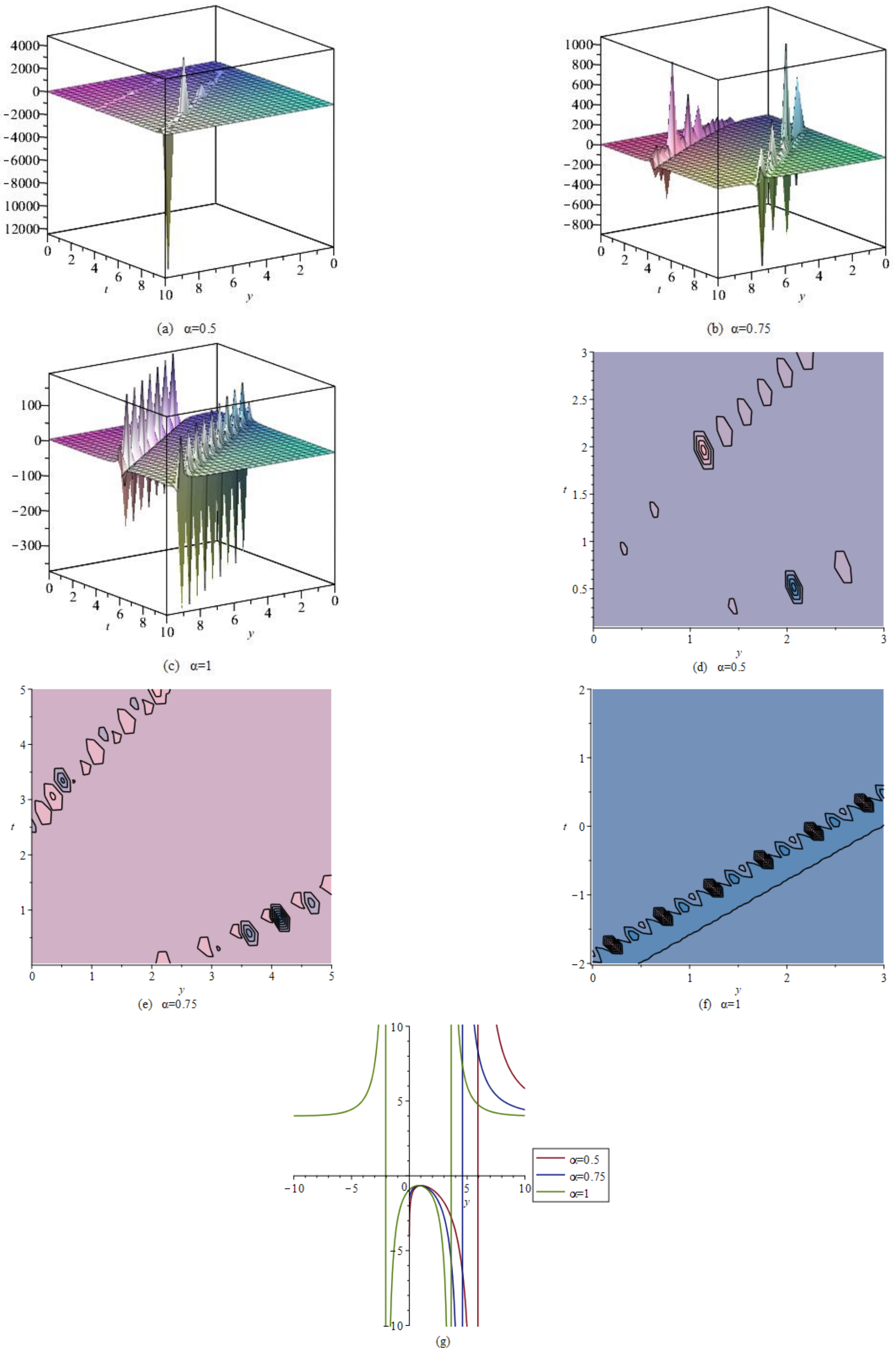


Figure 5. Plots (a) to (c) include 3D representation of $\psi_5(y,t)$, for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] will $0 \le y \le 5,\ 0 \le t \le 3,$ provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

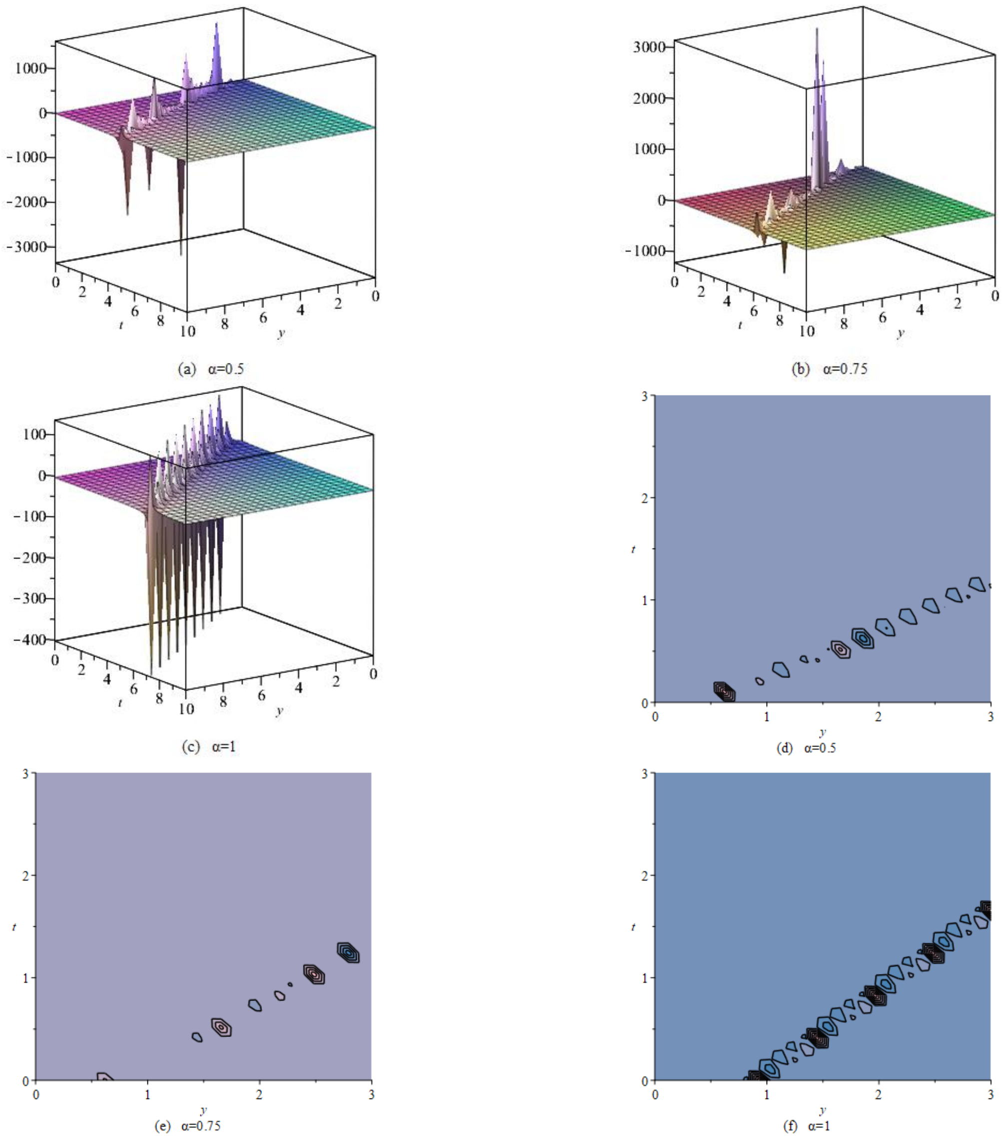
Case 6:

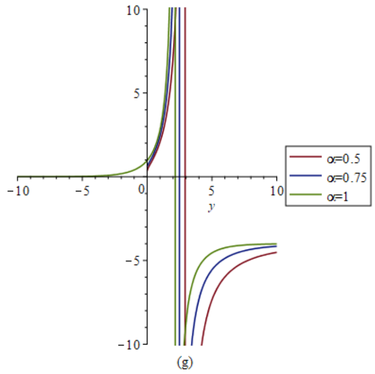
$$P = \frac{6rb_1}{a_1},\ Q = -\frac{6rb_1^2}{a_1^2},\ R = \frac{r}{k^2}, k = k, r = r, a_{-1} = 0, a_0 = a_0, a_1 = a_1, \tag{20}$$

$$b_{-1} = \frac{a_0\left(a_0 b_1 - a_1 b_0\right)}{a_1^2}, b_0 = b_0, b_1 = b_1,$$

where a_0, a_1, b_0 and b_1 are free parameters.

$$\psi_6(y,t) = \frac{a_0 + a_1 \exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}{\frac{a_0\left(a_0 b_1 - a_1 b_0\right)}{a_1^2}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + b_0 + b_1 \exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}. \tag{21}$$





(g)

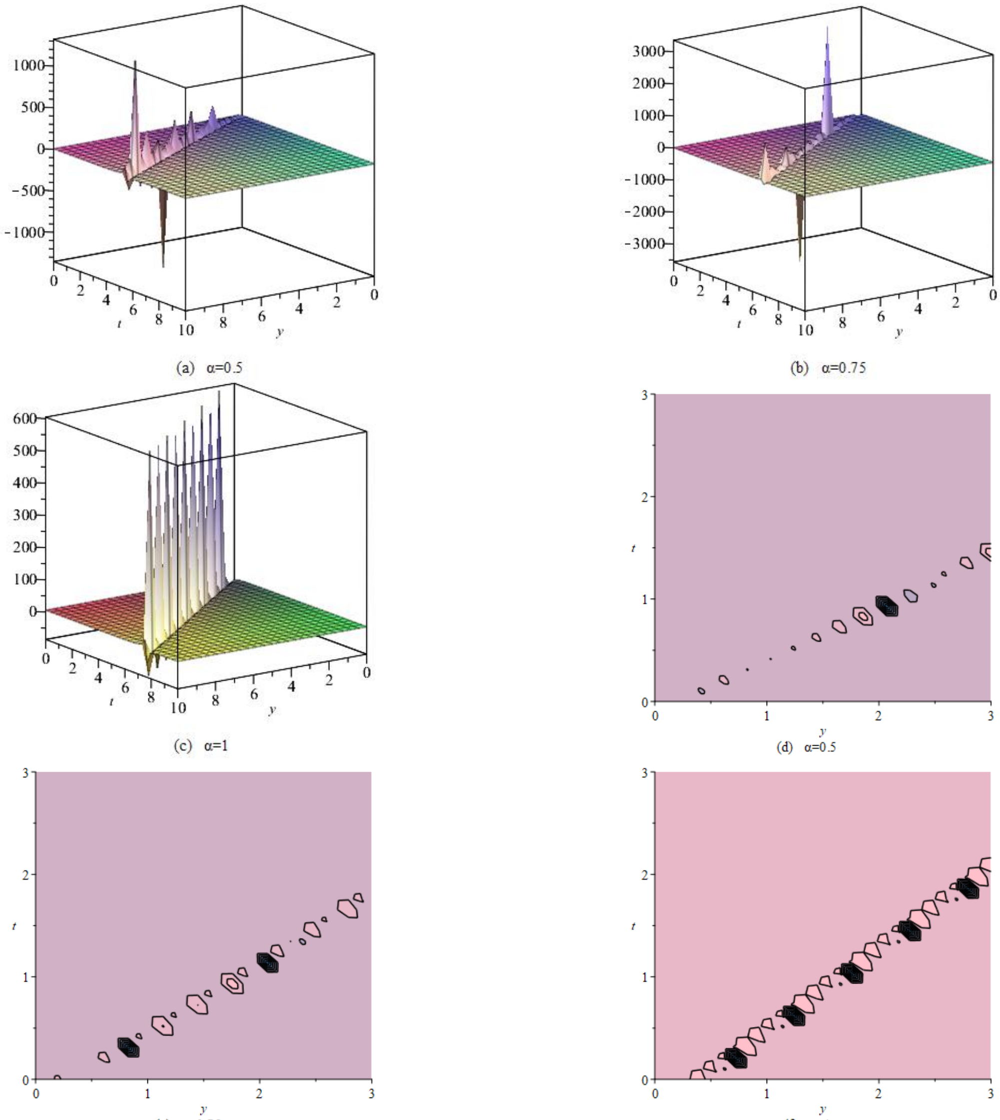
Figure 6. Plots (a) to (c) include 3D representation of $\psi_6(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] will $0 \le y \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 7:

$$P=\frac{6rb_1}{a_1},\ Q=-\frac{6rb_1^2}{a_1^2},\ R=\frac{r}{4k^2},k=k,r=r,a_{-1}=0,a_0=0,a_1=a_1,$$
$$b_{-1}=b_{-1},b_0=0,b_1=b_1, \tag{22}$$

where a_0,a_1,b_0 and b_1 are free parameters.

$$\psi_7(y,t)=\frac{a_1\exp\left[\frac{Ky^\alpha}{\alpha}-\frac{rkt^\alpha}{\alpha}\right]}{b_{-1}\exp\left[-\frac{Ky^\alpha}{\alpha}+\frac{rkt^\alpha}{\alpha}\right]+b_1\exp\left[\frac{Ky^\alpha}{\alpha}-\frac{rkt^\alpha}{\alpha}\right]}. \tag{23}$$



(a) α=0.5 (b) α=0.75 (c) α=1 (d) α=0.5 (e) α=0.75 (f) α=1

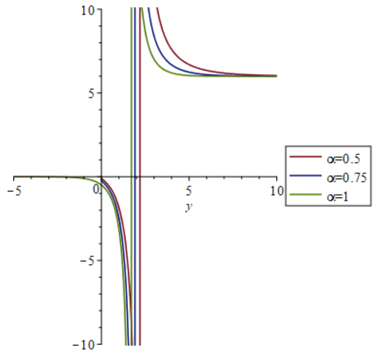


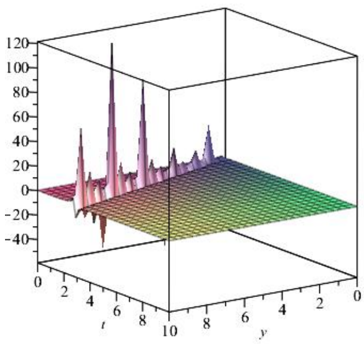
Figure 7. Plots (a) to (c) include 3D representation of $\psi_7(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10$. [(d), (e), and (f)] with $0 \le y \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 8:

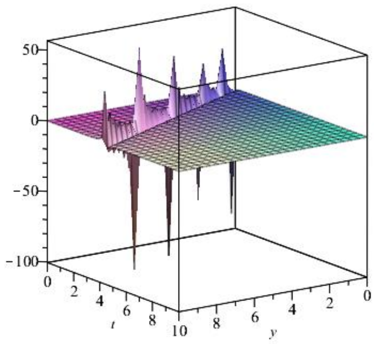
$$P = -\frac{6r(a_{-1}b_1 - a_0 b_0)}{a_0^2},\ Q = -\frac{6r(a_{-1}b_1 - a_0 b_0)^2}{a_0^4},\ R = \frac{r}{k^2}, k = k, r = r,$$
$$a_{-1} = a_{-1}, a_0 = a_0, a_1 = 0, b_{-1} = -\frac{a_{-1}(a_{-1}b_1 - a_0 b_0)}{a_0^2}, b_0 = b_0, b_1 = b_1, \quad (24)$$

where a_{-1}, a_0, b_0 and b_1 are free parameters.

$$\psi_8(y,t) = \frac{a_{-1}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + a_0}{-\frac{a_{-1}(a_{-1}b_1 - a_0 b_0)}{a_0^2}\exp\left[-\frac{Ky^\alpha}{\alpha} + \frac{rkt^\alpha}{\alpha}\right] + b_0 + b_1\exp\left[\frac{Ky^\alpha}{\alpha} - \frac{rkt^\alpha}{\alpha}\right]}. \quad (25)$$



(a) α=0.5



(b) α=0.75

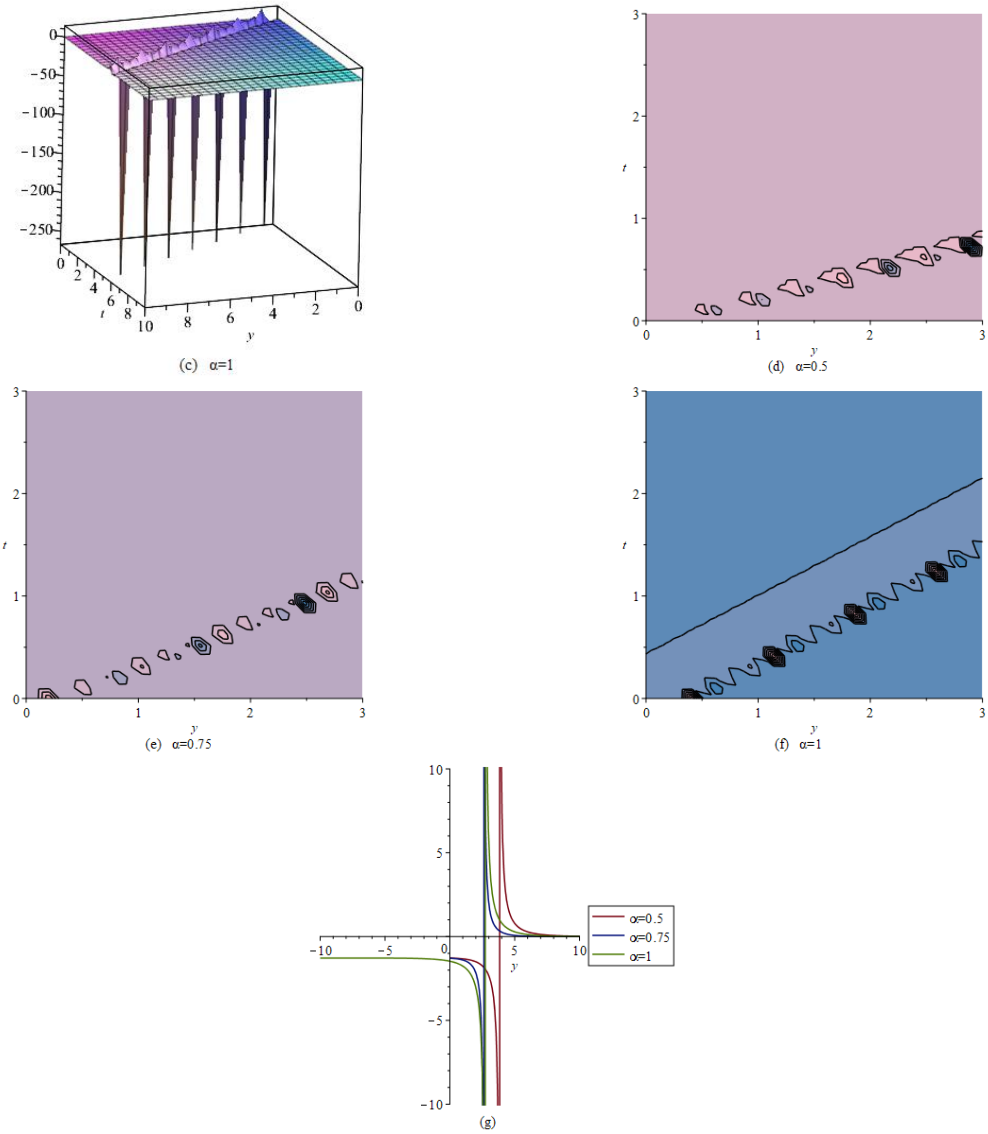


Figure 8. Plots (a) to (c) include 3D representation of $\psi_8(y,t)$ for the values $0 \le y \le 10, 0 \le t \le 10.$ [(d), (e), and (f)] will $0 \le y \le 3, 0 \le t \le 3,$ provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 9:

$$P=-\frac{6rb_{-1}}{a_{-1}},\ Q=-\frac{6rb_{-1}^{2}}{a_{-1}^{2}},\ R=\frac{r}{4k^{2}},k=k,r=r,a_{-1}=a_{-1},a_{0}=0,a_{1}=0,$$
$$b_{-1}=b_{-1},b_{0}=0,b_{1}=b_{1}, \tag{26}$$

where a_{-1},a_0,b_0 and b_1 are free parameters.

$$\psi_9(y,t)=\frac{a_{-1}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]}{b_{-1}\exp\left[-\frac{Ky^{\alpha}}{\alpha}+\frac{rkt^{\alpha}}{\alpha}\right]+b_{1}\exp\left[\frac{Ky^{\alpha}}{\alpha}-\frac{rkt^{\alpha}}{\alpha}\right]}. \tag{27}$$

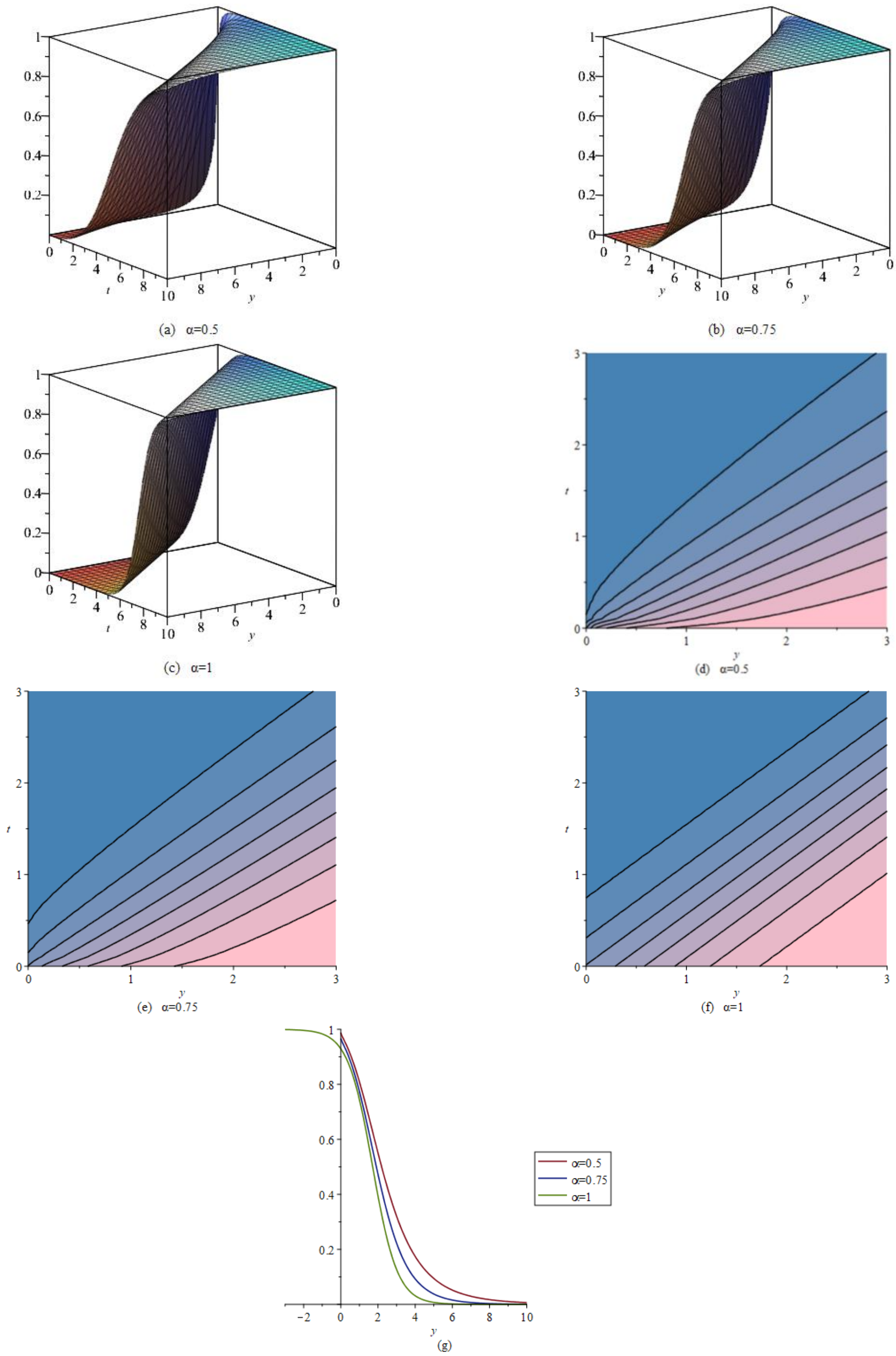


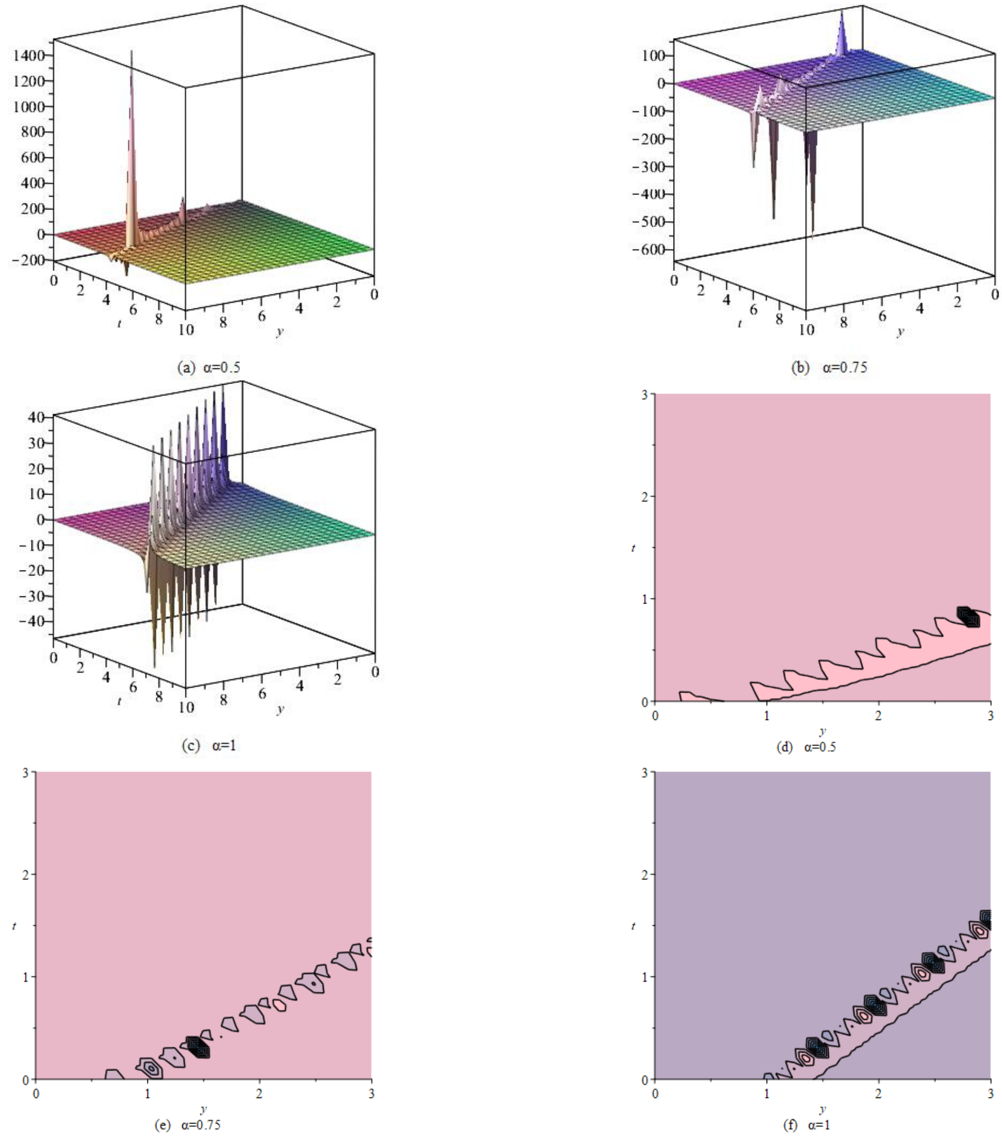
Figure 9. Plots (a) to (c) include 3D representation of $\psi_9(y,t)$ for the values $0 \le x \le 10, 0 \le t \le 10$. [(d), (e), and (f)] with $0 \le x \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

Case 10:

$$P=-\frac{6rb_0}{a_0},\ Q=-\frac{6r\left(4b_{-1}b_1-b_0^2\right)}{a_{-1}^2},\ R=\frac{r}{k^2},k=k,r=r,a_{-1}=0,$$
$$a_0=a_0,a_1=0,b_{-1}=b_{-1},b_0=b_0,b_1=b_1, \tag{28}$$

where a_{-1},a_0,b_0 and b_1 are free parameters.

$$\psi_{10}(y,t)=\frac{a_0}{b_{-1}\exp\left[-\frac{Ky^\alpha}{\alpha}+\frac{rkt^\alpha}{\alpha}\right]+b_0+b_1\exp\left[\frac{Ky^\alpha}{\alpha}-\frac{rkt^\alpha}{\alpha}\right]}. \tag{29}$$



(a) α=0.5 (b) α=0.75

(c) α=1 (d) α=0.5

(e) α=0.75 (f) α=1

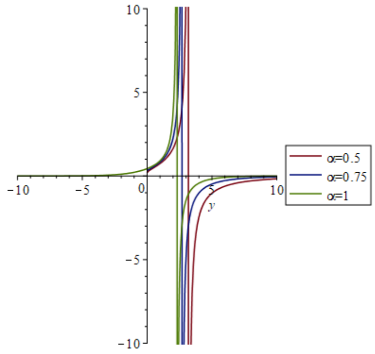


Figure 10. Plots (a) to (c) include 3D representation of $\psi_{10}(y,t)$ for the values $0 \le x \le 10, 0 \le t \le 10$. [(d), (e), and (f)] with $0 \le y \le 3, 0 \le t \le 3$, provide a visual comparison using contour plots. (g) presents the comparison as a 2D plot.

3. RESULTS AND DISCUSSION

The efficient Exp-function method is employed to derive soliton solutions for the nonlinear fractional KdV-mKdV equations. We evaluate our results against the solutions found in existing literature that have been derived using different methods. The primary focus of this research is to discover new and more general solutions to fractional order by exploring different parameter values. The literature review highlights the diversity of solutions achieved through a range of methods. This method is highly effective in handling the complexity of these equations and generating accurate solutions. The solutions derived using this method were systematically analyzed and compared with those previously reported in the literature. The successful application of the method to the combined KdV-mKdV equations underscores its potential as an effective method for addressing nonlinear fractional differential equations. The new and generalized solutions obtained in this study contribute to the existing body of knowledge and open new research avenues. Our findings demonstrate that the method can effectively handle the complexities inherent in nonlinear fractional systems, providing precise and comprehensive solutions. This makes it an invaluable method for researchers working in the area of nonlinear dynamics and mathematical physics. This study examined ten distinct cases with varying parameter values utilizing fractional derivatives. Our findings underscore the significant role of fractional derivatives in comprehending the form of the nonlinear evolution equation. These derivatives are essential for characterizing the system's dynamics and behavior. Compared to existing results in the literature, our findings are both novel and more general. In addition, we provide new insights into the behavior of the system under fractional-order conditions.

By analyzing various parameter conditions, the presence of different types of solitary wave solutions is identified. These solutions are illustrated through 3D plots, contour plots, and 2D plots, as shown in Figs. 1-10 at $\alpha = 0.5$, $\alpha = 0.75$ and: Fig. 1 demonstrates the solution of $\psi_1(y,t)$ for $a_{-1} = \frac{1}{2}, a_0 = \frac{2}{3}, b_{-1} = \frac{1}{2}, b_0 = \frac{1}{3}, k = 0.75, r = 1.25$; Fig. 2 demonstrates the solution of $\psi_2(y,t)$ at $a_1 = 1, a_0 = \frac{2}{3}, b_1 = \frac{1}{4}, b_0 = \frac{1}{3}, k = 1.75, r = 1.5$; Fig. 3 demonstrates the solution of $\psi_3(y,t)$ for $a_1 = 1, a_{-1} = \frac{1}{2}, b_1 = \frac{1}{4}, k = 0.75, r = 1.25$; Fig. 4 demonstrates the solution of $\psi_4(y,t)$ for $a_{-1}=\frac{1}{2}, a_0=\frac{2}{3}, b_0=\frac{1}{3}, k=0.75, r=1.25$; Fig. 5 demonstrates the solution of $\psi_5(y,t)$ for $a_0=-\frac{2}{3}, a_1=1, a_{-1}=\frac{1}{2}, b_1=\frac{1}{4}, k=1.75, r=1.25$; Fig. 6 demonstrates the solution of $\psi_6(y,t)$ for $a_1=1.5, a_0=\frac{2}{3}, b_0=\frac{1}{3}, b_1=-\frac{1}{4}, k=0.75, r=1.25$; Fig. 7 demonstrates the solution of $\psi_7(y,t)$ for $a_1=-1.5, b_1=-\frac{1}{4}, b_{-1}=\frac{1}{2}, k=1.75, r=1.75$; Fig. 8 demonstrates the solution of $\psi_8(y,t)$ for $a_{-1}=\frac{1}{2}, a_0=\frac{2}{3}, b_0=-\frac{1}{3}, b_1=\frac{1}{4}, k=1.75, r=1.75$ Fig. 9 demonstrates the solution of $\psi_9(y,t)$ for $a_{-1}=\frac{1}{2}, b_{-1}=\frac{1}{2}, b_1=\frac{1}{4}, k=0.75, r=1.25$ Fig. 10 demonstrates the solution of $\psi_{10}(y,t)$ for $a_0=\frac{2}{3}, b_1=-\frac{1}{4}, b_0=\frac{1}{3}, b_{-1}=\frac{1}{2}, k=0.75, r=1.25.$

As a result, the fractional calculus model used in this study is very flexible and well-suited for analyzing complex global systems. The solution of the model can be applied across various fields in science and engineering. The obtained solutions are more general, and innovative, and have not been previously detailed in the existing literature. The results further underscore the value and effectiveness of the methodologies applied.

4. MODULATION INSTABILITY ANALYSIS

To analyze the modulation instability (MI) of Eq. (5), we apply the standard linear stability analysis as outlined in references [58–60]. The perturbed solution for the KDV-mKDVequation can be expressed as follows.

$$\psi(y,t)=k_0+\rho K(y,t) \tag{30}$$

Let k_0 be the constant-state solution of Eq. (29). By inserting Eq. (5) into Eq. (29), the equation is transformed into the following form.

$$\begin{aligned}&\rho D_t^{\alpha}K+Pk_0^2\rho D_y^{\alpha}K+P\rho^2 KD_y^{\alpha}K+Qk_0^2\rho D_y^{\alpha}K+2Qk_0\rho^2 KD_y^{\alpha}K+Q\rho^3K^2D_y^{\alpha}K\\&+R\rho D_y^{3\alpha}K=0.\end{aligned} \tag{31}$$

Upon linearizing Eq. 30 ρ, we obtain.

$$D_t^{\alpha}Q+Pk_0D_y^{\alpha}K+Qk_0^2D_y^{\alpha}K+RD_y^{3\alpha}K=0 \tag{32}$$

$$\psi(y,t)=\lambda e^{i(l_2y+\Omega t)} \tag{33}$$

Let l_1 and l_2 be the normalized wave numbers, and let Ω represent the perturbation frequency. Substituting Eq. 32 into Eq. 31, we can obtain the following result.

$$\Omega=-((Pk_0+Qk_0^2)l_2+l_2^3) \tag{34}$$

The sign of Ω suggests whether the solution will grow or diminish over a certain time interval. Whenever Ω is negative for particular values of l_2, any combination of solutions will be vanish. Conversely, the superposition will appear large when Ω is positive for different values l_2. The first scenario corresponds to a stable case, while the second represents an unstable one. If $\Omega_{\max}$ is exactly zero then system remains in a state of marginal stability [60-62].

5. CONCLUSIONS

In the present article, the Exp- function method combined with a fractional traveling wave transformation has been successfully applied to derive exact solutions for the combined fractional KdV-mKdV equations. The transformation simplifies the analysis by reducing the number of independent variables, thus allowing us to focus on the dynamics of the traveling wave. The exact solutions obtained include a variety of wave structures, such as solitary wave solutions, kink wave solutions, and periodic wave solutions. These solutions reflect the complex interplay between nonlinear and dispersive effects in the fractional KdV-mKdV equations. Our solutions are more general and encompass a broader class of wave structures. The inclusion of fractional-order derivatives further enhances the model by capturing memory effects and nonlocal behaviors inherent in many physical systems. Furthermore, graphical representations of the solutions demonstrate how the fractional order of the derivative influences the shape and behavior of the waves, providing insights into the underlying physics.

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