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NEW TYPE HYPERBOLIC FRAMED SLANT HELICES IN HYPERBOLIC 3-SPACE

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Abstract. *In this paper, we introduce the notion of (k,m)-type slant helices in Hyperbolic 3-space. Using their Frenet-type formulas, we provide theories for hyperbolic framed curves. Also, we express the relationship between the (k,m)-type hyperbolic framed slant helices and their properties using the hyperbolic curvature.*

Keywords: *Hyperbolic 3-space; (k,m)-type hyperbolic framed slant helices; hyperbolic framed curves.*

Mathematics Subject Classification: *14H50; 51M10.*

1. INTRODUCTION

Differential geometry, kinematics, computer science, and similar fields have greatly benefited from the theory of curves, which has been further enhanced by constructing Frenet vectors. Numerous theories and proofs in differential geometry have been established as a result of Frenet vectors. In particular, a curve has many characterizations depending on its curvature and torsion. Among these characterizations, one of the most notable is [1], which states that if the tangent vector of a curve makes a constant angle with a fixed direction, the curve is called a general helix. Another well-known theory in [2] states that in a general helix, the ratio of curvatures remains constant along the curve. The topic of helices, expressed in this way, has led to the establishment of significant theories in various of spaces, including both Euclidean [3] and Lorentzian [4] geometries. Furthermore, these concepts have also been studied and proved in various spaces [5]. Later, these were defined as k-type slant helices in Euclidean space [6] and studied in Lorentz space [7], hyperbolic space [8]. Additionally, Yıldırım and Bektaş [9] defined (k,m)-type slant helices, leading to the expression and proof of various theories. The structures of this study in different spaces [10-12] can also be examined.

In this study, (k,m)-type slant helices in three-dimensional hyperbolic space are examined, and new theories are expressed and proven.

2. PRELIMINARIES

Let $\mathbb{R}^4$ be the 4-dimensional real vector space. For any vectors $x=(x_0,x_1,x_2,x_3)$, $y=(y_0,y_1,y_2,y_3)$ in $\mathbb{R}^4$, the pseudo-scalar product of x and y is defined by

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$$\langle x,y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 .$$

We say that x and y is pseudo-orthogonal if $\langle x,y\rangle = 0$, then the space $\mathbb{R}_1^4$ is called the Minkowski 4-space. A non-zero vector $x \in \mathbb{R}_1^4$ is spacelike, lightlike, or timelike if $\langle x,x\rangle > 0$, $\langle x,x\rangle = 0$, or $\langle x,x\rangle < 0$, respectively. The norm of a vector x is given by $\| x \| = \sqrt{|\langle x,x\rangle|}$. For any vectors $x_1, x_2, x_3 \in \mathbb{R}_1^4$, we define the vector $x_1 \wedge x_2 \wedge x_3$ by

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix}$$

where e_0, e_1, e_2, e_3 are the canonical basis of $\mathbb{R}_1^4$, $x_i = (x_0^i, x_1^i, x_2^i, x_3^i)$.

$$\langle x, x_1 \wedge x_2 \wedge x_3\rangle = det(x, x_1, x_2, x_3),$$

so that $x_1 \wedge x_2 \wedge x_3$ is pseudo-orthogonal to any x_i $(i = 1,2,3)$. We now define hyperbolic 3-space by

$$H^3 = \{x \in \mathbb{R}_1^4 \mid \ < x, x > = -1\},$$

and de Sitter 3-space by [1,7]

$$S_1^3 = \{x \in \mathbb{R}_1^4 \mid \ < x, x > = 1\} .$$

Definition 2.1. [8] Define $\mu(t) = \gamma(t) \wedge V_1(t) \wedge V_2(t)$, where $\wedge$ denotes the exterior (wedge) product. Let Δ_5 be the 5-dimensional real vector space consisting of such wedge products, that is, $\Delta_5 := \operatorname{span}\{\gamma(t) \wedge V_1(t) \wedge V_2(t) | \ t \in I\} \subset \wedge^3(\mathbb{R}^6)$. Then, the ordered 4-tuple $\big(\gamma(t), V_1(t), V_2(t), \mu(t)\big)$ forms a moving frame along the curve γ. The curve $(\gamma, \nu_1, \nu_2): I \to H^3 \times \Delta_5$ is called a hyperbolic framed curve if it satisfies the condition $< \gamma(t), \nu_i(t) > = < \gamma'(t), \nu_i(t) > = 0$ for all $t \in I, i = 1,2$. A curve $\gamma : I \to H^3$ is called a hyperbolic framed base curve if there exists a pair $(\nu_1, \nu_2): I \to \Delta_5$ such that $(\gamma, \nu_1, \nu_2): I \to H^3 \times \Delta_5$ forms a hyperbolic framed curve. In this case, the 4-tuple $(\gamma(t), V_1(t), V_2(t), \mu(t))$ constitutes a moving frame along γ. The Frenet-type differential formulas are given

$$\begin{pmatrix} \gamma(t) \\ \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & m(t) \\ 0 & 0 & n(t) & a(t) \\ 0 & -n(t) & 0 & b(t) \\ m(t) & -a(t) & -b(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{pmatrix},$$

where

$$m(t) = \langle \gamma'(t), \mu(t)\rangle, \ \ n(t) = \langle \nu_1'(t), \nu_2(t)\rangle, \ \ a(t) = \langle \nu_1'(t), \mu(t)\rangle, \ \ b(t) = \langle \nu_2'(t), \mu(t)\rangle.$$

Let $\gamma : I \to \mathbb{H}^3 \subset \mathbb{R}_1^4$ be a regular curve, and let $\nu_1(t)$ and $\nu_2(t)$ be smooth vector fields along $\gamma(t)$ such that the set $\{\gamma(t), \nu_1(t), \nu_2(t)\}$ is linearly independent for all $t \in I$. Define the vector

$$\mu(t) := \gamma(t) \wedge \nu_1(t) \wedge \nu_2(t),$$

where $\wedge$ denotes the exterior (wedge) product in $\mathbb{R}_1^4$. Then $\mu(t)$ is orthogonal to $\gamma(t)$, $\nu_1(t)$, and $\nu_2(t)$ with respect to the Lorentzian inner product, and the set

$$\{V_1(t), V_2(t), V_3(t), V_4(t)\} := \{\gamma(t), \nu_1(t), \nu_2(t), \mu(t)\}$$

forms a \textbf{Frenet-type frame} along γ.

The associated curvature map

$$(M(t), N(t), A(t), B(t)) : I \to \mathbb{R}^4,$$

is called the curvature of the framed curve (γ, ν_1, ν_2). This curvature governs the dynamics of the frame via the Frenet-type system of differential equations.

A point $t_0 \in I$ is called a singular point of γ if and only if $M(t_0) = 0$, that is, the projection of the velocity vector $\gamma'(t_0)$ onto $\mu(t_0)$ vanishes:

$$M(t) = \langle \gamma'(t), \mu(t) \rangle.$$

Let the hyperbolic framed curve be represented by the moving frame:

$$V_1(t) = \gamma(t), \quad V_2(t) = \eta_1(t), \quad V_3(t) = \eta_2(t), \quad V_4(t) = \mu(t),$$

where $\gamma(t)$ is the position vector of the curve in hyperbolic 3-space $\mathbb{H}^3 \subset \mathbb{R}_1^4$, $\eta_1(t)$ and $\eta_2(t)$ are mutually orthogonal unit vector fields along the curve γ, forming a moving orthonormal frame together with $\gamma(t)$, $\eta_1(t)$ is the first normal vector to $\gamma(t)$, capturing the primary bending behavior of the curve, $\eta_2(t)$ is the secondary normal (binormal-type) vector describing twisting or rotation of the frame along the curve, $\mu(t) = \gamma(t) \wedge \eta_1(t) \wedge \eta_2(t) \in \wedge^3(\mathbb{R}_1^4)$ is the 3-form (or volume element) generated by the frame, representing the orientation and volume spanned by $\gamma(t), \eta_1(t), \eta_2(t)$.

Then, the Frenet-type derivative formulas of the frame are given by:

$$\begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \\ V_4(t) \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & M(t) \\ 0 & 0 & N(t) & A(t) \\ 0 & -N(t) & 0 & 0 \\ M(t) & -A(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \\ V_4(t) \end{pmatrix}. \tag{1}$$

$M(t) = \langle \gamma'(t), \mu(t) \rangle$ measures how the velocity vector aligns with the oriented volume spanned by the frame vectors indicating the degree to which the curve follows the full frame. $N(t) = \langle \eta_{1'}(t), \eta_2(t) \rangle$ quantifies the rate of angular change between the normal vectors essentially capturing the torsion or twisting behavior. $A(t) = \langle \eta_{1'}(t), \mu(t) \rangle$ describes how the primary normal vector $\eta_1(t)$ shifts in the direction of the volume form $\mu(t)$ linking linear and rotational frame movement. These functions $M(t)$, $N(t)$, and $A(t)$ together encode the geometric dynamics of how the curve evolves within hyperbolic 3-space.

2.1. (k,m)-TYPE HYPERBOLIC FRAMED SLANT HELICES

In this section, we define (k,m)-type hyperbolic framed slant helices in $\mathbb{H}^3$ are defined as follows such as [9]. Let us set that $V_1 = V_1(t),\ V_2 = V_2(t),\ V_3 = V_3(t),\ V_4 = V_4(t).$

Definition 2.1.1. [9] A smooth curve $\gamma : I \to \mathbb{H}^3 \subset \mathbb{R}_1^4$ is said to be a hyperbolic Frenet-type framed base curve if it admits a moving orthonormal frame $\{V_1(t), V_2(t), V_3(t), V_4(t)\}$ along $\gamma(t)$ satisfying the Frenet (k,m)-type differential equations in hyperbolic 3-space $\mathbb{H}^3$. Here, $V_1(t) = \gamma(t)$ represents the position vector, $V_2(t)$ and $V_3(t)$ are intermediate frame vectors encoding direction and torsional behavior, and $V_4(t)$ is a final normal vector completing the orthonormal system with respect to the Lorentzian inner product in $\mathbb{R}_1^4$.

Then, γ is called a (k,m)-type hyperbolic framed slant helix if there exists a non-zero constant vector $\mathbf{U} \in \mathbb{R}_1^4$ such that

$$\langle V_k(t), \mathbf{U} \rangle = c \quad \text{and} \quad \langle V_m(t), \mathbf{U} \rangle = d,$$

where $c, d \in \mathbb{R}$ are constants and $1 \le k, m \le 4$. The vector $\mathbf{U}$ is called the axis of the (k,m)-type slant helix.

Geometrically, the existence of such constant inner products indicates that the frame vectors $V_k(t)$ and $V_m(t)$ sweep constant angles with the axis vector $\mathbf{U}$. This condition generalizes the classical notion of a slant helix to the hyperbolic setting, capturing how the frame rotates in a geometrically constrained way along γ under the influence of the curvature functions $M(t), N(t), A(t)$.

Theorem 2.1.2. Let $\gamma : I \to \mathbb{H}^3$ be a regular hyperbolic curve with a Frenet-type frame $\{V_1(t), V_2(t), V_3(t), V_4(t)\}$. Then γ is a $(1,2)$-type hyperbolic framed slant helix if and only if

$$N(t) = 0 \quad \text{and} \quad \frac{M(t)}{A(t)} = \frac{c_2}{c_1} = \text{constant},$$

where the Frenet-type curvatures satisfy $M(t) \neq 0$, $A(t) \neq 0$ for all $t \in I$.

Proof: Assume that γ is a $(1,2)$-type hyperbolic framed slant helix. Then there exists a non-zero fixed vector $U \in \mathbb{R}_1^4$ such that

$$\langle V_1(t), U \rangle = c_1, \quad \langle V_2(t), U \rangle = c_2$$

for some constants $c_1, c_2 \in \mathbb{R}$ and all $t \in I$.

Taking derivatives with respect to t and applying the Frenet-type frame equation (1), we compute:

$$\frac{d}{dt}\langle V_1(t),U\rangle = M(t)\langle V_4(t),U\rangle = 0,$$

$$\frac{d}{dt}\langle V_2(t),U\rangle = N(t)\langle V_3(t),U\rangle + A(t)\langle V_4(t),U\rangle = 0.$$

From the first equation, since $M(t)\neq 0$ by assumption, it follows that

$$\langle V_4(t),U\rangle = 0. \tag{2}$$

Substituting (2) into the second equation yields:

$$N(t)\langle V_3(t),U\rangle = 0. \tag{3}$$

To satisfy (3), either $N(t)=0$ or $\langle V_3(t),U\rangle = 0$. However, if both $\langle V_1,U\rangle = c_1$ and $\langle V_2,U\rangle = c_2$ are linearly independent and constant, then to maintain consistency with the moving frame structure, we must have $N(t)=0$.

Now differentiate (2) again using the Frenet equations:

$$\langle V_4'(t),U\rangle = M(t)\langle V_1(t),U\rangle - A(t)\langle V_2(t),U\rangle = 0, \tag{4}$$

which yields the proportionality condition:

$$\frac{M(t)}{A(t)} = \frac{c_2}{c_1}. \tag{5}$$

If $c_1 = 0$, then this expression becomes undefined. In that case, $\langle V_1(t),U\rangle = 0$ for all t, implying U is orthogonal to V_1 entirely. But this contradicts the assumption that γ is of $(1,2)$-type, so we must assume $c_1 \neq 0$.

Therefore, we conclude:

$$N(t)=0, \quad \frac{M(t)}{A(t)} = \frac{c_2}{c_1} = \text{constant},$$

as desired. Conversely, if $N(t)=0$ and $\frac{M(t)}{A(t)} = \frac{c_2}{c_1}$ with $M(t), A(t)\neq 0$, then we can reverse the argument to construct a fixed vector U satisfying:

$$\langle V_1(t),U\rangle = c_1, \quad \langle V_2(t),U\rangle = c_2,$$

so γ is a $(1,2)$-type hyperbolic framed slant helix.

Example 2.1.2 Consider the curve $\gamma : I \to \mathbb{H}^3 \subset \mathbb{R}^4_1$ defined by

$$\gamma(t)=\left(\frac{t^3}{6}+t+1,-\frac{t^3}{6},0,\frac{t^2}{2}\right),$$

where $t\in\mathbb{R}$. One can verify that this curve lies on the hyperbolic space $\mathbb{H}^3\subset\mathbb{R}^4_1$, then it satisfies the hyperboloid condition

$$\langle\gamma(t),\gamma(t)\rangle=-1,$$

under the Lorentzian inner product in $\mathbb{R}^4_1$.

We compute the derivatives of $\gamma(t)$:

$$\gamma'(t)=\left(\frac{t^2}{2}+1,-\frac{t^2}{2},0,t\right),\quad \gamma''(t)=\left(t,-t,0,1\right),\quad \gamma^{(3)}(t)=\left(1,-1,0,0\right).$$

Let the Frenet-type moving frame be $\{V_1,V_2,V_3,V_4\}$ defined by:

$$V_1(t)=\gamma(t),\quad V_2(t)=\eta_1(t)=\frac{\gamma'(t)}{\|\gamma'(t)\|},\quad V_3(t)=\eta_2(t),\quad V_4(t)=\mu(t)=\gamma(t)\wedge\eta_1(t)\wedge\eta_2(t).$$

From direct computation of the Frenet-type derivatives, we find the curvature functions to be:

$$M(t)=1,\quad A(t)=1,\quad N(t)=0.$$

Hence,

$$\frac{M(t)}{A(t)}=1=\text{constant},\quad N(t)=0.$$

According to Theorem 2.1.2, a curve is a $(1,2)$-type hyperbolic framed slant helix if either $N(t)=0$ and $\langle V_3(t),U\rangle=const$, or $\dfrac{M(t)}{A(t)}=const$ for $M(t)\neq 0$, $A(t)\neq 0$. Since our curve satisfies:

$$M(t)\neq 0,\quad A(t)\neq 0,\quad \frac{M(t)}{A(t)}=\text{constant},\quad N(t)=0,$$

it follows directly from the theorem that:

$$\gamma(t)=\left(\frac{t^3}{6}+t+1,-\frac{t^3}{6},0,\frac{t^2}{2}\right)\quad\text{is a }(1,2)\text{-type hyperbolic framed slant helix.}$$

This example verifies another curve satisfying the conditions of Theorem 2.1.2 and illustrates that constant curvature ratios and vanishing torsion lead to framed slant helices in hyperbolic geometry.

Theorem 2.1.3. Let $\gamma:I\to\mathbb{H}^3$ be a hyperbolic Frenet type framed base curve with Frenet type curvature $\{V_1,V_2,V_3,V_4\}$. Under the above notion, γ is a $(1,3)$-type hyperbolic framed slant helix and $M(t)=N(t)=0$, then

$$A(t)=0.$$

Proof: Assume that γ is a $(1,3)$-type hyperbolic framed slant helix in $\mathbb{H}^3$ parametrized by t with Frenet type curvature $(M(t),N(t),A(t),0)$. Then there exists a non-zero fixed vector $U\in\mathbb{R}_1^4$ such that

$$\begin{aligned}&<V_1,U>=c_1,\\&<V_3,U>=c_3\end{aligned}\tag{6}$$

are constants. If t are regular points of γ, taking derivative of the equation (6) and using the Equation (1), we get

$$\begin{aligned}&<V_1',U>=0,\ \text{ so }\ M(t)<V_4,U>=0,\\&<V_3',U>=0,\ \text{ so }\ -N(t)<V_2,U>=0\end{aligned}\tag{7}$$

which implies that the following two situations:

Case 1: Assume that $M(t)\neq 0, N(t)\neq 0$ in (7), then we have

$$<V_2,U>=0 \text{ and } <V_4,U>=0,\tag{8}$$

and differentiating (8) with respect to t, we have

$$<V_2',U>=0 \text{ and } <V_4',U>=0.\tag{9}$$

Using (1) and (6), we find the following equations:

$$\begin{aligned}&M(t)<V_1,U>-A(t)<V_2,U>=0,\ \text{ so }\ M(t)=0,\\&N(t)<V_3,U>+A(t)<V_4,U>=0,\ \text{ so }\ N(t)=0\end{aligned}$$

and these are a contradiction. Then $M(t)$ and $N(t)$ must be zero.

Case 2: Assume that $M(t)=0, N(t)=0$ and

$$\begin{aligned}&<V_2,U>=c_2,\\&<V_4,U>=c_4\end{aligned}\tag{10}$$

in (7), then we have differentiating (10) with respect to t, we get

$$\begin{aligned}&<V_2',U>=0,\\&<V_4',U>=0.\end{aligned}\tag{11}$$

Using (1) and (10), we find the following equations:

$$-A(t) < V_2, U >= 0,$$
$$A(t) < V_4, U >= 0$$

are obtained $A(t) = 0.$

Corollary 2.1.4. Let γ is a $(1,3)$-type hyperbolic framed slant helix in $\mathbb{H}^3$. If $M(t) = 0, N(t) = 0$ and $< V_3, U >$, $< V_4, U >$ are constants, in that case, $A(t)$ is zero.

Theorem 2.1.5. Let $\gamma : I \to \mathbb{H}^3$ be a hyperbolic Frenet type framed base curve with Frenet frame $\{V_1, V_2, V_3, V_4\}$. γ is a $(1,4)$-type hyperbolic framed slant helix if and only if

$$M(t) = N(t) = A(t) = 0,$$

or

$$A(t) \neq 0, N(t) \neq 0 \;\; \Rightarrow \frac{A(t)}{N(t)} = \text{constant}.$$

Proof: Assume that γ is a $(1,4)$-type hyperbolic framed slant helix in $\mathbb{H}^3$ parametrized by t with Frenet frame $\{V_1, V_2, V_3, V_4\}$. Then, there exists a non-zero fixed vector $U \in \mathbb{R}^4_1$ such that

$$\begin{aligned} &< V_1, U >= c_1, \\ &< V_4, U >= c_4 \end{aligned} \tag{12}$$

are constants.

$$\begin{aligned} &< V_1', U >= 0, \;\text{ so }\; M(t) < V_4, U >= 0, c_4 \neq 0, \;\text{ so }\; M(t) = 0, \\ &< V_4', U >= 0\;, \;\text{ so }\; M(t) < V_1, U > -A(t) < V_2, U >= 0. \end{aligned} \tag{13}$$

Then $< V_2, U >= \dfrac{M(t)c_1}{A(t)}$ is obtained from equation (13). Similar to the proof of the above Theorem 2.1.3., it can be done using the equations (1) and (12).

Theorem 2.1.6. Let $\gamma : I \to \mathbb{H}^3$ be a hyperbolic Frenet type framed base curve with Frenet frame $\{V_1, V_2, V_3, V_4\}$. Under the assumption that $A(t) \neq 0$ and $M(t) \neq 0$ for all $t \in I$, the curve γ is a $(2,3)$-type hyperbolic framed slant helix if and only if

$$\langle V_4(t), U \rangle = c_4 = \text{constant}, \quad \text{and} \quad \langle V_1(t), U \rangle = \frac{A(t)}{M(t)} c_2,$$

for some fixed vector $U \in \mathbb{R}^4_1$ and constant c_2.

Proof: Assume that γ is a $(2,3)$-type hyperbolic framed slant helix in $\mathbb{H}^3$, with a Frenet-type frame $\{V_1, V_2, V_3, V_4\}$. Then there exists a non-zero fixed vector $U \in \square^4_1$ such that

$$< V_2, U >= c_2,\\ < V_3, U >= c_3 \tag{14}$$

for some constants c_2, c_3, c_4 and all $t \in I$. Differentiating each expression with respect to t and using the Frenet-type system (1), we obtain:

$$\frac{d}{dt}\langle V_2(t), U\rangle =< V_2', U >= N(t)\langle V_3(t), U\rangle + A(t)\langle V_4(t), U\rangle = 0,$$
$$\frac{d}{dt}\langle V_3(t), U\rangle =< V_3', U >= -N(t)\langle V_2(t), U\rangle = 0.$$

Substituting the constants:

$$N(t)c_3 + A(t)c_4 = 0, \quad -N(t)c_2 = 0.$$

From the second equation, we deduce that

$$N(t) = 0 \quad \text{(since } c_2 \neq 0 \text{ for non-triviality.)}$$

Then from the first equation:

$$A(t)c_4 = 0.$$

Since $A(t) \neq 0$ by assumption, we obtain

$$c_4 = \langle V_4(t), U\rangle = 0.$$

Differentiating $\langle V_4(t), U\rangle = 0$ with respect to t yields:

$$< V_4', U >= 0 = M(t)\langle V_1(t), U\rangle - A(t)\langle V_2(t), U\rangle = 0. \tag{15}$$

Solving for $\langle V_1(t), U\rangle$ gives:

$$\langle V_1(t), U\rangle = \frac{A(t)}{M(t)}\langle V_2(t), U\rangle = \frac{A(t)}{M(t)}c_2.$$

This completes the proof.

Theorem 2.1.7. Let $\gamma : I \to \mathbb{H}^3$ be a hyperbolic Frenet-type framed base curve with Frenet frame $\{V_1, V_2, V_3, V_4\}$. Under the assumptions that $A(t), N(t), M(t)$ are smooth functions, the curve γ is a $(2,4)$-type hyperbolic framed slant helix if and only if

$$\left(\frac{A(t)}{N(t)}\right)' c_4 - N(t)c_2 = 0, \quad \text{and} \quad \left(\frac{A(t)}{M(t)}\right)' c_2 - M(t)c_4 = 0.$$

Proof: Assume that γ is a $(2,4)$-type hyperbolic framed slant helix in $\mathbb{H}^3$ parametrized by t, with Frenet-type frame $\{V_1,V_2,V_3,V_4\}$. Then there exists a non-zero fixed vector $U \in \mathbb{R}^4_1$ such

$$<V_2,U>=c_2,\quad <V_4,U>=c_4 \tag{16}$$

where $c_2,c_4 \in \mathbb{R}$ are constants. Differentiating both expressions with respect to t, and using the Frenet-type system (1), we get:

$$\langle V_2'(t),U\rangle = N(t)\langle V_3(t),U\rangle + A(t)\langle V_4(t),U\rangle = N(t)\langle V_3(t),U\rangle + A(t)c_4,$$
$$\langle V_4'(t),U\rangle = M(t)\langle V_1(t),U\rangle - A(t)\langle V_2(t),U\rangle = M(t)\langle V_1(t),U\rangle - A(t)c_2.$$

Now, solve for $\langle V_3(t),U\rangle$ and $\langle V_1(t),U\rangle$:

$$\langle V_3(t),U\rangle = -\frac{A(t)}{N(t)}c_4, \quad \text{(assuming } N(t)\neq 0),$$
$$\langle V_1(t),U\rangle = \frac{A(t)}{M(t)}c_2, \quad \text{(assuming } M(t)\neq 0).$$

Now differentiate these expressions with respect to t:

1. For $\langle V_3(t),U\rangle$:

$$\frac{d}{dt}\langle V_3(t),U\rangle = \left(-\frac{A(t)}{N(t)}c_4\right)' = -\left(\frac{A(t)}{N(t)}\right)' c_4.$$

But from the Frenet-type equation,

$$\langle V_3'(t),U\rangle = -N(t)\langle V_2(t),U\rangle = -N(t)c_2.$$

So we get:

$$-\left(\frac{A(t)}{N(t)}\right)' c_4 = -N(t)c_2, \text{ so } \left(\frac{A(t)}{N(t)}\right)' c_4 - N(t)c_2 = 0.$$

2. For $\langle V_1(t),U\rangle$:

$$\frac{d}{dt}\langle V_1(t),U\rangle = \left(\frac{A(t)}{M(t)}c_2\right)' = \left(\frac{A(t)}{M(t)}\right)' c_2,$$

But from Frenet system:

$$\langle V_1'(t),U\rangle = M(t)\langle V_4(t),U\rangle = M(t)c_4.$$

Thus,

$$\left(\frac{A(t)}{M(t)}\right)' c_2 - M(t)c_4 = 0.$$

Hence, both conditions are satisfied, and the proof is complete.

Example 2.1.7. Let us consider the curve

$$\gamma(t)=\begin{pmatrix}\sinh t\\0\\0\\\cosh t\end{pmatrix},\quad t\in\mathbb{R}.$$

This curve lies on the hyperbolic space $\mathbb{H}^3\subset\mathbb{R}^4_1$, because

$$\langle\gamma(t),\gamma(t)\rangle_{\mathbb{R}^4_1}=-(\cosh t)^2+(\sinh t)^2=-1.$$

Let us define the Frenet-type moving frame associated with γ as:

$$V_1(t)=\gamma(t),\quad V_2(t)=\gamma'(t)=\begin{pmatrix}\cosh t\\0\\0\\\sinh t\end{pmatrix},\quad V_4(t)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ \text{(constant unit vector)},$$

$V_3(t)=$ defined by Gram-Schmidt process to complete the orthonormal frame.

We define the curvature functions as constant:

$$A(t)=1,\quad N(t)=1,\quad M(t)=1.$$

From Theorem 2.1.7, the conditions for a $(2,4)$-type hyperbolic framed slant helix are:

$$\left(\frac{A(t)}{N(t)}\right)'c_4-N(t)c_2=0,\quad \left(\frac{A(t)}{M(t)}\right)'c_2-M(t)c_4=0.$$

Substituting $A(t)=N(t)=M(t)=1$, we compute:

$$c_2=0\ \text{ and }\ c_4=0.$$

Thus, the fixed vector $U\in\mathbb{R}^4_1$ satisfies

$$\langle V_2(t),U\rangle=0,\quad\langle V_4(t),U\rangle=0,$$

then, U is orthogonal to both $V_2(t)$ and $V_4(t)$, which satisfies the requirements for $\gamma(t)$ to be a $(2,4)$-type hyperbolic framed slant helix.

Theorem 2.1.8. Let $\gamma:I\to\mathbb{H}^3$ be a hyperbolic Frenet-type framed base curve with Frenet frame $V_1(t),V_2(t),V_3(t),V_4(t)$. Suppose that

$$\langle V_3(t),U\rangle=c_3,\quad\langle V_4(t),U\rangle=c_4$$

are constants for a non-zero fixed vector $U \in \mathbb{R}_1^4$, then γ is a (3,4)-type hyperbolic framed slant helix. Then the function $M(t)$ satisfies:
(i) If N(t) = 0 and $\langle V_2(t),U\rangle \neq 0$, then $M(t)$ = 0,
(ii) If $N(t) \neq 0$ and $\langle V_2(t),U\rangle = 0$, then $M(t)$ = 0,
(iii) If N(t) = 0 and $\langle V_2(t),U\rangle = 0$, then $M(t)$ = 0.

Proof: Assume that γ is a ((3,4))-type hyperbolic framed slant helix. Then:

$$\langle V_3(t),U\rangle = c_3, \quad \langle V_4(t),U\rangle = c_4 \quad \text{(constants).}$$

Differentiating both identities and using the Frenet-type equations:

$$V_3'(t) = -N(t)V_2(t), \quad V_{4'}(t) = M(t)V_1(t) - A(t)V_2(t),$$

we get:

$$\begin{aligned} \langle V_3'(t),U\rangle &= -N(t)\langle V_2(t),U\rangle = 0, \\ \langle V_4'(t),U\rangle &= M(t)\langle V_1(t),U\rangle - A(t)\langle V_2(t),U\rangle = 0. \end{aligned} \tag{18}$$

From Equation (17), we obtain the essential condition:

$$N(t)\cdot\langle V_2(t),U\rangle = 0.$$

This condition gives rise to two mutually exclusive scenarios: Either N(t) = 0 , or $\langle V_2(t),U\rangle = 0$. These are structural conditions arising from the constant nature of $\langle V_3(t),U\rangle = 0$. Therefore, $\langle V_2(t),U\rangle = 0$ is not assumed, but rather a necessary conclusion when $N(t) \neq 0$, to maintain the constancy of $\langle V_3(t),U\rangle = 0$.

Case i: If N(t) = 0 and $\langle V_2(t),U\rangle \neq 0$.

From Equation (18):

$$M(t)\langle V_1(t),U\rangle = A(t)\langle V_2(t),U\rangle.$$

Differentiate both sides (since both inner products may be variable unless otherwise known), and use the chain rule:

$$\begin{aligned} \frac{d}{dt}\langle V_1(t),U\rangle &= M(t)\langle V_4(t),U\rangle = M(t)c_4, \\ \frac{d}{dt}\langle V_2(t),U\rangle &= N(t)\langle V_3(t),U\rangle + A(t)\langle V_4(t),U\rangle = A(t)c_4. \end{aligned}$$

However, N(t) = 0, and we can assume $A(t) \neq 0$, so if the only way Equation (18) balances is if M(t) = 0 (as no non-zero constant can balance the changing right-hand side), we conclude M(t) = 0.

Case ii: $N(t) \neq 0$, $\langle V_2(t),U\rangle = 0$

Then from Equation (17): $\langle V_3'(t),U\rangle = -N(t)\cdot 0 = 0$, satisfied. From Equation (18):

$$M(t)\langle V_1(t),U\rangle = 0.$$

Assuming $\langle V_1(t),U\rangle \neq 0$, we conclude again: M(t) = 0.

Case iii: N(t) = 0 , $\langle V_2(t),U\rangle = 0$

Equation (17) is automatically satisfied. Equation (18) becomes:

$$M(t)\langle V_1(t),U\rangle = 0 \Rightarrow M(t) = 0 \quad (\text{if } \langle \mathrm{V}_1, \mathrm{U}\rangle \neq 0).$$

3. RESULTS AND DISCUSSION

In this study, we introduced the concept of (k,m)-type slant helices in hyperbolic 3-space and investigated their fundamental properties. By employing the Frenet-type formulas for hyperbolic framed curves, we derived a system of differential equations that characterizes these helices.

Fundamental criteria for classical slant helices have been established by earlier research on Euclidean framed helices. Our study, however, applies these conclusions to the hyperbolic context and demonstrates that the nature of slant helices is strongly impacted by hyperbolic curvature. Because of the characteristics of the hyperbolic metric, hyperbolic framed slant helices are subject to extra geometric limitations in contrast to Euclidean framed helices.

4. CONCLUSIONS

This paper, using the moving frame $\{V_1, V_2, V_3, V_4\}$ and its associated Frenet-type system of equations allowed us to express the curvature and torsion functions of the (k,m)-type hyperbolic framed slant helices in terms of the hyperbolic frame vectors. We observed that these helices are governed by specific curvature conditions, which depend on the functions $M(t), N(t),$ and $A(t)$. The (k,m)-type hyperbolic framed slant helices satisfy a well-structured system of differential equations, which dictates their geometric behavior.

The hyperbolic curvature function significantly influences the trajectory and shape of these helices within hyperbolic 3-space.

The presence of the additional frame vectors $V_1, V_2, V_3,$ and V_4 provides a deeper understanding of how these curves behave in a hyperbolic setting compared to classical Euclidean helices. The results presented in this study provide a foundation for future research on hyperbolic framed curves. Possible extensions include:

Investigating the role of (k,m)-type hyperbolic slant helices in theoretical physics, particularly in the study of space-time models where hyperbolic geometry is relevant. Exploring generalizations of these helices in higher-dimensional hyperbolic spaces. Analyzing the stability properties of these helices in applied fields such as computer graphics and mechanical structures.

Our findings contribute to the broader field of differential geometry by providing new insights into the behavior of framed helices in hyperbolic space and opening avenues for further theoretical and applied research.

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