ORIGINAL PAPER

TWO PARAMETRIC GENERALIZATIONS OF HARVDA-CHARVAT'S ENTROPY AND ITS APPLICATION IN SOURCE CODING

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Abstract. *In this manuscript, we present a new two-parametric generalization of Harvda-Charvat's measure of entropy $H_\alpha^\beta(P)$ and its salient characteristics. We also obtain it's most significant entropies that are widely known and have a sway in the literature of information and coding theory. Furthermore, we also present a new generalized mean code-word length $L_\alpha^\beta(P)$, and we additionally ascertain how $H_\alpha^\beta(P)$ and $L_\alpha^\beta(P)$ are interrelated in terms of the source coding theorem and also demonstrate it through the Shannon-Fano Coding Algorithm. Finally, we check it's monotonicity through a dataset on precipitation.*
Keywords: *Shannon's entropy; Harvda-Charvat's entropy; Average length; L'Hospital's rule.*

1. INTRODUCTION

Information Theory is a mathematical field founded by Claude E. Shannon [1] aimed at analyzing the mechanisms involved in the transmission, storage, and measurement of information. It establishes systematic techniques for assessing how effectively messages can be communicated from a source to a destination through a communication channel, while accounting for challenges such as noise, interference, and signal degradation. Consider a discrete random variable X $=\{x_1, x_2, x_3, \dots, x_n\}$ with its respective probabilities P = $\{p_1, p_2, p_3, \dots, p_n\}$, then the concept of entropy is defined as:

$$H(P) = -\sum_{i=1}^{n} p_i \, log_D p_i \tag{1}$$

The unit of entropy is taken to the base of the logarithm D, if D = 2, then entropy measure is known as a bit; D = e, then entropy measure is known as Nat; and if D = 10, then entropy measure is known as Hartley. Numerous generalized measures of Shannon's entropy under a discrete random variable have been presented in the literature of information theory. Harvda-Charvat [2] presented an idea of parametric entropy and defined the entropy of order β as:

$$H^\beta(P) = \frac{1}{1-\beta}\left[\sum_{i=1}^{n} p_i{}^\beta - 1\right], \beta > 0, \beta \neq 1 \tag{2}$$

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Apart from Harvda-Charvat [2], various other researchers, viz., Rényi [3], Campbell, L. L. [4], Sharma and Mittal [5], Hooda, D. S., Bhaker, U. S. [6], Bhat and Baig [7-12], etc., have also developed some generalized measures in the theory of information.

2. A NEW TWO-PARAMETRIC GENERALIZATION OF HARVDA-CHARVAT'S MEASURE OF ENTROPY

Let X = $\{x_1, x_2, x_{3,\dots}, x_n\}$ with their respective probabilities P = $\{p_1, p_2, p_{3j} \dots, p_n\}$ then we derived a new two-parametric generalization of Harvda-Charvat's entropy H_α^β(P) is given by

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^n p_i^{\frac{\beta}{\alpha}} - 1\right], \alpha > 0, \beta > 0, \alpha \neq \beta \tag{3}$$

where α and β represent scaling factors that adjust how the entropy responds to different probability distributions. Let us now explicate α and β. From the application point

- Harvda-Charvat's entropy can be tuned to reduce the influence of rare or extreme events, unlike Shannon's entropy, which is very sensitive to small probabilities.
- Moreover, it is used to enhance signal extraction in a noisy environment by reducing the effect of random outliers.

2.1. PARTICULAR CASES OF OUR PROPOSED GENERALIZED HARVDA-CHARVAT'S MEASURE OF ENTROPY

a. When α = 1, equation (3) reduces to Harvda-Charvat's [2] entropy of order β i.e.,

$$H_{\alpha=1}^\beta(P) = H^\beta(\mathrm{P}) = \frac{1}{1-\beta}\left[\sum_{i=1}^n p_i^\beta - 1\right]$$

b. When β = 1, equation (3) reduces to Harvda-Charvat's [2] entropy of order $\frac{1}{\alpha}$ i.e.,

$$H_\alpha^{\beta=1}(P) = H_{\frac{1}{\alpha}}(\mathrm{P}) = \frac{1}{1-\frac{1}{\alpha}}\left[\sum_{i=1}^n p_i^{\frac{1}{\alpha}} - 1\right]$$

c. When $\beta = 2\alpha$, equation (3) reduces to Harvda-Charvat's quadratic entropy, i.e.

$$H^{\beta=2\alpha}(P) = H^2(P) = 1 - \sum_{i=1}^n p_i^2$$

d. When α = 1 and β → 1, then by applying L'Hospital's rule, equation (3) reduces to the entropy given by Shannon [1], i.e.,

$$H_{\alpha=1}^{\beta\to1}(P) = \mathrm{H(P)} = -\sum_{i=1}^n p_i log_D p_i$$

e. When $\beta = 1$ and α → 1, then applying L'Hospital's rule, equation (3) reduces to the entropy given by Shannon [1], i.e.,

$$H_{\alpha\to 1}^{\beta=1}(P) = \mathrm{H(P)} = -\sum_{i=1}^{n} p_i log_D p_i$$

f. When β→ α, then by applying L'Hospital's rule, equation (3) reduces to the entropy given by Shannon [1], i.e.,

$$H^{\beta\to\alpha}(P) = \mathrm{H(P)} = -\sum_{i=1}^{n} p_i log_D p_i$$

g. When β > 0, α > 0 and β ≠ α, and if all the events are equally likely, i.e., pi = $\frac{1}{n}, \forall i = 1,2,3,\dots,n$, then we have

$$H_{\alpha}^{\beta}\left(\frac{1}{n}\right) = \mathrm{H}\left(\frac{1}{n}\right) = log_D n\ ,$$

which is maximum entropy.

2.2. PROPERTIES OF OUR PROPOSED GENERALIZED HARVDA-CHARVAT'S MEASURE OF ENTROPY

Some significant features of our generalized entropy measure $H_{\alpha}^{\beta}(P)$ have been scrutinized in this section:

Property 1. $H_{\alpha}^{\beta}(P) > 0$, when α>0 and β>0.

Proof: We have

$$H_{\alpha}^{\beta}(P) = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^{n} {p_i}^{\frac{\beta}{\alpha}} - 1\right],\ \alpha > 0,\ \beta > 0, \neq \alpha\ .$$

Case *i*: For $\beta > \alpha$.

For $\beta > \alpha$, we have $\frac{\beta}{\alpha} > 1$. Since 0≤ $p_i \le 1, \forall i = 1,2,3,\dots,n$ and $\sum_{i=1}^{n} p_i = 1$, which implies that

$${p_i}^{\frac{\beta}{\alpha}} < p_i$$

After some mathematical operation, it follows that:

$$\left[\sum_{i=1}^{n} {p_i}^{\frac{\beta}{\alpha}} - 1\right] < 0 \tag{4}$$

We have $\beta > \alpha$, which implies $\beta - \alpha < 0$. Also, for $\alpha > 0$, so we have

$$\frac{\alpha}{\alpha-\beta} < 0 \tag{5}$$

On combining equations (4) and (5), we have for $\beta > \alpha$

$$H_{\alpha}^{\beta}(P) = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^{n} {p_i}^{\frac{\beta}{\alpha}} - 1\right] > 0 \tag{6}$$

Case $\boldsymbol{ii}$: When $< \alpha$.

When $\beta < \alpha$, we have $\frac{\beta}{\alpha} < 1$. Since, $0 \le p_i \le 1, \forall i = 1, 2, 3, \dots, n$ and $\sum_{i=1}^{n} p_i = 1$, which implies that

$$p_i^{\frac{\beta}{\alpha}} > p_i$$

After some mathematical operation, it follows that:

$$\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right] > 0 \tag{7}$$

As we have $\beta < \alpha$, which implies that $\alpha - \beta > 0$. Also for $\beta > 0$, so we have

$$\frac{\alpha}{\alpha - \beta} > 0 \tag{8}$$

On combining equation (7) and (8), we get

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha - \beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right] > 0 \tag{9}$$

From equations (6) and (9), we noticed that $H_\alpha^\beta(P)$ is positive for the defined values of the parameters α and β i.e.,

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha - \beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right] > 0$$

For $\beta > 0, \alpha > 0, \beta \ne \alpha$.

Property 2. $H_\alpha^\beta(P)$ is a symmetric function on every p_i, i = 1, 2, 3,…, n

Proof: This property is obviously true, i.e.,

$$H_\alpha^\beta(p_1, p_2, \dots, p_{n-1}, p_n) = H(p_n, p_1, p_2, \dots, p_{n-1})$$

Property 3. The maximum value of $H_\alpha^\beta(P)$ is achieved when the choice of the occurrence of all the events is equal.

Proof: We have

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha - \beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right], \beta > 0, \alpha > 0, \beta \ne \alpha$$

Assume the choice of the occurrence of all the events is equal, i.e., $pi = \frac{1}{n}, \forall i = 1, 2, 3, \dots, n$, then we have

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha - \beta}\left[\sum_{i=1}^{n}\left(\frac{1}{n}\right)^{\frac{B}{\alpha}} - 1\right]$$

After some mathematical computations, it follows that

$$H_\alpha^\beta\left(\frac{1}{n}\right) = H\left(\frac{1}{n}\right) = log_D n$$

Which is the maximum entropy.

Property 4. The additive property is satisfied by $H_\alpha^\beta(P)$ in the following mathematical perspective:

$$H_\alpha^\beta(P * Q) = H_\alpha^\beta(P) + H_\alpha^\beta(Q)$$

where,

$$(P * Q) = \{p_1 q_1, \dots, p_1 q_{m,}\, p_{2,} q_1, \dots,\, p_n q_1, \dots, p_n q_m\}$$

is the joint probability mass function of two independent discrete random variables.

Proof: Suppose$(P * Q) = \{p_1 q_1, \dots, p_1 q_{m,}\, p_{2,} q_1, \dots,\, p_n q_1, \dots, p_n q_m\}$, be the joint probability mass function of two independent discrete random variables, then

We have

$$H_\alpha^\beta(P * Q) = \frac{\alpha}{\alpha - \beta}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i q_j)^{\frac{\beta}{\alpha}} - 1\right)\right] \tag{10}$$

We can rewrite $(p_i q_j)^{\frac{\beta}{\alpha}}$ as

$$(p_i q_j)^{\frac{\beta}{\alpha}} = p_i^{\frac{\beta}{\alpha}} q_j^{\frac{\beta}{\alpha}}$$

Thus, the summation becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i q_j)^{\frac{\beta}{\alpha}} = \sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} \sum_{j=1}^{m} q_j^{\frac{\beta}{\alpha}}$$

The entropy

$$H_\alpha^\beta(P) = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right]$$

and

$$H_\alpha^\beta(Q) = \frac{\alpha}{\alpha-\beta}\left[\sum_{j=1}^{m} q_j^{\frac{\beta}{\alpha}} - 1\right]$$

On simplifying the above mathematical computation, we have

$$H_\alpha^\beta(P*Q) = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right] + \frac{\alpha}{\alpha-\beta}\left[\sum_{j=1}^{m} q_j^{\frac{\beta}{\alpha}} - 1\right]$$

Which implies,

$$H_\alpha^\beta(P*Q) = H_\alpha^\beta(P) + H_\alpha^\beta(Q)$$

This completes the proof.

3. SOURCE CODING THEOREMS

Consider a finite input source symbol $X = \{x_1, x_2, x_3, \dots, x_n\}$ with their respective probabilities of transmission $P = \{p_1, p_2, p_3, \dots, p_n\}$. Suppose we have code-words which have lengths $l_1, l_2, l_3, \dots, l_n$ and then the expected length of the coded message is defined by Shannon [1] as:

$$L(P) = \sum_{i=1}^{n} p_i l_i \tag{11}$$

A code is said to be a uniquely decipherable code over an alphabet of D symbols with length $L = \{l_1, l_2, l_3, \dots, l_n\}$ if and only if the Kraft's inequality holds, i.e.,

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{12}$$

For all codes satisfying the inequality (12), the mean code-word length $L(P)$ defined at (11), lies between $H(P)$ and $H(P) + 1$ i.e.,

$$H(P) < L(P) < H(P) + 1$$

This is also called Shannon's noiseless coding theorem. Kapur [13] defined his mean code-word length for a discrete channel as:

$$L^{\beta}(P) = \frac{1}{1-\beta}\left\{\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-1}{\beta}\right)}\right]^{\beta} - 1\right\}, \beta > 0, \beta \neq 1 \tag{13}$$

and also showed that $L^{\beta}(P)$ lies between $H^{\beta}(P)$ and $H^{\beta}(P)+1$ under the condition if the codes satisfy inequality (12), i.e.,

$$H^{\beta}(P) < L^{\beta}(P) < H^{\beta}(P) + 1$$

Numerous generalized source coding theorems under the condition of unique decipherability have been introduced by numerous scholars over the past few decades, for example, publications [14-16].

We presented a new generalized mean code-word length $L_{\alpha}^{\beta}(P)$ in this manuscript as:

$$L_{\alpha}^{\beta}(P) = \frac{\alpha}{\alpha-\beta}\left\{\left[\left(\sum_{i=1}^{n} p_i\, D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)}\right)\right]^{\beta} - 1\right\}, \alpha > 0, \beta > 0, \alpha \neq \beta \tag{14}$$

where D is the number of alphabets used to code input source symbols.

3.1. PARTICULAR CASES OF NEW GENERALIZED MEAN CODE-WORD LENGTH

Case i: When $\alpha = 1$, (14) reduces to code-word length corresponding to Kapur's mean code-word length, i.e.,

$$L_{\alpha=1}^{\beta}(P) = L^{\beta}(P) = \frac{1}{1-\beta}\left\{\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-1}{\beta}\right)}\right]^{\beta} - 1\right\} \tag{15}$$

Case ii: When $\alpha = 1$, and $\beta \to 1$, then, by applying L'Hospital's rule (14) reduces to the optimal mean code-word length corresponding to Shannon entropy i.e.,

$$L_{\alpha=1}^{\beta\to 1}(P) = L(P) = \sum_{i=1}^{n} p_i l_i$$

Case iii: When $\alpha \to \beta$, then, by applying L'Hospital's rule (14) reduces to the optimum mean code-word length given by Shannon [1], i.e.,

$$L^{\alpha\to\beta} = L(P) = \sum_{i=1}^{n} p_i l_i$$

Now we will derive the relationship between (3) and (14) in terms of source coding theorems.

3.2. RELATIONSHIP OF HARVDA-CHARVAT'S GENERALIZED MEASURE AND ITS CORRESPONDING CODE-WORD LENGTH

The two theorems below show the relationship between the generalized measure and its corresponding code-word length.

Theorem 1. For all alphabets with $D > 1$ symbols, suppose the set of code-word lengths is $L = \{l_{1,} l_{2,} l_{3,}, \dots, l_n\}$, which satisfy Kraft's inequality. The relationship between $H_\alpha^\beta(P)$ and $L_\alpha^\beta(P)$ is given by:

$$H_\alpha^\beta(P) \leq L_\alpha^\beta(P)$$

and equality, i.e., $H_\alpha^\beta(P) = L_\alpha^\beta(P)$ holds if and only if

$$l_i = -log_D \left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^n p_i^{\frac{\beta}{\alpha}}} \right] \tag{16}$$

Proof: For all $x_i, y_i > 0, i = 1, 2, 3, \dots, n$ and $\frac{1}{\theta} + \frac{1}{\delta} = 1, \theta < 1(\neq 0), \delta < 1(\neq 0), \theta < 0$, then by the reverse of Holder's inequality, we have

$$\left(\sum_{i=1}^n x_i^\theta \right)^{\frac{1}{\theta}} \left(\sum_{i=1}^n y_i^\delta \right)^{\frac{1}{\delta}} \leq \left(\sum_{i=1}^n x_i y_i \right) \tag{17}$$

The equality of (17) holds if $\exists\ c > 0$, such that

$$x_i^\theta = c y_i^\delta \tag{18}$$

Let

$$x_i = p_i^{\frac{\beta}{\beta-\alpha}} D^{-l_i}\ y_i = p_i^{\frac{\beta}{\alpha(\beta-\alpha)}}$$

$$\theta = \frac{\beta - \alpha}{\beta}\ \delta = \beta - \alpha$$

Substituting the above values in (17) and after some mathematical calculation, we get

$$\left[\sum_{i=1}^n p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)} \right]^{\frac{\beta}{\beta-\alpha}} \left[\sum_{i=1}^n p_i^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta-\alpha}} \leq \sum_{i=1}^n D^{-l_i}$$

By using inequality (17) and after some mathematical calculation, we get

$$\left[\sum_{i=1}^n p_i^{\frac{\beta}{\alpha}} \right]^{\frac{1}{\beta-\alpha}} \leq \left[\sum_{i=1}^n p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)} \right]^{\frac{\beta}{\beta-\alpha}} \tag{19}$$

Again, after some mathematical calculations on inequality (19) and subtracting 1 from both sides, we get

$$\frac{\alpha}{\beta-\alpha}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right] \le \frac{\alpha}{\beta-\alpha}\left\{\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)}\right]^{\beta} - 1\right\}$$

Or we can rewrite the above inequality as:

$$H_\alpha^\beta(P) \le L_\alpha^\beta(P)$$

Now we will show the equality i.e., $H_\alpha^\beta(P) = L_\alpha^\beta(P)$ holds if and only if

$$l_i = -log_D\left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right]$$

After some mathematical calculations, we get

$$D^{-l_i} = \left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right]$$

Now, after multiplying throughout by p_i to the above equation, and applying appropriate mathematical calculations, it follows that:

$$\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)} = \left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \tag{20}$$

After some mathematical calculations, we have

$$\frac{\alpha}{\beta-\alpha}\left\{\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)}\right]^{\beta} - 1\right\} = \frac{\alpha}{\alpha-\beta}\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}} - 1\right]$$

Or we can rewrite the above equality as

$$H_\alpha^\beta(P) = L_\alpha^\beta(P)$$

Theorem 2. For a code-word with lengths $L = \{l_1, l_2, l_3, \dots, l_n\}$ satisfying Kraft's inequality, and then $H_\alpha^\beta(P)$ and $L_\alpha^\beta(P)$ are related as follows:

$$L_\alpha^\beta(P) < H_\alpha^\beta(P)D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}\left(D^{\alpha-\beta} - 1\right), \text{when } \alpha > 0, \beta > 0, \beta \neq \alpha.$$

Proof: From Theorem 1, we see that $H_\alpha^\beta(P) = L_\alpha^\beta(P)$ is satisfied if and only if

$$l_i = -log_D\left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right]$$

The above expression can also be written as

$$l_i = -log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right]$$

Let code-word lengths $L = \{l_1, l_2, l_3, \dots, l_n\}$ be such that they satisfy the following inequalities:

$$-log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] \le l_i < -log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] + 1 \tag{21}$$

Consider the interval

$$\theta_i = \left[-log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right], -log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] + 1\right]$$

of length 1. In every θ_i, there lies exactly one positive integral l_i, such that

$$0 < -log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] \le l_i < -log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] + 1 \tag{22}$$

We will first show that sequence $l_1, l_2, l_3, \dots, l_n$, defined satisfies the Kraft's inequality. From the left-hand side of (23), we have

$$-log_D p_i^{\frac{\beta}{\alpha}} + log_D\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] \le l_i$$

or equivalently

$$D^{-l_i} \le \left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right] \tag{23}$$

Taking the summation over $i = 1, 2, 3, \dots, n$ on both sides of (23), we have

$$\sum_{i=1}^{n} D^{-l}i \le 1$$

This is Kraft's (1949) inequality. Now the last inequality of (23) gives:

$$D^{l_i} < \left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right]^{-1} D \tag{24}$$

After some mathematical calculations, we have

$$D^{l_i\left(\frac{\alpha-\beta}{\beta}\right)} < \left[\frac{p_i^{\frac{\beta}{\alpha}}}{\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}}\right]^{\frac{\beta-\alpha}{\beta}} D^{\frac{\alpha-\beta}{\beta}} \tag{25}$$

Now multiplying inequality (25) both sides by p_i , then summing over $i = 1, 2, 3, \dots, n$ and after suitable simplifications, we have

$$\sum_{i=1}^{n} p_i D^{l_i\left(\frac{\alpha-\beta}{\beta}\right)} < \left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} D^{\frac{\alpha-\beta}{\beta}} \tag{26}$$

Raising both sides to the power β and subtracting 1 throughout the inequality (26), we have

$$\frac{\alpha}{\alpha-\beta}\left\{\left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\beta-\alpha}{\beta}\right)}\right]^{\beta} - 1\right\} < \frac{\alpha}{\alpha-\beta}\left\{\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] D^{\alpha-\beta} - 1\right\} \tag{27}$$

From right hand side of (27) we have

$$\frac{\alpha}{\alpha-\beta}\left\{\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] D^{\alpha-\beta} - 1\right\}$$
$$= \frac{\alpha}{\alpha-\beta}\left\{\left[\sum_{i=1}^{n} p_i^{\frac{\beta}{\alpha}}\right] - 1\right\} D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}\left(D^{\alpha-\beta} - 1\right)$$

or we can write that

$$L_\alpha^\beta(P) < H_\alpha^\beta(P) D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}\left(D^{\alpha-\beta} - 1\right).$$

Hence from above two coding theorems, we conclude that

$$H_\alpha^\beta(P) \le L_\alpha^\beta(P) < H_\alpha^\beta(P) D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}\left(D^{\alpha-\beta} - 1\right)$$

when $\alpha > 0, \beta > 0, \alpha \neq \beta$.

4. SHANNON-FANO CODING ALGORITHM

In this portion, we will demonstrate the validity of Theorems 1 and 2 by taking empirical data from [17] given in Table 1. The probability values of p_1, p_2, p_4, p_5 are given as $0.25, 0.5, 0.125, 0.125$ respectively. Now, by using the Shannon-Fano Coding Algorithm, we have got the binary codes and code-word lengths corresponding to each probability given in Table 1.

Table 1. Shannon-Fano Coding Algorithm.

p_i	Shannon-Fano Codes	l_i	α	β	$H_\alpha^\beta(P)$	$L_\alpha^\beta(P)$	$D^{\alpha-\beta}$	$(D^{\alpha-\beta} - 1)$	$H_\alpha^\beta(P)D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}(D^{\alpha-\beta} - 1)$
0.25	001	3	0.9	0.6	1.5603	2.4527	1.2311	0.6933	2.6142
0.5	10	2							
0.125	1101	4							
0.125	1111	4							

Now the table allows us to deduce that the validity of Theorem 1 and Theorem 2 extends to the Shannon-Fano coding scheme as $H_\alpha^\beta(P) \leq L_\alpha^\beta(P)$ and $H_\alpha^\beta(P) \leq L_\alpha^\beta(P) < H_\alpha^\beta(P)D^{\alpha-\beta} + \frac{\alpha}{\alpha-\beta}\left(D^{\alpha-\beta} - 1\right)$ when , $\alpha > 0, \beta > 0, \alpha \neq \beta$.

5. REAL LIFE APPLICATION

Hinkley [18] provided a dataset comprising thirty consecutive years of March precipitation measurements, recorded in inches, which has also been used by [19]. The dataset captures variations in rainfall over this period and serves as a valuable resource for statistical and climatological analysis. The recorded values are as follows:

0.77 1.74 0.81 1.20 1.95 1.20 0.47 1.43 3.37 2.20
3.00 3.09 1.51 2.10 0.52 1.62 1.31 0.32 0.59 0.81
2.81 1.87 1.18 1.35 4.75 2.48 0.96 1.89 0.90 2.05

Table 2. The behaviour of our proposed Harvda-Charvat's measure, when α is fixed at 3.5, and β varies for the given Database.

β	$H_\alpha^\beta(P)$	β	$H_\alpha^\beta(P)$	β	$H_\alpha^\beta(P)$
0.10	27.64168	0.20	27.32464	0.30	27.04876
0.11	27.60812	0.21	27.29520	0.31	27.02344
0.12	27.57498	0.22	27.26617	0.32	26.99854
0.13	27.54224	0.23	27.23755	0.33	26.97405
0.14	27.50992	0.24	27.20934	0.34	26.94997
0.15	27.47801	0.25	27.18155	0.35	26.92631
0.16	27.44651	0.26	27.15417	0.36	26.90306
0.17	27.41543	0.27	27.12720	0.37	26.88023
0.18	27.38475	0.28	27.10064	0.38	26.85781
0.19	27.35449	0.29	27.07450	0.39	26.83581

Table 3. The behaviour of our proposed Harvda-Charvat's measure when β=0.02 is fixed and α varies for the given Dataset.

α	$H_\alpha^\beta(P)$	α	$H_\alpha^\beta(P)$	α	$H_\alpha^\beta(P)$
0.10	26.36485	0.20	26.92631	0.30	27.22810
0.11	26.42747	0.21	26.96598	0.31	27.24950
0.12	26.49334	0.22	27.00304	0.32	27.26977
0.13	26.55863	0.23	27.03770	0.33	27.28900
0.14	26.62148	0.24	27.07018	0.34	27.30727
0.15	26.68105	0.25	27.10064	0.35	27.32464
0.16	26.73701	0.26	27.12926	0.36	27.34117
0.17	26.78937	0.27	27.15618	0.37	27.35693
0.18	26.83823	0.28	27.18155	0.38	27.37196
0.19	26.88380	0.29	27.20549	0.39	27.38632

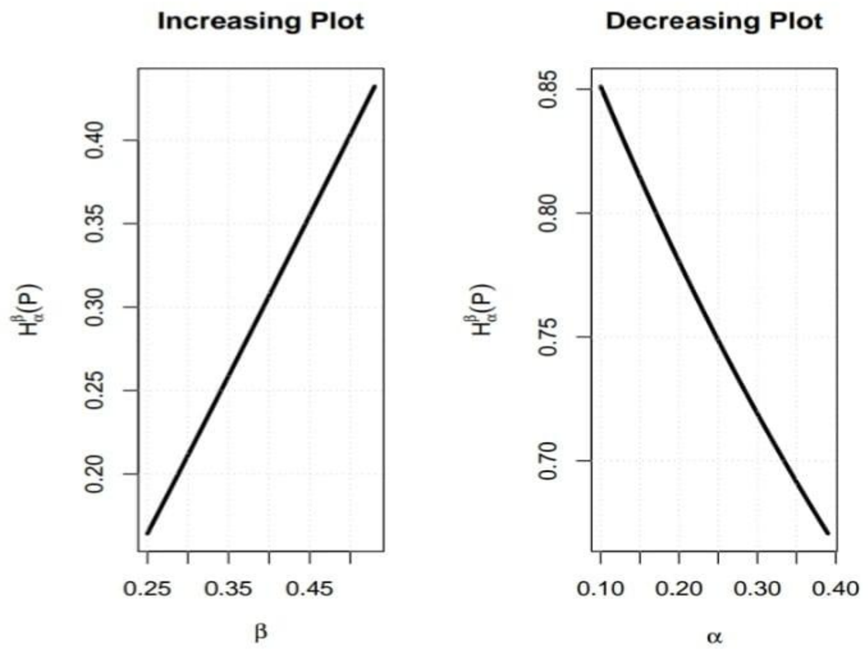

Figure 1. Monotonicity of our Proposed Measure.

6. CONCLUSIONS

This study presents a new generalization of Harvda-Charvat's measure of entropy, highlighting it's importance in information theory and applied mathematics. The paper explores the key properties of this novel generalization. Furthermore, a new generalized mean code-word length is introduced, and the relationship between $H_\alpha^\beta(P)$ and $L_\alpha^\beta(P)$is established through the source coding theorems presented in this paper, and also demonstrated through the Shannon-Fano Coding Algorithm. Finally, we checked it's monotonicity through a precipitation dataset through which we can better anticipate the future changes to changing climatic patterns.

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