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MULTIVARIABLE *k*-GENERALIZED MITTAG-LEFFLER FUNCTION WITH ITS DIFFERENTIATION AND INTEGRATION PROPERTIES

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Abstract. *In this current paper, we are introducing a new multivariate k-generalized Mittag-Leffler function (Mk GMLf)* $E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,x_2,...,x_s)$. *The paper is organized into five sections containing some properties of the above-mentioned function, like recurrence relations, derivative properties, and integral transforms. A relation of this Mk GMLf with MGMLf has also been derived. We also derive k-fractional integration and k-fractional differentiation of the Mk GMLf. Numerous former results studied by many researchers can also be derived as special cases of our results.*

Keywords: *k-Beta function; k-Gamma function; k-Fractional derivative; k-Fractional integral;k-Pochhammer symbol;k-generalized Mittag-Leffler function.*

1. INTRODUCTION

Nowadays, the Mittag-Leffler function has become a more interesting area of research due to its wide area of applications in solving fractional differential and integral equations, fractional boundary layer equations, telegraph equations, etc. [1-4]. At first, in 1903, the following series representation of the Mittag-Leffler function was discussed by Gosta [5]:

$$E_l(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(nl+1)}, \tag{1}$$

where $x\in\mathbb{C}, l>0$.

Generalization of the above MLf was described by Wiman [6], in 1905 as follows:

$$E_{l,m}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(nl+m)}, \tag{2}$$

where $x,l,m\in\mathbb{C};\Re(l)>0,\Re(m)>0$.

In 1971, Prabhakar [7], generalized the MLf defined in (2) in the following manner:

$$E_{l,m}^{\rho}(x)=\sum_{n=0}^{\infty}\frac{(\rho)_n}{\Gamma(nl+m)}\frac{x^n}{n!}, \tag{3}$$

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where $x,l,m,\rho \in \mathbb{C}; \Re(l) > 0, \Re(m) > 0, \Re(\rho) > 0.$

In 2011, R. K. Saxena et al. [8], investigated the generalization of MLf for several variables as follows:

$$E_{(l_j),m}^{(\rho_j)}(x_1,x_2,...,x_s)=\sum_{n_1,n_2,...,n_s=0}^{\infty}\frac{(\rho_1)_{n_1}...(\rho_s)_{n_s}}{\Gamma(m+n_1l_1+n_2l_2+...+n_sl_s)}\frac{x_1^{n_1}...x_s^{n_s}}{n_1!...n_s!}, \tag{4}$$

where $m, l_j, \rho_j \in \mathbb{C}; \Re(l_j) > 0; j = 1,2,...,s.$

Further, the generalization of the multivariate MLf defined in (4) was also given in the same paper of R. K. Saxena et al. [8], as follows:

$$E_{(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)=\sum_{n_1,...,n_s=0}^{\infty}\frac{(\rho_1)_{n_1\sigma_1}...(\rho_s)_{n_s\sigma_s}}{\Gamma(m+n_1l_1+...+n_sl_s)}\frac{x_1^{n_1}...x_s^{n_s}}{n_1!...n_s!}, \tag{5}$$

where $\rho_j, \sigma_j, l_j, m \in \mathbb{C}; \Re(l_j) > 0, \Re(\sigma_j) > 0 : m \notin Z_0^-; j = 1,2,...,s.$

In 2012, the concept of k -MLf was developed by G.A. Dorrego et al. [9]. They extended definition of MLf as follows:

$$E_{k,l,m}^{\rho}(x)=\sum_{n=0}^{\infty}\frac{(\rho)_{n,k}}{\Gamma_k(nl+m)}\frac{x^n}{n!}, \tag{6}$$

where $(\rho)_{n,k}; \Gamma_k(x)$ represent the *k*-Pochhammer symbol and the *k*-Gamma function respectively [10], defined and represented as follows:

$$(x)_{n,k}=x(x+k)...(x+(n-1)k), \tag{7}$$

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{8}$$

and $k > 0; x,l,m,\rho \in \mathbb{C}; \Re(l) > 0, \Re(m) > 0, \Re(\rho) > 0.$

Here, in this paper, we consider the following multivariate generalization of k -MLf, defined and represented as follows:

$$E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)=\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1}^{s}n_j!}, \tag{9}$$

where $k > 0; \rho_j, \sigma_j, l_j, m \in \mathbb{C}; \Re(l_j) > 0, \Re(\sigma_j); m \notin Z_0^-$ and $j = 1,2,...,s$.

1.1. SPECIAL CASES

(i) Considering $k = 1$ in eq. (9), we have the multivariate MLf, defined in (5).
(ii) Taking $k = 1$, $\sigma_j = 1$ for $j = 1,2,\ldots,s$ in eq. (9), we get another multivariate MLf, defined in (4).
(iii) Assuming $s = 1$, $x_1 = x$, $\rho_1 = \rho$, $l_1 = l$, and $\sigma_1 = 1$ in eq. (9), we have k-MLf as defined in (6).

Now, to obtain the main results, we require some earlier established relations as follows [10, 11]:

$$\Gamma_k(x+k)=x\Gamma_k(x). \tag{10}$$

$$B_k(x,y)=\frac{1}{k}\int_0^1 u^{\frac{x}{k}-1}(1-u)^{\frac{y}{k}-1}du=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{11}$$

$$(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{12}$$

$$(a)_{m+n,k}=(a)_{m,k}(a+mk)_{n,k}. \tag{13}$$

$$(\gamma)_{n,k}=k^n\left(\frac{\gamma}{k}\right)_n. \tag{14}$$

We now establish certain relations, derivatives, integral transforms, fractional order differentiation, and integration formulas for multivariable k-GMLf as our main results.

2. RELATION WITH MULTIVARIABLE GMLF

Theorem 2.1. The multivariable k-GMLf defined in (9) satisfies the following relation

$$E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s)=k^{1-\frac{m}{k}}E_{\left(\frac{l_j}{k}\right),\frac{m}{k}}^{\left(\frac{\rho_j}{k}\right),(\sigma_j)}\left(k^{\sigma_1-\frac{l_1}{k}}x_1,\ldots,k^{\sigma_s-\frac{l_s}{k}}x_s\right), \tag{15}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin Z_0^-$ and $j=1,2,\ldots,s.$

Proof: Applying the equations (14) and (8) to the R.H.S. of equation (9), we at once arrive at our required result (15).

3. RECURRENCE RELATIONS

(i)

$$E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)=mE_{k,(l_j),m+k}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)+\sum_{i=1}^{s} l_i x_i \frac{\partial}{\partial x_i} E_{k,(l_j),m+k}^{(\rho_j),(\sigma_j)}(x_1,...,x_s), \tag{16}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin Z_0^-$ and $j=1,2,...,s.$

Proof: R. H. S. of Eq. (16) can be written in the following form using (9) and differentiating w. r. t. x_i's:

$$\begin{aligned} & mE_{k,(l_j),m+k}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)+\sum_{i=1}^{s} l_i x_i \frac{\partial}{\partial x_i} E_{k,(l_j),m+k}^{(\rho_j),(\sigma_j)}(x_1,...,x_s) \\ &= m \sum_{n_1,...,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+k+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!} \\ &+\sum_{i=l}^{s} l_i x_i \sum_{n_1,...,n_s=0}^{\infty} n_i x_i^{-1} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+k+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!}. \end{aligned} \tag{17}$$

Now, on using the eq. (10), and interpreting with the help of (9), we get the required result (16).

(ii)

$$\begin{aligned} & E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)-\sum_{n_1,...,n_s=0}^{r-1} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!} \\ &= \Pi_{j=1}^{s}(\rho_j)_{r\sigma_j,k} \Pi_{j=1}^{s} x_j^{r} \sum_{n_1,..,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j+r\sigma_j k)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} r l_j+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s}(n_j+1)_r\, n_j!} \end{aligned} \tag{18}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin Z_0^-$ and $j=1,2,...,s.$

Proof: L. H. S. of eq. (18) can be written in the following manner using (9),

$$E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,...,x_s)-\sum_{n_1,...,n_s=0}^{r-1} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!} \tag{19}$$

$$= \sum_{n_1,\dots,n_s=r}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!}$$

$$= \sum_{n_1,\dots,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{(n_j+r)\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} (n_j+r) l_j)} \frac{\Pi_{j=1}^{s} x_j^{(n_j+r)}}{\Pi_{j=1}^{s} (n_j+r)!} \tag{20}$$

$$= \sum_{n_1,\dots,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{(n_j+r)\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} r l_j+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}\Pi_{j=1}^{s} x_j^{r}}{\Pi_{j=1}^{s} (n_j+r)\dots(n_j+1)n_j!}. \tag{21}$$

Now, using eqs. (13) and (7) for k = 1, we get the required result (18).

(iii)

$$sE_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1-k,\rho_2,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1,\rho_2-k,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)$$
$$-\dots E_{k,(l_j),m}^{(\rho_1,\rho_2,\dots,\rho_s-k),(\sigma_j)}(x_1,\dots,x_s)=k^{\sigma_1}\sigma_1\left(\frac{\rho_1}{k}\right)_{\sigma_1-1} x_1 E_{k,(l_j),m+l_1}^{(\rho_1+\sigma_1 k-k,\rho_2,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s) \tag{22}$$
$$+\dots+k^{\sigma_s}\sigma_s\left(\frac{\rho_s}{k}\right)_{\sigma_s-1} x_s E_{k,(l_j),m+l_s}^{(\rho_1,\rho_2,\dots,\rho_s+\sigma_s k-k),(\sigma_j)}(x_1,\dots,x_s),$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin Z_0^-$ and $j=1,2,\dots,s;\ s\in\mathbb{N}.$

Proof: L. H. S. of qu. (22) can be written using (9) as follows,

$$sE_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1-k,\rho_2,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1,\rho_2-k,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)$$
$$-\dots E_{k,(l_j),m}^{(\rho_1,\rho_2,\dots,\rho_s-k),(\sigma_j)}(x_1,\dots,x_s)=E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\dots,x_s)$$
$$-E_{k,(l_j),m}^{(\rho_1-k,\rho_2,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)+E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1,\rho_2-k,\dots,\rho_s),(\sigma_j)}(x_1,\dots,x_s)$$
$$+\dots+E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\dots,x_s)-E_{k,(l_j),m}^{(\rho_1,\rho_2,\dots,\rho_s-k),(\sigma_j)}(x_1,\dots,x_s) \tag{23}$$
$$= \sum_{n_1,\dots,n_s=0}^{\infty} \frac{\{(\rho_1)_{n_1\sigma_1,k}-(\rho_1-k)_{n_1\sigma_1,k}\}\prod_{j=2,\,j\neq1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s} x_j^{n_j}}{\Pi_{j=1}^{s} n_j!}$$

$$+\ldots+\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\{(\rho_s)_{n_s\sigma_s,k}-(\rho_s-k)_{n_s\sigma_s,k}\}\prod_{j=1}^{s-1}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1}^{s}n_j!}.$$

Using eq. (7), we have

$$sE_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s)-E_{k,(l_j),m}^{(\rho_1-k,\rho_2,\ldots,\rho_s),(\sigma_j)}(x_1,\ldots,x_s)-E_{k,(l_j),m}^{(\rho_1,\rho_2-k,\ldots,\rho_s),(\sigma_j)}(x_1,\ldots,x_s)$$

$$-\ldots E_{k,(l_j),m}^{(\rho_1,\rho_2,\ldots,\rho_s-k),(\sigma_j)}(x_1,\ldots,x_s)=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\{n_1\sigma_1k(\rho_1)_{n_1\sigma_1-1,k}\}\prod_{j=2,\,j\neq1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}$$

$$\times\frac{\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1}^{s}n_j!}+\ldots+\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\{n_s\sigma_sk(\rho_s)_{n_s\sigma_s-1,k}\}\prod_{j=1}^{s-1}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1}^{s}n_j!}\tag{24}$$

$$=\sum_{n_1,..,n_s=0}^{\infty}\frac{\{(n_1+1)\sigma_1k(\rho_1)_{(n_1+1)\sigma_1-1,k}\}\prod_{j=2,\,j\neq1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j+l_1)}\frac{\Pi_{j=2}^{s}x_j^{n_j}}{\Pi_{j=2}^{s}n_j!}\frac{x_1^{n_1+1}}{(n_1+1)!}$$

$$+\ldots+\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\{(n_s+1)\sigma_sk(\rho_s)_{(n_s+1)\sigma_s-1,k}\}\prod_{j=1}^{s-1}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j+l_s)}\frac{\Pi_{j=1}^{s-1}x_j^{n_j}}{\Pi_{j=1}^{s-1}n_j!}\frac{x_s^{n_s+1}}{(n_s+1)!}$$

Now, using the eqs. (13) and (14), and in view of (9), we obtain the required result in eq. (22).

4. DERIVATIVE PROPERTIES

(i)

$$k^r\frac{d^r}{du^r}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s)\right\}=u^{\frac{m}{k}-1-r}E_{k,(l_j),m-rk}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s),\tag{25}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin Z_0^-$ and $j=1,2,\ldots,s;\ r\in\mathbb{N}$.

Proof: With the help of eq. (9), we have

$$E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s\right)=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}x_j^{n_j}u^{\sum_{j=1}^{s}\frac{n_jl_j}{k}}}{\Pi_{j=1}^{s}n_j!}. \tag{26}$$

Now, $k^r\frac{d}{du}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s\right)\right\}$

$$=k^r\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}x_j^{n_j}\left(\frac{m+\sum_{j=1}^{s}n_jl_j}{k}-1\right)u^{\frac{m+\sum_{j=1}^{s}n_jl_j}{k}-2}}{\Pi_{j=1}^{s}n_j!}. \tag{27}$$

Similarly, on differentiating the above eq. (27) upto'r' times w. r. t. 'u' and using eq. (10), we get our required eq. (25).

(ii)

$$\frac{d^r}{dx_i^r}\left\{E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s)\right\}=(\rho_i)_{r\sigma_i,k}E_{k,(l_j),m+l_ir}^{(\rho_1,\rho_2,\ldots,\rho_i+r\sigma_ik,\ldots,\rho_s),(\sigma_j)}(x_1,\ldots,x_s), \tag{28}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$ and $j=1,2,\ldots,s;\ r\in\mathbb{N}$.

Proof: We have the following equation in view of (9), and as differentiation w. r. t. 'x_i',

$$\frac{d}{dx_i}\left\{E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s)\right\}=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{n_ix_i^{(-1)}\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1}^{s}n_j!}. \tag{29}$$

Similarly,

$$\frac{d^r}{dx_i^r}\left\{E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s)\right\}=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{x_i^{(-r)}\Pi_{j=1}^{s}x_j^{n_j}}{\Pi_{j=1,j\neq i}^{s}n_j!(n_j-r)!}. \tag{30}$$

Applying $n_i\to n_i+r$ and using eq. (13), eq. (9), we get the required result in eq. (28).

5. INTEGRAL TRANSFORMS

5.1. k-BETA TRANSFORM

i.

$$\frac{1}{\Gamma_k(\mu)}\int_0^1 u^{\frac{m}{k}-1}(1-u)^{\frac{\mu}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s\right)du = kE_{k,(l_j),m+\mu}^{(\rho_j),(\sigma_j)}(x_1,\ldots,x_s), \tag{31}$$

ii.

$$\begin{aligned}&\frac{1}{\Gamma_k(\mu)}\int_t^z (z-r)^{\frac{\mu}{k}-1}(r-t)^{\frac{m}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left((r-t)^{\frac{l_1}{k}}x_1,\ldots,(r-t)^{\frac{l_s}{k}}x_s\right)dr\\ &= k(z-t)^{\frac{\mu+m}{k}-1} E_{k,(l_j),m+\mu}^{(\rho_j),(\sigma_j)}\left((z-t)^{\frac{l_1}{k}}x_1,\ldots,(z-t)^{\frac{l_s}{k}}x_s\right),\end{aligned} \tag{32}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0; m\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$ and $j=1,2,\ldots,s$.

Proof of (31)*:* L. H. S. of equ. (31) can be written using (9) in the following manner

$$\begin{aligned}&\frac{1}{\Gamma_k(\mu)}\int_0^1 u^{\frac{m}{k}-1}(1-u)^{\frac{\mu}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s\right)du\\ &=\frac{1}{\Gamma_k(\mu)}\int_0^1 u^{\frac{m}{k}-1}(1-u)^{\frac{\mu}{k}-1}\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^s n_j l_j)}\frac{\Pi_{j=1}^s x_j^{n_j}u^{\sum_{j=1}^s\frac{n_j l_j}{k}}}{\Pi_{j=1}^s n_j!}du\end{aligned} \tag{33}$$

On interchanging order of integration and summation, we have

$$\begin{aligned}&\frac{1}{\Gamma_k(\mu)}\int_0^1 u^{\frac{m}{k}-1}(1-u)^{\frac{\mu}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s\right)du\\ &=\frac{1}{\Gamma_k(\mu)}\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^s n_j l_j)}\frac{\Pi_{j=1}^s x_j^{n_j}}{\Pi_{j=1}^s n_j!}\int_0^1 u^{\frac{m+\sum_{j=1}^s n_j l_j}{k}-1}(1-u)^{\frac{\mu}{k}-1}du.\end{aligned} \tag{34}$$

Now, using eqs. (11) and (9), we get our desired result in eq. (31).

Proof of (32): Taking $\frac{r-t}{z-t}=u, dr=(z-t)du$ in L.H.S.of equ. (32), and in view of (9), we obtain

$$\frac{1}{\Gamma_k(\mu)}\int_t^z (z-r)^{\frac{\mu}{k}-1}(r-t)^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left((r-t)^{\frac{l_1}{k}}x_1,\ldots,(r-t)^{\frac{l_s}{k}}x_s\right)dr$$

$$=\frac{1}{\Gamma_k(\mu)}\int_0^1 (z-t)^{\frac{\mu}{k}+\frac{m}{k}-1}(1-u)^{\frac{\mu}{k}-1}u^{\frac{m}{k}-1}\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k\left(m+\sum_{j=1}^s n_j l_j\right)} \tag{35}$$

$$\times\frac{\Pi_{j=1}^s x_j^{n_j}\{u(z-t)\}^{\sum_{j=1}^s\frac{n_j l_j}{k}}}{\Pi_{j=1}^s n_j!}du$$

On interchanging order of summation and integration, we obtain

$$\frac{1}{\Gamma_k(\mu)}\int_t^z (z-r)^{\frac{\mu}{k}-1}(r-t)^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left((r-t)^{\frac{l_1}{k}}x_1,\ldots,(r-t)^{\frac{l_s}{k}}x_s\right)dr$$

$$=\frac{(z-t)^{\frac{\mu}{k}+\frac{m}{k}-1}}{\Gamma_k(\mu)}\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k\left(m+\sum_{j=1}^s n_j l_j\right)}\frac{\Pi_{j=1}^s\{(z-t)^{\frac{l_j}{k}}x_j\}^{n_j}}{\Pi_{j=1}^s n_j!} \tag{36}$$

$$\times\int_0^1 u^{\frac{m+\sum_{j=1}^s n_j l_j}{k}-1}(1-u)^{\frac{\mu}{k}-1}du\,.$$

Now, on using eqs. (11) and (9), we arrive at our desired result in eq. (32).

5.2. LAPLACE TRANSFORM

$$\mathcal{L}\left\{u^{\alpha-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\xi_1}x_1,\ldots,u^{\xi_s}x_s\right);p\right\}=\frac{k^{1-\frac{m}{k}}\Gamma(\alpha)}{p^{\alpha}\Gamma\left(\frac{m}{k}\right)}$$

$$\times F_{1:0,\ldots,0}^{1\;:1,\ldots,1}\left[\begin{matrix}(\alpha:\xi_1,\ldots,\xi_s) & :\left(\frac{\rho_1}{k},\sigma_1\right),\ldots,\left(\frac{\rho_s}{k},\sigma_s\right); & \\ & & \frac{k^{\sigma_1-\frac{l_1}{k}}x_1}{p^{\xi_1}},\ldots,\frac{k^{\sigma_s-\frac{l_s}{k}}x_s}{p^{\xi_s}} \\ \left(\frac{m}{k}:\frac{l_1}{k},\ldots,\frac{l_s}{k}\right) & : \quad -;\ldots;-; & \end{matrix}\right], \tag{37}$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$; $F[x_1,\ldots,x_s]$ represents Lauricella function [13, p.37, eqs.(21)-(23)] and $j=1,2,\ldots,s$.

Proof: Using the definition of Laplace Transform [12], and in view of (9), the L. H. S. of (37) takes the following form

$$\mathcal{L}\left\{u^{\alpha-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\xi_1}x_1,\ldots,u^{\xi_s}x_s);p\right\}=\int_0^\infty e^{-pu}u^{\alpha-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\xi_1}x_1,\ldots,u^{\xi_s}x_s)du$$

$$=\int_0^\infty e^{-pu}u^{\alpha-1}\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^s n_jl_j)}\frac{\Pi_{j=1}^s(u^{\xi_j}x_j)^{n_j}}{\Pi_{j=1}^s n_j!}du. \tag{38}$$

Now, on interchanging the order of summation and integration, we obtain

$$\mathcal{L}\left\{u^{\alpha-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\xi_1}x_1,\ldots,u^{\xi_s}x_s);p\right\}$$

$$=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^s n_jl_j)}\frac{\Pi_{j=1}^s x_j^{n_j}}{\Pi_{j=1}^s n_j!}\int_0^\infty e^{-pu}u^{\alpha+\sum_{j=1}^s n_j\xi_j-1}du$$

$$=\sum_{n_1,\ldots,n_s=0}^{\infty}\frac{\Pi_{j=1}^s(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^s n_jl_j)}\frac{\Pi_{j=1}^s x_j^{n_j}}{\Pi_{j=1}^s n_j!}\frac{\Gamma(\alpha+\sum_{j=1}^s n_j\xi_j)}{p^{(\alpha+\sum_{j=1}^s n_j\xi_j)}}. \tag{39}$$

Now using the eqs. (12), (8), and in view of the definition of generalized Lauricella series [13, p.37, eqs. (21)-(23)], we at once arrive at the desired result in eq. (37).

5.3. HANKEL TRANSFORM

$$\int_0^\infty u^{\eta-1}J_\nu(au)E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(bu^{l_1}x_1,\ldots,bu^{l_s}x_s)du$$

$$=\frac{2^{\eta-1}}{a^\eta}\frac{k^{1-\frac{m}{k}}}{\Gamma(\frac{m}{k})}\frac{\Gamma(\frac{\nu+\eta}{2})}{\Gamma(1+\frac{\nu-\eta}{2})} \tag{1}$$

$$\times F_{2:0,\ldots,0}^{1\;:1,\ldots,1}\left[\begin{matrix}\left(\frac{\nu+\eta}{2};\frac{l_1}{2},\ldots,\frac{l_s}{2}\right) : \left(\frac{\rho_1}{k},\sigma_1\right),\ldots,\left(\frac{\rho_s}{k},\sigma_s\right); \\ \left(\frac{m}{k};\frac{l_1}{k},\ldots,\frac{l_s}{k}\right),\left(1+\frac{\nu-\eta}{2};-\frac{l_1}{2},\ldots,-\frac{l_s}{2}\right) : -;\ldots;-;\end{matrix}\; k^{\sigma_1-\frac{l_1}{k}}b\left(\frac{2}{a}\right)^{l_1}x_1,\ldots,k^{\sigma_s-\frac{l_s}{k}}b\left(\frac{2}{a}\right)^{l_s}x_s\right],$$

where $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}$ and $j=1,2,\ldots,s,a>0,\Re(\nu)>0,\Re(x)>0.$

Proof: L. H. S. of eq. (40) can be written in the following manner using (9),

$$\int_0^\infty u^{\eta-1} J_\nu(au) E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(bu^{l_1}x_1,\ldots,bu^{l_s}x_s)\,du$$
$$=\int_0^\infty u^{\eta-1} J_\nu(au) \sum_{n_1,\ldots,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{\Pi_{j=1}^{s}(bu^{l_j}x_j)^{n_j}}{\Pi_{j=1}^{s} n_j!}\,du. \tag{41}$$

Now, on interchanging the order of summation and integration, and using the following identity [14] therein,

$$\int_0^\infty u^{\eta-1} J_\nu(au)\,du = \frac{2^{\eta-1}}{a^\eta} \frac{\Gamma(\frac{\nu+\eta}{2})}{\Gamma(1+\frac{\nu-\eta}{2})}, \tag{42}$$

we have

$$\int_0^\infty u^{\eta-1} J_\nu(au) E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(bu^{l_1}x_1,\ldots,bu^{l_s}x_s)\,du$$
$$= \sum_{n_1,\ldots,n_s=0}^{\infty} \frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s} n_j l_j)} \frac{2^{\eta-1}}{a^\eta} \frac{\Gamma\left(\frac{\nu+\eta+\sum_{j=1}^{s} n_j l_j}{2}\right)}{\Gamma\left(1+\frac{\nu-\eta-\sum_{j=1}^{s} n_j l_j}{2}\right)} \frac{\Pi_{j=1}^{s}\left(b\left(\frac{2}{a}\right)^{l_j} x_j\right)^{n_j}}{\Pi_{j=1}^{s} n_j!}. \tag{43}$$

Again using the eqs. (12), (8), and interpreting in view of definition of generalized Lauricella series, we at once obtain our required relation (40).

6. FRACTIONAL CALCULUS

6.1. FRACTIONAL INTEGRATION

$$I_k^\eta\left\{u^{\frac{m}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s)\right\} = u^{\frac{\eta+m}{k}-1} E_{k,(l_j),m+\eta}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,\ldots,u^{\frac{l_s}{k}}x_s), \tag{44}$$

where I_k^α is the k-R-L fractional integral, defined by S. Mubeen et al. [15] as:

$$I_k^{\alpha} f(z)=\frac{1}{k\Gamma_k(\alpha)}\int_0^z (z-t)^{\frac{\alpha}{k}-1} f(t)dt, \tag{45}$$

and $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin\mathbb{Z}_0^-=\{0,-1,-2,...\};\ j=1,2,...,s,$ and $\eta\in\mathbb{R}$.

Proof: On using the above definition (45), we get the L. H. S. of (44)

$$I_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s)\right\}$$

$$=\frac{1}{k\Gamma_k(\eta)}\int_0^u (u-t)^{\frac{\eta}{k}-1} t^{\frac{m}{k}-1} E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(t^{\frac{l_1}{k}}x_1,...,t^{\frac{l_s}{k}}x_s)dt$$

$$=\frac{1}{k\Gamma_k(\eta)}\int_0^u (u-t)^{\frac{\eta}{k}-1} t^{\frac{m}{k}-1}\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}(t^{\frac{l_j}{k}}x_j)^{n_j}}{\Pi_{j=1}^{s}n_j!}dt. \tag{46}$$

On interchanging order of summation and integration, and taking $t=uz, dt=udz,$ we get

$$I_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s)\right\}=\frac{u^{\frac{\eta+m}{k}-1}}{k\Gamma_k(\eta)}\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}$$

$$\times\frac{\Pi_{j=1}^{s}(u^{\frac{l_j}{k}}x_j)^{n_j}}{\Pi_{j=1}^{s}n_j!}\int_0^1 z^{\frac{m+\sum_{j=1}^{s}n_jl_j}{k}-1}(1-z)^{\frac{\eta}{k}-1}dz. \tag{47}$$

Now using eqs. (11) and (9), we obtain our desired result (44).

6.2. FRACTIONAL DERIVATIVE

$$D_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s)\right\}=\frac{u^{\frac{m-\eta+1}{k}-2}}{k}E_{k,(l_j),1+m-\eta-k}^{(\rho_j),(\sigma_j)}(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s), \tag{48}$$

where D_k^{η} is the k -R-L Fractional derivative, introduced by L. G. Romero et al. [16] as:

$$D_k^{\eta} f(z)=\frac{d}{dz}I_k^{1-\eta} f(t)dt=\frac{1}{k\Gamma_k(1-\eta)}\frac{d}{dz}\left\{\int_0^z (z-t)^{\frac{1-\eta}{k}-1} f(t)dt\right\}, \tag{49}$$

and $k>0;\rho_j,\sigma_j,l_j,m\in\mathbb{C};\Re(l_j)>0,\Re(\sigma_j)>0;m\notin\mathbb{Z}_0^-=\{0,-1,-2,...\}$ and $j=1,2,...,s$ and $\eta\in\mathbb{R}$.

Proof: On using the above definition (49) in L. H. S. of (48), we have

$$D_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s\right)\right\}$$
$$=\frac{1}{k\Gamma_k(1-\eta)}\frac{d}{du}\left\{\int_0^u(u-t)^{\frac{1-\eta}{k}-1}t^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(t^{\frac{l_1}{k}}x_1,...,t^{\frac{l_s}{k}}x_s\right)dt\right\}$$
$$=\frac{1}{k\Gamma_k(1-\eta)}\frac{d}{du}\left\{\int_0^u(u-t)^{\frac{1-\eta}{k}-1}t^{\frac{m}{k}-1}\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}(t^{\frac{l_j}{k}}x_j)^{n_j}}{\Pi_{j=1}^{s}n_j!}dt\right\}. \tag{50}$$

On interchanging the order of summation and integration, and taking $t=uz,dt=udz,$ we get

$$D_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s\right)\right\}=\frac{1}{k\Gamma_k(1-\eta)}$$
$$\times\frac{d}{du}\left\{\begin{array}{l}u^{\frac{1-\eta+m}{k}-1}\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j)}\frac{\Pi_{j=1}^{s}(u^{\frac{l_j}{k}}x_j)^{n_j}}{\Pi_{j=1}^{s}n_j!}\\ \int_0^1 z^{\frac{m+\sum_{j=1}^{s}n_jl_j}{k}-1}(1-z)^{\frac{1-\eta}{k}-1}dz\end{array}\right\}. \tag{51}$$

Using definition of the k -beta function (11), we get

$$D_k^{\eta}\left\{u^{\frac{m}{k}-1}E_{k,(l_j),m}^{(\rho_j),(\sigma_j)}\left(u^{\frac{l_1}{k}}x_1,...,u^{\frac{l_s}{k}}x_s\right)\right\}$$
$$=\frac{d}{du}\left[u^{\frac{1-\eta+m}{k}-1}\sum_{n_1,...,n_s=0}^{\infty}\frac{\Pi_{j=1}^{s}(\rho_j)_{n_j\sigma_j,k}}{\Gamma_k(m+\sum_{j=1}^{s}n_jl_j+1-\eta)}\frac{\Pi_{j=1}^{s}(u^{\frac{l_j}{k}}x_j)^{n_j}}{\Pi_{j=1}^{s}n_j!}\right]. \tag{52}$$

After differentiation and using equation (10), and interpreting in view of (9), we obtain the desired eq. (48).

7. CONCLUSIONS

In this paper, a simple approach is used to establish the main results for multivariable MLf considered here. By considering appropriate values of parameters, several former established results can be derived from our results, a few are as follows:

i. Considering $s = 1$, $\rho_1 = \gamma$, $\sigma_1 = q$, $l_1 = \alpha$, $m = \beta$, $x_1 = z$ in equations (15), (16), and (22), we get the earlier established result of K. S. Gehlot [17], respectively, which is turned at $\sigma_1 = 1$ into the results of G. A. Dorrego et al. [9], respectively.

ii. Assuming $s = 1$, $\rho_1 = \gamma$, $\sigma_1 = q$, $l_1 = \alpha$, $m = \beta$, $x_1 = z$, $r = j$ in eq. (28), we obtain the known result of K. S. Gehlot [17, p. 2217, eq. (17)] which is turned at $\sigma_1 = 1$ into the results of G. A. Dorrego et al. [9, p.710, eq. (II.15)].

iii. Taking $s = 1$, $\rho_1 = \gamma$, $\sigma_1 = q$, $l_1 = \alpha$, $m = \beta$, $x_1 = z$, $u = \mu$, $\mu = \delta$ in equation (31), the eq. (31) reduced to the earlier result given in [9].

Similarly, for appropriate values of parameters, we can obtain some earlier established results of [1,18-20].

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