ORIGINAL PAPER

EXPLORING NEW SOLITONS TO THE NONLINEAR GENERALIZED (3+1)-DIMENSIONAL GAS BUBBLES LIQUID MODEL

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Abstract. *The cornerstone of this article builds various new concepts of traveling solutions to the nonlinear generalized (3+1)-dimensional gas bubbles liquid model. The suggested model, which plays a significant role in fluid mechanics, describes nonlinear wave propagation in the gas-bubble-liquid model. Three famous universal techniques, namely, the Generalized Kudryashov Technique, the Paul-Painleve Technique, and the (G'/G)-Expansion Technique. Many new arising soliton types in the form of bright soliton solutions, dark soliton solutions, and other new rational solutions by using the three suggested techniques have been constructed. Our realized solitons are new compared with [2-5], who solved this model by other techniques. Also, the two-dimensional (2D) and the three-dimensional (3D) simulations that show the dynamical behaviors of the arising solitons have been represented. Furthermore, the physical descriptions of some achieved solutions are documented.*

Keywords: *Nonlinear Generalized (3+1)-Dimensional Gas Bubbles Liquid Model; Generalized Kudryashov Technique; Paul-Painleve Approach Technique; (G'/G)- Technique; Arising Solitons.*

1. INTRODUCTION

The nonlinear physical phenomena can be described by differential equations with nonlinear partial derivatives. The wave fields that describe mechanical or electromagnetic waves are called dynamical wave equations and are described by partial differential equations of second order. The acoustics, electromagnetism, and fluid equations are three kinds in these fields.

The Nonlinear Generalized (3 +1)-Dimensional Gas Bubbles Liquid Model (NG G(3+1)-DGLM) refers to a variety of nonlinear operations in liquids containing gas bubbles that will be proposed. When the suggested model surrenders to mathematical adjustments, this NG G(3+1)-DGLM naturally converts into either one of the following equations, which are the (3+1)-Dimensional Kadomtsev-Petviashvili Equation, the (3+1)-Dimensional Nonlinear Wave Equation, and the Korteweg-De Vries Equation. Some authors investigate the suggested model through various methods see for example, Tu et al. [1] who derived the Backlund transformation, infinite conservation laws and periodic wave soliton solutions of the NG G(3+1)-DGLM, Wang et al. [2] who obtained Lump, mixed Lump-stripe and rogue wave

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stripe solutions of the suggested model, Zhao et al. [3] who used the new generalized exponential rational function method to derive the various kinds of new accurate soliton solutions to the NG G(3+1)-DGLM, Shen et al. [4] who discussed G(3+1)-DNLWE by using Hirota's bilinear method to obtain abundant soliton wave solutions and the linear superposition principle Sabiu et al. [5] who utilized the extended auxiliary equation scheme to explore the newly traveling wave solutions of the NG G(3+1)-DGLM, Adem et al. [6] who used the Lie symmetry, multiplier, and simplest equation manners to satisfy innovative symmetry reductions, group invariant solutions, conservation laws to the NG G(3+1)-DGLM.

There are several semi-analytical techniques as the solitary wave ansatz method, Hirota's bilinear method, tanh and sine-cosine methods, extended tanh-function method, the extended simple equation method, the modified simple equation method, the extended direct algebraic method, $e^{-\varphi}$-expansion method, the Ricatti-Bernolli-Sub ODE method, (G'/G)-expansion method, the Paul-Painleve Approach Method (PPAM), so on… that are used to construct the traveling wave solutions of different nonlinear science problems have been mentioned in several published articles [7-26].

The NG G(3+1)-DGLM [3] that describes nonlinear wave propagation in the liquid with a gas bubble can be written as

$$(Q_\tau + \mu_1 Q Q_x + \mu_2 Q_{xxx} + \mu_3 Q_x)_x + \mu_4 Q_{yy} + \mu_5 Q_{zz} = 0. \tag{1}$$

where $\mu_i, i = 1, 2, 3, 4, 5$ are arbitrary constants $Q = Q(x, y, z, \tau)$ denotes the differentiable function with the longitudinal parameters x, y, z and temporal parameter τ. Now, with the aid of the following transformation

$$Q(x, y, z, \tau) = R(\zeta), \zeta = s_1 x + s_2 y + s_3 z - w\tau. \tag{2}$$

Equation (1) will be converted to

$$(-ws_1 + \mu_3 s_1^2 + \mu_4 s_2^2 + \mu_5 s_3^2)R'' + \mu_1 s_1^2 (R'^2 + RR'') + \mu_2 s_1^4 R^4 = 0. \tag{3}$$

By integrating Eq. (3) twice, neglecting the constants of integration, we get

$$2\mu_2 s_1^4 R'' + \mu_1 s_1^2 R^2 + 2(-ws_1 + \mu_3 s_1^2 + \mu_4 s_2^2 + \mu_5 s_3^2)R = 0. \tag{4}$$

By applying the homogeneous balance between R'' and R^2 we get $M = 2$.

This research was prepared in the following successive sections. In section 1, the introduction and the synopsis of reducing the suggested model by the suitable wave transformation are considered. In section 2, the generalized Kudryashov Technique (GKT) [27-28] and its concepts have been included. In section 3, the PPAM [29-31] and its concepts have been proposed. In section 4, the (G'/G)-Expansion Method [32, 33] and its concepts are introduced. In section 5, brief conclusions for the entire body of work are considered.

2. GENERALIZED KUDRYASHOV TECHNIQUE

To discuss the general formulation of any nonlinear partial differential equation, we proceed with the function H

$$H(R,R_x,R_y,R_z,R_\tau,R_{xx},R_{yy},,R_{zz},R_{xy},R_{xz},R_{yz},R_{\tau\tau},...)=0. \tag{5}$$

where H is a function in $R(x,\tau)$, its partial derivatives; with the help of the transformation $R(x,t)=R(\zeta),\zeta=\mu_1 x+\mu_2 y+\mu_3 z-w\tau.$ Eq. (5) becomes

$$\Gamma(R,\frac{dR}{d\zeta},\frac{d^2R}{d\zeta^2},...)=0. \tag{6}$$

Eq. (6) denotes to ODE in $R,R',R'',....$. The GKM proceeded with the solution to Eq. (6) as

$$R(\zeta)=\frac{\sum_{i=0}^{N}a_iZ^i(\zeta)}{\sum_{j=0}^{M}b_jZ^j(\zeta)}=\frac{a_0+a_1Z(\zeta)+a_2Z^2(\zeta)+...}{b_0+b_1Z(\zeta)+b_2Z^2(\zeta)+...}. \tag{7}$$

where M is the balance number, $N=M+1,$ $(a_1,a_2,a_3,...,a_N),(b_1,b_2,b_3,...,b_M),$ and $a_N\neq 0,b_M\neq 0$ are later determined parameters, while $Z(\zeta)$ can be obtained from the following equation

$$\frac{dZ(\zeta)}{d\zeta}=Z^2(\zeta)-Z(\zeta). \tag{8}$$

When Eq. (8) is integrated with respect to ζ, put C as the integration constant, we obtain

$$Z(\zeta)=\frac{1}{1+Ce^{\zeta}}. \tag{9}$$

Now, for Eq. (4), the balance number is $M=2,$ hence $N=M+1=3,$ and the solution is:

$$R(\zeta)=\frac{a_0+a_1Z(\zeta)+a_2Z^2(\zeta)+a_3Z^3(\zeta)}{b_0+b_1Z(\zeta)+b_2Z^2(\zeta)}. \tag{10}$$

When R,R^2,R'' are proceeding at Eq. (4), by setting the equivalence for various powers of a_iZ^i implies some algebraic equations, by solving them, two results will be explored. We will construct an identical solution for only one of them, which is:

$$\begin{aligned} &a_0=\frac{1}{6}(a_2+a_3),a_1=\frac{-1}{6}(6a_2+5a_3),b_1=\frac{a_3b_0}{a_2+a_3},\mu_2=\frac{-(a_2+a_3)\mu_1}{12b_0s_1^2},\\ &\mu_3=\frac{12wb_0s_1-a_2s_1^2\mu_1-a_3s_1^2\mu_1-12b_0s_2^2\mu_4-12b_0s_2^2\mu_5}{12b_0s_1^2},b_2=0. \end{aligned} \tag{11}$$

This result can be simplified to be

$$a_0 = 0.3,\ a_1 = -1.8,\ a_2 = a_3 = b_0 = w = s_1 = s_2 = s_3 = C = \mu_1 = \mu_4 = \mu_5 = 1, \\ b_1 = 0.5,\ \mu_2 = -0.3,\ \mu_3 = -1.2,\ b_2 = 0. \tag{12}$$

So, the solution is:

$$R(\zeta) = \frac{0.3 - 1.8\left(\frac{1}{1+e^{\zeta}}\right) + \left(\frac{1}{1+e^{\zeta}}\right)^2 + \left(\frac{1}{1+e^{\zeta}}\right)^3}{1 + 0.5\left(\frac{1}{1+e^{\zeta}}\right)}. \tag{13}$$

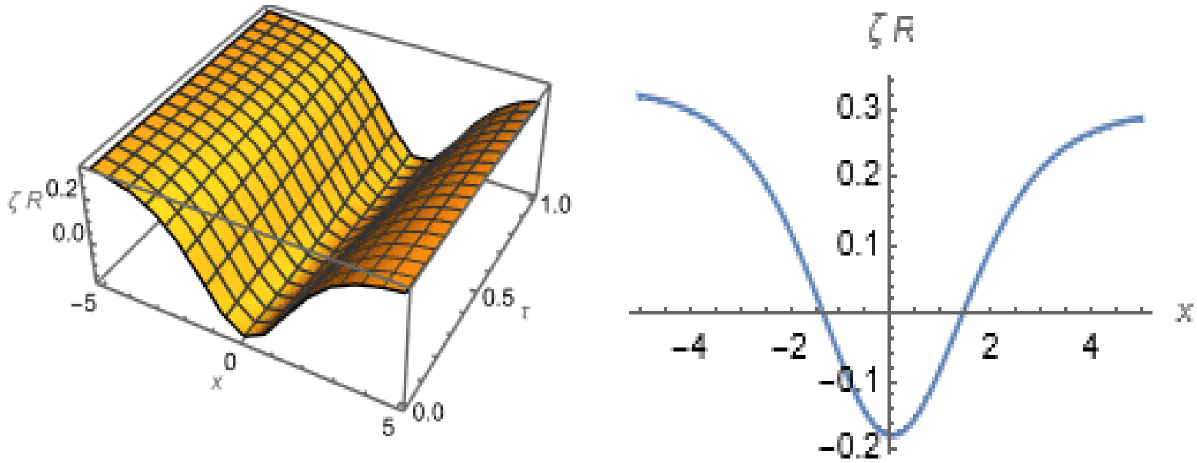

Figure 1. The two and three-dimensional soliton dynamical behaviors of Eq. (13).

3. PAUL-PAINLEVE APPROACH TECHNIQUE

This manner introduces the solution of Eq. (6) to be

$$R(\zeta) = A_0 + \sum_{i=1}^{M} A_i S^i(\chi) e^{-N\zeta}, \chi = E(\zeta). \tag{14}$$

The function $S(\chi)$ that appears in Eq. (14) is the solution of the Riccati equation:

$$S_{\chi} - AS^2 = 0, \text{and} \quad \chi = C_1 - \frac{e^{-N\zeta}}{N}, \\ S(\chi) = \frac{1}{A\chi + \chi_0}. \tag{15}$$

when R, R^2, R'' are proceeding at Eq. (4), by setting the equivalence for various powers of $S^i(\zeta)e^{-iN\zeta}$ implies some algebraic equations, by solving them, eight results will be explored, we will construct an identical solution for only one of them, which is:

$$A_0 = \frac{-2[s_2^2(\mu_4+\mu_5)+s_1^2\mu_3 - ws_1]}{s_1^2\mu_1}, A_1 = \frac{-2i\sqrt{3A_2}\sqrt{s_2^2(\mu_4+\mu_5)+s_1^2\mu_3 - ws_1}}{s_1\sqrt{\mu_1}},$$
$$A = \frac{-i\sqrt{A_2\mu_1}}{2\sqrt{3}\,s_1\sqrt{\mu_2}}, N = \frac{-\sqrt{s_2^2(\mu_4+\mu_5)+s_1^2\mu_3 - ws_1}}{s_1^2\sqrt{\mu_2}}. \tag{16}$$

With certain choices of parameter values, Eq. (16) can be reduced to

$$A = 0.3,\ N = -1.4,\ A_0 = -4,\ A_1 = 5,\ A_2 = -1,$$
$$w = s_1 = s_2 = s_3 = C_1 = \mathrm{X}_0 = \mu_1 = \mu_2 = \mu_4 = \mu_5 = 1. \tag{17}$$

The solution is

$$R(\zeta) = -4 + 5\left(\frac{e^{-N\zeta}}{A\left(C_1 - \frac{e^{-N\zeta}}{N}\right)+X_0}\right) - \left(\frac{e^{-N\zeta}}{A\left(C_1 - \frac{e^{-N\zeta}}{N}\right)+X_0}\right)^2. \tag{18}$$

$$R(\zeta) = -4 + 5\left(\frac{1.4e^{1.4\zeta}}{1.8+0.3e^{1.4\zeta}}\right) - \left(\frac{1.4e^{1.4\zeta}}{1.8+0.3e^{1.4\zeta}}\right)^2. \tag{19}$$

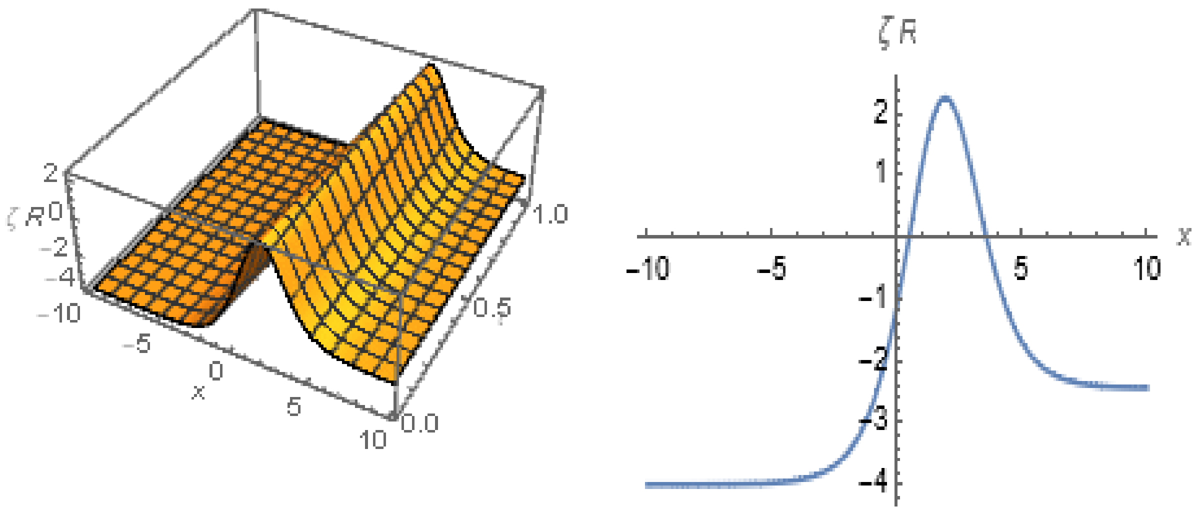

Figure 2. The two and three-dimensional soliton dynamical behaviors of Eq. (19)

4. THE (G'/G)-EXPANSION METHOD

This technique precedes the solution of Eq. (6) as

$$R(\zeta)=A_0+\sum_{i=1}^{M}A_i\left[\frac{G'}{G}\right]^i, A_M\neq 0. \tag{20}$$

where M is the calculated balance number, while the function $G(\zeta)$ satisfies the condition $G''+\mu G'+\lambda G=0$, whose expected solution is one of the following three forms

(I): if $\mu^2-4\lambda\succ 0$, the first form is:

$$(\frac{G'}{G})=\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{l_1\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{l_1\cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right)-\frac{\mu}{2}. \tag{21}$$

(II): if $\mu^2-4\lambda\prec 0$, the second form is:

$$(\frac{G'}{G})=\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{-l_1\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{l_1\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right)-\frac{\mu}{2}. \tag{22}$$

(III): if $\mu^2-4\lambda=0$, the third form is:

$$(\frac{G'}{G})=(\frac{l_2}{l_1+l_2\zeta})-\frac{\mu}{2}. \tag{23}$$

The (G'/G) concept introduces the solution of Eq. (4) according to its balance number in the form

$$R(\zeta)=A_0+A_1\left(\frac{G'}{G}\right)+A_2\left(\frac{G'}{G}\right)^2. \tag{24}$$

When R, R^2, R'' are proceeding at Eq. (4), by setting the equivalence for various powers of $(\frac{G'}{G})^i$ implies some algebraic equations, by solving them, eight results will be explored, we will construct an identical solution for only one of them, which is:

$$\begin{aligned}&A_0=\frac{\mu_3 s_1^2+\mu_4 s_2^2+\mu_5 s_3^2-ws_1-3\mu^2\mu_2 s_1^4}{\mu_1 s_1^2}, A_1=\frac{-12\mu s_1^2\mu_2}{\mu_1}, A_2=\frac{-12\mu_2 s_1^2}{\mu_1},\\ &\lambda=\frac{\mu_3 s_1^2+\mu_4 s_2^2+\mu_5 s_3^2-ws_1-\mu^2\mu_2 s_1^4}{4\mu_2 s_1^4}.\end{aligned} \tag{25}$$

This result can be simplified to be

$$\begin{aligned}&A_0=-1, A_1=4, A_2=4, \lambda=3, \mu_2=0.3, \mu_1=-1,\\ &w=\mu=s_1=s_2=s_3=\mu_3=\mu_4=\mu_5=1.\end{aligned} \tag{26}$$

According to the above values, of constants $\left(\frac{G'}{G}\right)$ will be in the form Eq. (22), which is:

$$(\frac{G'}{G})=\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{-l_1\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{l_1\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+l_2\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right)-\frac{\mu}{2}.$$

From which, by simple calculations, we get

$$(\frac{G'}{G})=\left(\frac{3.4\sinh 1.7\zeta-\cosh 1.7\zeta+i(5.5\cosh 1.7\zeta-2\sin 1.7\zeta)}{2(\cosh 1.7\zeta+2i\sin 1.7\zeta)}\right). \tag{27}$$

Thus

$$R(\zeta)=-1+4\left\{\frac{\left(6\sinh^2 1.7\zeta-15\sinh 1.7\zeta\cosh 1.7\zeta-2\right)+i\left(3\sinh^2 1.7\zeta-8\sinh 1.7\zeta\cosh 1.7\zeta+11\right)}{4+12\sinh^2 1.7\zeta}\right\}$$
$$+4\left\{\frac{\left(6\sinh^2 1.7\zeta-15\sinh 1.7\zeta\cosh 1.7\zeta-2\right)+i\left(3\sinh^2 1.7\zeta-8\sinh 1.7\zeta\cosh 1.7\zeta+11\right)}{4+12\sinh^2 1.7\zeta}\right\}^2. \tag{28}$$

$$\mathrm{Re}.R(\zeta)=-1+\left\{\frac{24\sinh^2 1.7\zeta-60\sinh 1.7\zeta\cosh 1.7\zeta-8}{4+12\sinh^2 1.7\zeta}\right\}$$
$$+4\left\{\frac{\left(6\sinh^2 1.7\zeta-15\sinh 1.7\zeta\cosh 1.7\zeta-2\right)^2-\left(3\sinh^2 1.7\zeta-8\sinh 1.7\zeta\cosh 1.7\zeta+11\right)^2}{\left(4+12\sinh^2 1.7\zeta\right)^2}\right\}. \tag{29}$$

$$\operatorname{Im}.R(\zeta)=\left\{\frac{12\sinh^{2}1.7\zeta-32\sinh 1.7\zeta\cosh 1.7\zeta+44}{4+12\sinh^{2}1.7\zeta}\right\}+8\frac{\left\{\begin{array}{l}\left(6\sinh^{2}1.7\zeta-15\sinh 1.7\zeta\cosh 1.7\zeta-2\right)\\ \times\left(3\sinh^{2}1.7\zeta-8\sinh 1.7\zeta\cosh 1.7\zeta+11\right)\end{array}\right\}}{\left(4+12\sinh^{2}1.7\zeta\right)^{2}}. \tag{30}$$

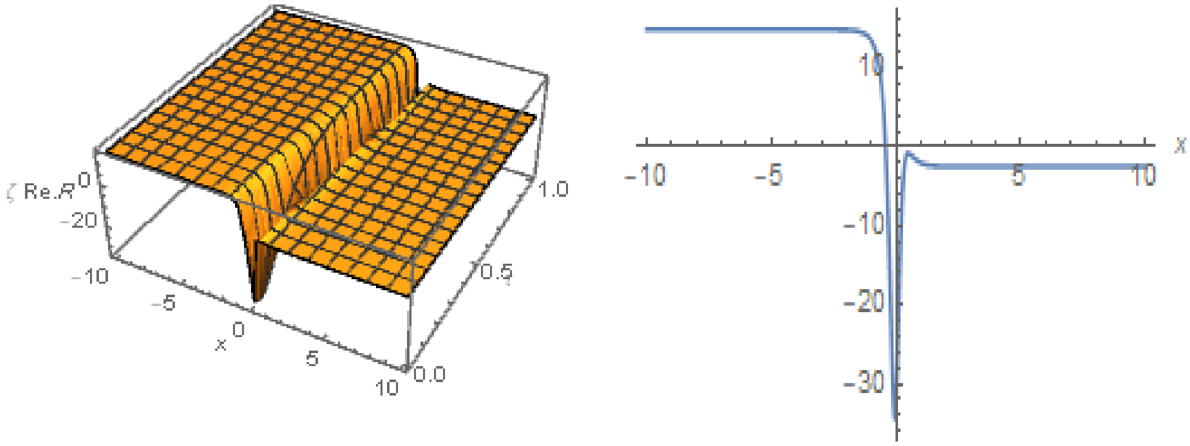

Figure 3. The two and three-dimensional soliton dynamical behaviors of Eq. (29).

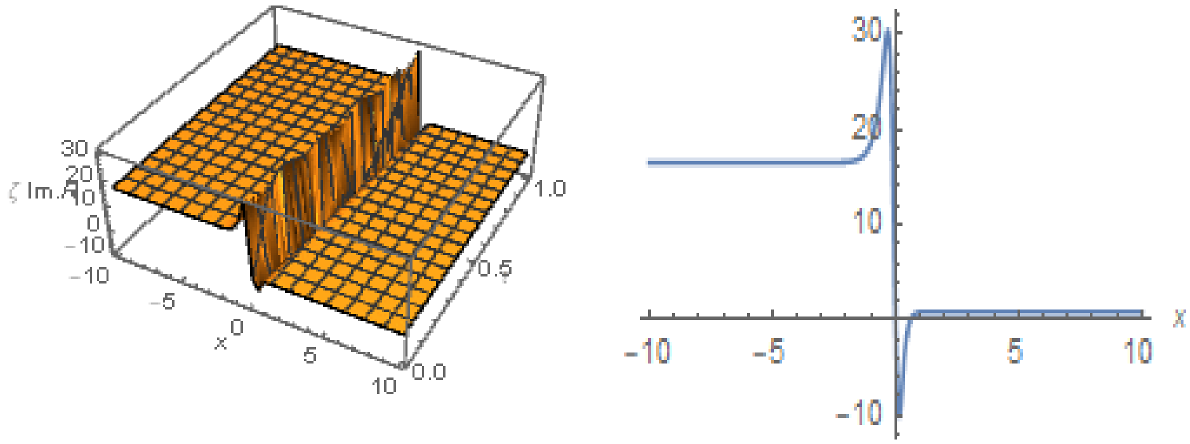

Figure 4. The two and three-dimensional soliton dynamical behaviors of Eq. (30).

6. CONCLUSIONS

In this article, the GKT, the PPAM, and the (G'/G)-concept have been successfully employed for the first time to explore the new forms of the soliton arising in the NG G(3+1)-DGLM. This model plays an important role in fluid mechanics and ocean engineering**.** The suggested model represents the nonlinear wave propagation behaviors in liquids with gas bubbles.

The new solitons arising that are extracted by using the three suggested methods have been documented in forms like bright soliton, dark soliton, and other new rational solitons. Our proceeding solutions are considered new compared with [2-5], who solved this model by various other methods. In addition, to show the range of real balance influence between the nonlinearity and dispersive wave propagation effect, we plot the 2D, 3D representations of the dynamics of the realized traveling wave solution behaviors via the Mathematica program.

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