ORIGINAL PAPER

ON SOME ALGEBRAIC PROPERTIES OF THE HYPER-DUAL BICOMPLEX JACOBSTHAL QUATERNIONS

CENNET ÇİMEN[1]

Manuscript received: 13.12.2024; Accepted paper: 20.11.2025;

Published online: 30.12.2025.

Abstract. *In this article, we define the hyper-dual bicomplex Jacobsthal quaternions. We give some properties of these quaternions. Moreover, we investigate the Binet formula and calculate Vajda's identity, Catalan's identity, Cassini's identity, d'Ocagne's identity, and generating functions for hyper-dual bicomplex Jacobsthal quaternions.*

Keywords: *Quaternion numbers; Dual numbers; Hyper-Dual numbers; Bicomplex numbers; Jacobsthal numbers.*

Mathematics Subject Classification (2020): *15A66; 13N10; 16S80; 30G35; 11B39*

1. INTRODUCTION

Clifford (1873) [1] defined the set of dual numbers as

$$D = \{D = x + x^*\varepsilon | x, \quad x^* \in \mathbb{R}\} \tag{1}$$

where ϵ is the dual unit with $\varepsilon \neq 0$, $\varepsilon^2 = 0$. For $d_1 = x + x^*\varepsilon$ and $d_2 = y + y^*\varepsilon$ are given as $d_1 + d_2 = (x + y) + (x^* + y^*)\varepsilon$ and $d_1 d_2 = (xy) + (x^*y +)\varepsilon$ respectively.

The set of hyper dual numbers given as

$$\mathbb{H}D = \{D = a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2 | a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where the dual units $\varepsilon_1, \varepsilon_2$ satisfy the following rules:

$$\varepsilon_1{}^2 = \varepsilon_2{}^2 = (\varepsilon_1\varepsilon_2)^2 = 0, \qquad \varepsilon_1 \neq 0,\ \varepsilon_2 \neq 0,\ \varepsilon_1 \neq \varepsilon_2, \qquad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0. \tag{2}$$

Hyper-dual numbers form a 4-dimensional vector space over $\mathbb{R}$, with basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2\}$. Also, a hyper-dual number D can be written as $D = d + d^*\varepsilon^*$ where $d = a_0 + a_1\varepsilon, d^* = a_2 + a_3\varepsilon \in D$ and $\varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^*$. For any two hyper-dual numbers $D_1 = d_1 + d_1^*\varepsilon^*$ and $D_2 = d_2 + d_2^*\varepsilon^*$ defined as:

$$D_1 + D_2 = (d_1 + d_2) + (d_1^* + d_2^*)\varepsilon^* \text{ and } D_1D_2 = d_1d_2 + (d_1d_2^* + d_1^*d_2)\varepsilon^*$$

respectively. For applications of hyper-dual numbers, see [2-6].

Aslan [7] defined the hyper-dual split quaternions as

[1] Hacettepe University, Hacettepe Ankara Chamber of Industry 1st Organized Industrial Zone Vocational School, 06938 Ankara, Turkey. Email: cennet.cimen@hacettepe.edu.tr.

https://doi.org/10.46939/J.Sci.Arts-25.4-a04

$$q = D_0 + D_1 i + D_2 j + D_3 k$$

where $D_0, D_1, D_2, D_3 \in \mathbb{H}D$ and i, j, k have the split quaternion multiplication property $i^2 = -1,\ j^2 = k^2 = ijk = 1$. Bicomplex numbers were introduced by Segre [8]. Price [9] presented bicomplex numbers based on multi-complex spaces and functions. The set of $B\mathbb{C}$ (Bicomplex numbers) is a two-dimensional algebra over complex numbers ($\mathbb{C}$) since the set of complex numbers ($\mathbb{C}$) is two-dimensional algebra over real numbers ($\mathbb{R}$). Therefore, the set of bicomplex numbers is a four-dimensional algebra over $\mathbb{R}$.
The set of $B\mathbb{C}$ (Bicomplex numbers) is given with

$$B\mathbb{C} = \{x = (x_1 + ix_2) + j(x_3 + ix_4) | x_1, x_2, x_3 x_4 \in \mathbb{R}\}$$

where i, j and ij satisfy the conditions

$$i^2 = -1, \qquad j^2 = -1, \qquad ij = ji. \tag{3}$$

The set of $B\mathbb{C}$ (Bicomplex numbers) is a real vector space for scalar multiplication and addition. The vector space $B\mathbb{C}$ equipped with a bicomplex product is a real associative algebra. Also, A set of bicomplex numbers $B\mathbb{C}$ together with the properties of multiplication, is a commutative algebra, see the works [10-12], among others. Furthermore, there are three different conjugations of bicomplex numbers [13-14] as follows:

$$\begin{gathered} x = x_1 + ix_2 + jx_3 + ijx_4 = (x_1 + ix_2) + j(x_3 + ix_4) \in B\mathbb{C}, \\ x_i^* = x_1 - ix_2 + jx_3 - ijx_4 \\ x_j^* = x_1 + ix_2 - jx_3 - ijx_4 \\ x_{ij}^* = x_1 - ix_2 - jx_3 + ijx_4 \end{gathered}$$

Therefore, the norms of these numbers are given by

$$\begin{aligned} N_{x_i} &= \|x \times x_i{}^*\| = \sqrt{|x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2j(x_1 x_3 + x_2 x_4)|} \\ N_{x_j} &= \|x \times x_j{}^*\| = \sqrt{|x_1^2 - x_2^2 + x_3^2 - x_4^2 + 2i(x_1 x_2 + x_3 x_4)|} \\ N_{x_{ij}} &= \|x \times x_{ij}{}^*\| = \sqrt{|x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2ij(x_1 x_4 + x_2 x_3)|} \end{aligned}$$

Quaternions were defined by Hamilton [15] as an extension of the complex numbers. In recent years, there has been an increasing interest in problems of some algebras, especially in the quaternion field. Since many algebraic problems on quaternion fields were encountered in some applied sciences, like quantum physics, analysis, differential geometry, and geostatics. In general, a quaternion is given by the following Equation;

$$q = a_0 + a_1 i + a_2 j + a_3 k = (a_0, a_1, a_2, a_3)$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and $1,\ i,\ j,\ k$ that satisfy the following multiplication rules:

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \\ ki = -ik = j. \end{gathered} \tag{4}$$

Similar to the quaternions, a dual quaternion $\tilde{q}$ is defined as

$$\tilde{q} = d_0 + d_1 i + d_2 j + d_3 k$$

where $d_0, d_1, d_2, d_3 \in D$ and the elements $i,\ j,\ k$ satisfies the quaternion multiplication rule:

$$i^2 = j^2 = k^2 = ijk = -1. \quad (5)$$

Kilic [16] introduced the dual Horadam quaternions as

$$\widetilde{Q_{w,n}} = \widetilde{w_n} + \widetilde{w_{n+1}}\, i + \widetilde{w_{n+2}}\, j + \widetilde{w_{n+3}}\, k,$$

where $\widetilde{w_n} = w_n + w_{n+1}\,\varepsilon$ is the nth dual Horadam number. Recently, Omur et al. [17] defined the hyper-dual numbers whose coefficients are from the sequences $U_{k,n}$ and $V_{k,n}$ which reduces to the sequences $\{w_n\,(0,1;\,p,1)\}$ and $\{w_n\,(2,p;\,p,1)\}$ for $k = 1$, respectively. Tan et al. [18] defined the hyper-dual Horadam quaternion numbers. They investigated the algebraic properties of these quaternions. Horadam [19-20] defined the Jacobsthal sequences $\{J_n\}$ with the recurrence relation:

$$J_{n+1} = J_n + 2J_{n-1}, \qquad n \geq 1 \quad (6)$$

with initial conditions $J_0 = 0,\ J_1 = 1.$ The Binet-like formula for Jacobsthal sequences is given as:

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (7)$$

where $\alpha = 2$ and $\beta = -1$, are the roots of the characteristic Equation $x^2 - x - 2 = 0$. Thus, note that $\alpha + \beta = 1,\ \alpha\beta = -2$ and $\alpha - \beta = 3$. Horadam [21] introduced and studied the Fibonacci quaternion sequence; also, these sequences have been studied in several papers (see, for example, [22-26]). Moreover, generalizations of the Fibonacci quaternions have been given in the literature (see, for example, [27-30]). Szynal-Liana and Wloch [31] investigated the Jacobsthal quaternions and obtained some properties of these quaternions. Aydin and Yüce [32] defined some properties of Jacobsthal quaternions. Also, they investigated the relations between the Jacobsthal quaternions, which are connected with Jacobsthal and Jacobsthal-Lucas numbers. Aydın [33] defined dual Jacobsthal quaternions. Also, the relations between dual Jacobsthal quaternions, which are connected with Jacobsthal and Jacobsthal-Lucas numbers investigated, and some identities for these quaternions. Halıcı [34] introduced a sequence of bicomplex numbers whose coefficients are chosen from the sequence of Jacobsthal-Lucas numbers and presented some identities of the bicomplex Jacobsthal-Lucas sequences. Aydın [35] defined the bicomplex Fibonacci quaternions and presented some identities of these quaternions. Catarino [36] investigated the bicomplex k-Pell quaternions. Kızılateş¸ et al. [37] investigated the bicomplex generalized tribonacci quaternions. They obtained Binet's formula, generating functions, and the summation formula for these quaternions. Also, they obtained the determinant of a special matrix. Gül [38] gave the dual bicomplex horadam quaternions and presented some properties of this sequence. In this study, we aim to contribute to the development of bicomplex numbers and hyper dual numbers. We investigate a new generalization of hyper dual quaternions called hyper dual bicomplex Jacobsthal quaternions. We define the hyper-dual bicomplex Jacobsthal quaternions and present Binet-like formula, Cassini's identity, Catalan's identity, Vajda's identity, d'Ocagne's identity, and generating functions for hyper-dual bicomplex Jacobsthal quaternions.

2. ALGEBRAIC PROPERTIES OF THE HYPER-DUAL BICOMPLEX JACOBSTHAL QUATERNIONS

In this section, we introduce a different generalization of hyper dual quaternions and give some properties for the hyper-dual bicomplex Jacobsthal quaternion; thus, we give some necessary definitions and concepts.

Definition 2.1. The nth hyper-dual Jacobsthal number is defined as

$$\mathbb{H}D_n^J = J_n + J_{n+1}\varepsilon_1 + J_{n+2}\varepsilon_2 + J_{n+3}\varepsilon_1\varepsilon_2 \quad (8)$$

where J_n is the nth Jacobsthal number and $\varepsilon_1, \varepsilon_2$ satisfies the multiplication rule in (2).

Theorem 2.1. For $n \geq 0$, we have

$$\mathbb{H}D_n^J = \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}, \quad (9)$$

where $\underline{\alpha} = 1 + 2\varepsilon_1 + 4\varepsilon_2 + 8\varepsilon_1\varepsilon_2$, $\underline{\beta} = 1 - \varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2$; α, β are roots of the Equation

$$x^2 - x - 2 = 0, \text{ i.e } \alpha = 2, \ \beta = -1.$$

Proof: From the equalities (7) and (8), we obtain

$$\begin{aligned} \mathbb{H}D_n^J &= J_n + J_{n+1}\varepsilon_1 + J_{n+2}\varepsilon_2 + J_{n+3}\varepsilon_1\varepsilon_2 \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\varepsilon_1 + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\varepsilon_2 + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\varepsilon_1\varepsilon_2 \\ &= \frac{1}{\alpha - \beta}[\alpha^n(1 + 2\varepsilon_1 + 4\varepsilon_2 + 8\varepsilon_1\varepsilon_2) - \beta^n(1 - \varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2)] \\ &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta}. \end{aligned}$$

Definition 2.2. The nth hyper-dual Jacobsthal quaternion number is defined as

$$\mathbb{H}D_n^{JQ} = \mathbb{H}D_n^J + \mathbb{H}D_{n+1}^J i + \mathbb{H}D_{n+2}^J j + \mathbb{H}D_{n+3}^J k \quad (10)$$

here $\mathbb{H}D_n^J$ is the nth hyper-dual Jacobsthal number, and i, j, k are basic quaternions in (4).

Theorem 2.2. For $n \geq 0$, we have

$$\mathbb{H}D_n^{JQ} = \frac{\underline{\alpha}\alpha^n A - \underline{\beta}\beta^n B}{\alpha - \beta} \quad (11)$$

where $A = 1 + 2i + 4j + 8k$, $B = 1 - i + j - k$.

Proof: From the equalities (9) and (10), the proof is easily seen.

Definition 2.3. The nth bicomplex Jacobsthal quaternion number is defined as

$$B\mathbb{C}_n^{JQ} = J_n + J_{n+1}i + J_{n+2}j + J_{n+3}ij \tag{12}$$

where $\mathrm{J_n}$ is the nth Jacobsthal number and $\mathrm{i, j, ij}$ are properties of the multiplication property in Equation (3).

Theorem 2.3. For $\mathrm{n} \geq 0$, we have

$$B\mathbb{C}_n^{JQ} = \frac{\alpha^n \alpha^* - \beta^n \beta^*}{\alpha - \beta} \tag{13}$$

where $\alpha^* = 1 + 2i + 4j + 8ij,\ \ \beta^* = 1 - i + j - ij.$

Proof: By using the equalities (7) and (12), we obtain

$$\begin{aligned} B\mathbb{C}_n^{JQ} &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} i + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} j + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} ij \\ &= \tfrac{1}{\alpha-\beta}\left[\alpha^n(1 + 2i + 4j + 8ij) - \beta^n(1 - i + j - ij)\right]. \end{aligned}$$

Definition 2.4. The nth hyper-dual bicomplex Jacobsthal quaternions $\left\{\mathbb{H}\mathrm{DB}\mathbb{C}_\mathrm{n}^\mathrm{JQ}\right\}_{\mathrm{n=0}}^{\infty}$ are defined by the following:

$$\mathbb{H}DB\mathbb{C}_n^{JQ} = \mathbb{H}D_n^{JQ} + \mathbb{H}D_{n+1}^{JQ} i + \mathbb{H}D_{n+2}^{JQ} j + \mathbb{H}D_{n+3}^{JQ} ij \tag{14}$$

where $\mathbb{H}D_n^{JQ}$ is the nth hyper-dual Jacobsthal quaternion number and where $i^2 = -1,\ j^2 = -1,\ ij = ji.$

By using hyper dual Jacobsthal numbers $\mathbb{H}D_n^{JQ}$, we get

$$\begin{aligned} \mathbb{H}DB\mathbb{C}_n^{JQ} &= \mathbb{H}D_n^{JQ} + \mathbb{H}D_{n+1}^{JQ} i + \mathbb{H}D_{n+2}^{JQ} j + \mathbb{H}D_{n+3}^{JQ} ij \\ &= (J_n + J_{n+1}\varepsilon_1 + J_{n+2}\varepsilon_2 + J_{n+3}\varepsilon_1\varepsilon_2) \\ &\quad +(J_{n+1} + J_{n+2}\varepsilon_1 + J_{n+3}\varepsilon_2 + J_{n+4}\varepsilon_1\varepsilon_2)i \\ &\quad +(J_{n+2} + J_{n+3}\varepsilon_1 + J_{n+2}\varepsilon_2 + J_{n+3}\varepsilon_1\varepsilon_2)j \\ &\quad +(J_{n+3} + J_{n+4}\varepsilon_1 + J_{n+5}\varepsilon_2 + J_{n+6}\varepsilon_1\varepsilon_2)ij. \end{aligned}$$

From Equation (14), it is obvious that $\mathbb{H}DB\mathbb{C}_n^{JQ} = \mathbb{H}DB\mathbb{C}_{n-1}^{JQ} + 2\mathbb{H}DB\mathbb{C}_{n-2}^{JQ}$ with the initial conditions

$$\begin{aligned} \mathbb{H}DB\mathbb{C}_0^{JQ} &= (\varepsilon_1 + \varepsilon_2 + 3\varepsilon_1\varepsilon_2) + (1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2)i \\ &\quad +(1 + 3\varepsilon_1 + 5\varepsilon_2 + 11\varepsilon_1\varepsilon_2)j + (3 + 5\varepsilon_1 + 11\varepsilon_2 + 21\varepsilon_1\varepsilon_2)ij, \\ \mathbb{H}DB\mathbb{C}_1^{JQ} &= (1 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_1\varepsilon_2) + (1 + 3\varepsilon_1 + 5\varepsilon_2 + 11\varepsilon_1\varepsilon_2)i \\ &\quad +(3 + 5\varepsilon_1 + 11\varepsilon_2 + 21\varepsilon_1\varepsilon_2)j + (5 + 11\varepsilon_1 + 21\varepsilon_2 + 43\varepsilon_1\varepsilon_2)ij. \end{aligned}$$

For two hyper-dual bicomplex Jacobsthal quaternions $\mathbb{H}DB\mathbb{C}_n^{JQ}$ and $\mathbb{H}DB\mathbb{C}_m^{JQ}$ addition and subtraction as follows:

$$\mathbb{H}DB\mathbb{C}_n^{JQ} \pm \mathbb{H}DB\mathbb{C}_m^{JQ} = \left(\mathbb{H}D_n^{JQ} \pm \mathbb{H}D_m^{JQ}\right) + \left(\mathbb{H}D_{n+1}^{JQ} \pm \mathbb{H}D_{m+1}^{JQ}\right)i$$

$$+\left(\mathbb{H}D_{n+2}^{JQ} \pm \mathbb{H}D_{m+2}^{JQ}\right)j + \left(\mathbb{H}D_{n+3}^{JQ} \pm \mathbb{H}D_{m+3}^{JQ}\right)ij,$$

and multiplication by

$$\begin{aligned}
&\mathbb{H}DB\mathbb{C}_n^{JQ} \times \mathbb{H}DB\mathbb{C}_m^{JQ} = \\
&= (\mathbb{H}D_n^{JQ} + \mathbb{H}D_{n+1}^{JQ}i + \mathbb{H}D_{n+2}^{JQ}j + \mathbb{H}D_{n+3}^{JQ}ij) \times (\mathbb{H}D_m^{JQ} + \mathbb{H}D_{m+1}^{JQ}i + \mathbb{H}D_{m+2}^{JQ} + \mathbb{H}D_{m+3}^{JQ}ij) \\
&= \mathbb{H}D_n^{JQ}\mathbb{H}D_m^{JQ} - \mathbb{H}D_{n+1}^{JQ}\,\mathbb{H}D_{m+1}^{JQ} - \mathbb{H}D_{n+2}^{JQ}\mathbb{H}D_{m+2}^{JQ} + \mathbb{H}D_{n+3}^{JQ}\mathbb{H}D_{m+3}^{JQ} \\
&+\left(\mathbb{H}D_n^{JQ}\,\mathbb{H}D_{m+1}^{JQ} + \mathbb{H}D_{n+1}^{JQ}\,\mathbb{H}D_m^{JQ} - \mathbb{H}D_{n+3}^{JQ}\mathbb{H}D_{m+2}^{JQ} - \mathbb{H}D_{n+2}^{JQ}\mathbb{H}D_{m+3}^{JQ}\right)i \\
&+\left(\mathbb{H}D_n^{JQ}\,\mathbb{H}D_{m+2}^{JQ} + \mathbb{H}D_{n+2}^{JQ}\,\mathbb{H}D_m^{JQ} - \mathbb{H}D_{n+1}^{JQ}\mathbb{H}D_{m+3}^{JQ} - \mathbb{H}D_{n+3}^{JQ}\mathbb{H}D_{m+1}^{JQ}\right)j \\
&+\left(\mathbb{H}D_n^{JQ}\,\mathbb{H}D_{m+3}^{JQ} + \mathbb{H}D_{n+1}^{JQ}\,\mathbb{H}D_{m+2}^{JQ} + \mathbb{H}D_{n+2}^{JQ}\mathbb{H}D_{m+1}^{JQ} + \mathbb{H}D_{n+3}^{JQ}\mathbb{H}D_m^{JQ}\right)ij.
\end{aligned}$$

The multiplication by the real scalar λ of the hyper-dual bicomplex Jacobsthal quaternions is given by the following:

$$\lambda\mathbb{H}DB\mathbb{C}_n^{JQ} = \lambda\mathbb{H}D_n^{JQ} + \lambda\mathbb{H}D_{n+1}^{JQ}i + \lambda\mathbb{H}D_{n+2}^{JQ}j + \lambda\mathbb{H}D_{n+3}^{JQ}ij.$$

The different conjugations for hyper-dual bicomplex Jacobsthal quaternions are defined as follows:

$$\begin{aligned}
\left(\mathbb{H}DB\mathbb{C}_n^{JQ}\right)_i^* &= \mathbb{H}D_n^{JQ} - \mathbb{H}D_{n+1}^{JQ}i + \mathbb{H}D_{n+2}^{JQ}j - \mathbb{H}D_{n+3}^{JQ}ij \\
\left(\mathbb{H}DB\mathbb{C}_n^{JQ}\right)_j^* &= \mathbb{H}D_n^{JQ} + \mathbb{H}D_{n+1}^{JQ}i - \mathbb{H}D_{n+2}^{JQ}j - \mathbb{H}D_{n+3}^{JQ}ij \\
\left(\mathbb{H}DB\mathbb{C}_n^{JQ}\right)_{ij}^* &= \mathbb{H}D_n^{JQ} - \mathbb{H}D_{n+1}^{JQ}i - \mathbb{H}D_{n+2}^{JQ}j + \mathbb{H}D_{n+3}^{JQ}ij.
\end{aligned}$$

The norm of a hyper-dual bicomplex Jacobsthal quaternions is defined as

$$\begin{aligned}
&\mathbb{H}DB\mathbb{C}_n^{JQ}\overline{HDBC_n^{JQ}} = \overline{HDBC_n^{JQ}}\mathbb{H}DB\mathbb{C}_n^{JQ} \\
&= \left(\mathbb{H}D_n^{JQ}\right)^2 + \left(\mathbb{H}D_{n+1}^{JQ}\right)^2 + \left(\mathbb{H}D_{n+2}^{JQ}\right)^2 + \left(\mathbb{H}D_{n+3}^{JQ}\right)^2
\end{aligned}$$

Theorem 2.4. For $n \geq 2$, the hyper-dual bicomplex Jacobsthal quaternions satisfy the following relation:

$$\mathbb{H}DB\mathbb{C}_n^{JQ} = \mathbb{H}DB\mathbb{C}_{n-1}^{JQ} + 2\mathbb{H}DB\mathbb{C}_{n-2}^{JQ} \tag{15}$$

Proof: By using of the equalities (6) and (10), we obtain

$$\begin{aligned}
\mathbb{H}DB\mathbb{C}_n^{JQ} &= \mathbb{H}D_{n-1}^{JQ} + \mathbb{H}D_n^{JQ}i + \mathbb{H}D_{n+1}^{JQ}j + \mathbb{H}D_{n+2}^{JQ}ij \\
&\quad +2(\mathbb{H}D_{n-2}^{JQ} + \mathbb{H}D_{n-1}^{JQ}i + \mathbb{H}D_n^{JQ}j + \mathbb{H}D_{n+1}^{JQ}ij) \\
&= \mathbb{H}D_n^{JQ} + \mathbb{H}D_{n+1}^{JQ}i + \mathbb{H}D_{n+2}^{JQ}j + \mathbb{H}D_{n+3}^{JQ}ij.
\end{aligned}$$

The following results give the Binet-like formula for the hyper-dual bicomplex Jacobsthal quaternions.

Theorem 2.5. For $n \geq 0$, we have

$$\mathbb{H}DB\mathbb{C}_n^{JQ} = \frac{\underline{\alpha}\alpha^n\alpha^* - \underline{\beta}\beta^n\beta^*}{\alpha - \beta} \tag{16}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are hyper dual numbers defined by $\underline{\alpha} = 1 + \alpha\varepsilon_1 + \alpha^2\varepsilon_2 + \alpha^3\varepsilon_1\varepsilon_2$, $\beta = 1 + \beta\varepsilon_1 + \beta^2\varepsilon_2 + \beta^3\varepsilon_1\varepsilon_2$, $\alpha^*$and $\beta^*$are bicomplex number is defined by $\alpha^* = 1 + \alpha i + \alpha^2 j + \alpha^3 ij$, $\beta^* = 1 + \beta i + \beta^2 j + \beta^3 ij$.

Proof: From the equalities (10) and (11), the proof is easily seen.

$$\begin{aligned}\mathbb{H}DB\mathbb{C}_n^{JQ} &= \frac{\underline{\alpha}\alpha^n - \underline{\beta}\beta^n}{\alpha - \beta} + \frac{\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}}{\alpha - \beta} i + \frac{\underline{\alpha}\alpha^{n+2} - \underline{\beta}\beta^{n+2}}{\alpha - \beta} j + \frac{\underline{\alpha}\alpha^{n+3} - \underline{\beta}\beta^{n+3}}{\alpha - \beta} ij \\ &= \frac{1}{\alpha - \beta}\left[\underline{\alpha}\alpha^n(1 + \alpha i + \alpha^2 j + \alpha^3 ij) - \underline{\beta}\beta^n(1 + \beta i + \beta^2 j + \beta^3 ij)\right] \\ &= \frac{\underline{\alpha}\alpha^n\alpha^* - \underline{\beta}\beta^n\beta^*}{\alpha - \beta}.\end{aligned}$$

Since the set of hyper-dual numbers and bicomplex numbers is a commutative algebra, $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$ and $\alpha^*\beta^* = \beta^*\alpha^*$. As a result of the Binet-like formula given in (16), we obtain Vajda's identity, d'Ocagne's identity, Cassini's, and Catalan's identities for the sequence of hyper-dual bicomplex Jacobsthal quaternions.

Theorem 2.6 (Vajda's identity)**.** For integers n, r and s, the following identity is true:

$$\mathbb{H}DB\mathbb{C}_{n+r}^{JQ}\mathbb{H}DB\mathbb{C}_{n+s}^{JQ} - \mathbb{H}DB\mathbb{C}_n^{JQ}\mathbb{H}DB\mathbb{C}_{n+r+s}^{JQ} = \frac{\underline{\alpha}\underline{\beta}\alpha^*\beta^*(-2)^n(\alpha^r - \beta^r)(\alpha^s - \beta^s)}{(\alpha - \beta)^2}. \tag{17}$$

Proof: From the equalities (11), the proof is obvious.

From the Vajda's identity, we have the following results:
By replacing r with $-s$ in Equation (17), we get Catalan's identity for hyper-dual Jacobsthal quaternions:

$$\mathbb{H}DB\mathbb{C}_{n-s}^{JQ}\mathbb{H}DB\mathbb{C}_{n+s}^{JQ} - \left(HDBC_n^{JQ}\right)^2 = \frac{-\underline{\alpha}\underline{\beta}\alpha^*\beta^*(-2)^{n-s}(\alpha^s - \beta^s)^2}{(\alpha - \beta)^2}. \tag{18}$$

By setting $s = -r = 1$ in Equation (17), we obtain Cassini's identity for hyper-dual Jacobsthal quaternions:

$$\mathbb{H}DB\mathbb{C}_{n-1}^{JQ}\mathbb{H}DB\mathbb{C}_{n+1}^{JQ} - \left(\mathbb{H}DB\mathbb{C}_n^{JQ}\right)^2 = -\underline{\alpha}\underline{\beta}\alpha^*\beta^*(-2)^{n-1}. \tag{19}$$

If we take $m - n$ instead of s and $r = 1$ in Equation (17), we obtain d'Ocagne's identity for hyper-dual Jacobsthal quaternions:

$$\mathbb{H}DB\mathbb{C}_{n+1}^{JQ}\mathbb{H}DB\mathbb{C}_m^{JQ} - \mathbb{H}DB\mathbb{C}_n^{JQ}\mathbb{H}DB\mathbb{C}_{m+1}^{JQ} = \frac{\underline{\alpha}\underline{\beta}\alpha^*\beta^*(-2)^n(\alpha^{m-n} - \beta^{m-n})}{\alpha - \beta}.$$

Theorem 2.7. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n-1} \mathbb{H}DB\mathbb{C}_r^{JQ} = \frac{1}{2}[-\mathbb{H}DB\mathbb{C}_n^{JQ} - 2\mathbb{H}DB\mathbb{C}_{n-1}^{JQ} + 2\mathbb{H}DB\mathbb{C}_0^{JQ} + \mathbb{H}DB\mathbb{C}_1^{JQ}].$$

Proof: From the Binet-like formula of the hyper-dual bicomplex Jacobsthal quaternions, the proof is easily seen.

We now shall obtain the generating functions for the hyper-dual bicomplex Jacobsthal quaternion sequence. By using the definition of a generating function, $g_{\mathbb{H}DB\mathbb{C}_{n+1}^{JQ}}(x)$ is given by the following:

$$g_{\mathbb{H}DB\mathbb{C}_{n+1}^{JQ}}(x) = \sum_{n=0}^{\infty} \mathbb{H}DB\mathbb{C}_n^{JQ} x^n \tag{20}$$

Therefore, using (20), we have the following result.

Theorem 2.8. The generating function for the hyper-dual bicomplex Jacobsthal quaternion sequence is given by the following:

$$g_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x) = \frac{\mathbb{H}DB\mathbb{C}_0^{JQ} + (\mathbb{H}DB\mathbb{C}_1^{JQ} - \mathbb{H}DB\mathbb{C}_0^{JQ})x}{1 - x - 2x^2}$$

Proof: Firstly, we write a generating function for the hyper-dual bicomplex Jacobsthal quaternions;

$$g_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x) = \mathbb{H}DB\mathbb{C}_0^{JQ} x^0 + \mathbb{H}DB\mathbb{C}_1^{JQ} x^1 + \cdots + \mathbb{H}DB\mathbb{C}_n^{JQ} x^n + \cdots \tag{21}$$

Secondly, we calculate $-xg_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x)$ and $-2x^2 g_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x)$ as the following Equations;

$$-xg_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x) = -\mathbb{H}DB\mathbb{C}_0^{JQ} x - \mathbb{H}DB\mathbb{C}_1^{JQ} x^2 - \cdots - \mathbb{H}DB\mathbb{C}_n^{JQ} x^{n+1} + \cdots \tag{22}$$

and

$$-2x^2 g_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x) = -2\mathbb{H}DB\mathbb{C}_0^{JQ} x^2 - 2\mathbb{H}DB\mathbb{C}_1^{JQ} x^3 - \cdots - 2\mathbb{H}DB\mathbb{C}_n^{JQ} x^{n+2} + \cdots \tag{23}$$

Finally, adding Equations (21), (22), (23), and using Equation (15), we obtain

$$(1 - x - 2x^2) g_{\mathbb{H}DB\mathbb{C}_n^{JQ}}(x) = \mathbb{H}DB\mathbb{C}_0^{JQ} + (\mathbb{H}DB\mathbb{C}_1^{JQ} - \mathbb{H}DB\mathbb{C}_0^{JQ})x.$$

3. CONCLUSIONS

In this work, we define hyper dual bicomplex Jacobsthal quaternions. We give many identities that take an important place in the literature for hyper dual bicomplex Jacobsthal quaternions. We also obtain some well-known important identities for these quaternions. In this way, we also think that we will contribute to the literature on hyper dual bicomplex Jacobsthal quaternion numbers.

Acknowledgements. *We are thankful to the referees for their careful reading of the manuscript and for valuable comments and suggestions that greatly improved the presentation of this work.*

REFERENCES

[1] Clifford, W. K, *Proceedings of the London Mathematical Society*, **4**, 361, 1873.
[2] Cohen, A., Shoham, M., *J. Mech. Rob.*, **8(**1), 011005, 2015.
[3] Cohen, A., Shoham, M., *Mech. Mach.Theory.*, **111**, 76, 2017.
[4] Fike, J. A., *3rd Annual Student Joint Workshop in Simulation-Based 26 Engineering and Design*, 2009.
[5] Fike, J. A., Alonso J. J., *49th AIAA Aerodspace Sciences Meeting 28, including the New Horizons Forum and Aerospace Exposition*, Orlando, Florida, **4**, 2011.
[6] Fike, J. A., Alonso, J. J., *Mechanism and Machine Theory*, **125**, 101, 2018.
[7] Aslan, S., Bekar, M., Yayli, Y., *Journal of Geometry and Physics*, **158,**.2020.
[8] Segre, C., *Math. Ann.*, **40**, 413, 1892.
[9] Price, G. B., *An Introduction to Multicomplex Spaces and Functions. New York: Marcel Dekker Inc.*, 1991.
[10] Luna-Elizarrarás, M. E., Shapiro, M., Struppa, D. C., Vajiac, A., *CUBO A Math. J.*, **44**(2), 61, 2012.
[11] Rochon, D., Shapiro, M., *An. Univ. Oradea Fasc. Mat.*, **11**, 71, 2004.
[12] Dişkaya, O., Menken, H., *Acta Universitatis Apulensis, **73,*** 17, *2023.*
[13] Rochon, D., *Fractals*, **8**, 355, 2000.
[14] Karakus, S. O., Aksoyak, K. F., *Adv. Appl. Clifford Algebra*, **25**, 943, 2015.
[15] Hamilton, W. R., *Elements of quaternions. Longman,* Green and Co., London, 1866.
[16] Kilic, N., *An. Stiint. Univ. Al. I. Cuza Iasi. Mat.*, **101**, 2020.
[17] Omur, N., Koparal, S., *Notes on Number Theory and Discrete Mathematics*, **26**(1), 191, 2020.
[18] Tan, E., Ait-Amrane, N. R., Gök, I., *Miskolc Mathematical Notes*, **22(**2), 903, 2021.
[19] Horadam, A. F., *Fibonacci Quart.*, **26**, 79, 1988.
[20] Horadam, A. F., *The Fibonacci Quarterly*, **34**(1), 40, 1996.
[21] Horadam, A. F., *The American Mathematical Monthly*, **70**(3), 289, 1963.
[22] Horadam, A. F., *Duke Math J.*, **32**, 437, 1965.
[23] Halıcı, S., *arXiv:1611.07660*, 2016.
[24] Halıcı, S., *Adv. Appl. Clifford Algebras*, **22**(2), 321, 2012.
[25] Iyer, M. R., *Fibonacci Quart.*, **7(**2), 201, 1969.
[26] Iyer, M. R., *Fibonacci Quart.*, **7**(3), 225–229, 1969.
[27] Flaut, C., Shpakivskyi, V., *Adv. Appl. Clifford Algebras*, **23**(3), 673, 2013.
[28] Ramirez, J., *An. St. Univ. Ovidius Constanta*, **23**(2), 201, 2015.
[29] Swamy, M. N., *Fibonacci Quart.*, **11**, 547, 1973.
[30] Akyigit, M., Köksal, H. H., Tosun, M., *Adv. Appl. Clifford Algebras*, **24**, 631, 2014.
[31] Szynal-Liana, A. Wloch, I., *Adv. Appl. Clifford Algebras*, **26**, 441, 2016.
[32] Aydın, F. T., Yüce, S., *Filomat*, **31**,18, 5567, 2017.
[33] Aydın, F. T., *Communications in Advanced Mathematical Sciences*, **3(**3), 130, 2020.

[34] Halıcı, S., *Journal of Mathematical Sciences and Modelling*, **3**(3), 139, 2020.
[35] Aydın, F. T., *Chaos Solitons Fract.*, **106**, 147, 2018.
[36] Catarino, P., *Computational Methods and Function Theory*, **19**, 65, 2019.
[37] Kızılates, C., Catarino, P., Tuğlu, N., *Mathematics*, **7**(80), 2019.
[38] Gül, K., *Notes on Number Theory and Discrete Mathematics*, **26**(4), 187, 2020.