ORIGINAL PAPER

ON RADÓ'S AND POPOVICIU'S INEQUALITIES

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Abstract. *This paper aims to give new forms of Radó's and Popoviciu's inequalities. Some applications are given.*

Keywords: *inequalities; means; AM-GM inequality; Radó's inequality; Popoviciu's inequality.*

1. INTRODUCTION

For all non-negative real numbers, denote by

$$A_n[a_1, a_2, \dots, a_n] := \frac{1}{n} \cdot \sum_{k=1}^{n} a_k$$

and

$$G_n[a_1, a_2, \dots, a_n] := \sqrt[n]{\prod_{k=1}^{n} a_k}$$

the arithmetic mean, respective the geometric mean of $a_1, a_2, \dots, a_n$. The following inequality, also known as *the AM-GM inequality*, holds true:

$$A_n[a_1, a_2, \dots, a_n] \geq G_n[a_1, a_2, \dots, a_n].$$

This means is the basis of discovering new inequalities of several types, so it has attracted the attention of many researchers. In consequence, plenty of refinements and extensions were given in the recent past. See, e.g., [1-3,6,7].

We refer in this work to the following inequalities:

$$n(A_n[a_1, a_2, \dots, a_n] - G_n[a_1, a_2, \dots, a_n]) \\ \geq (n-1)(A_{n-1}[a_1, a_2, \dots, a_{n-1}] - G_{n-1}[a_1, a_2, \dots, a_{n-1}]) \qquad \text{(Radó)}$$

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and

$$\left(\frac{A_n[a_1,a_2,\dots,a_n]}{G_n[a_1,a_2,\dots,a_n]}\right)^n \geq \left(\frac{A_{n-1}[a_1,a_2,\dots,a_{n-1}]}{G_{n-1}[a_1,a_2,\dots,a_{n-1}]}\right)^{n-1} \qquad \text{(Popoviciu)}$$

Radó's and Popoviciu's inequalities are fundamental results in the field of mathematical inequalities, particularly in the study of convex functions and majorization theory. Both inequalities have been extensively studied and generalized, leading to a rich body of research that explores their properties, extensions, and applications.

The original statement of Radó's inequality in 1965 can be found in the work of Beckenbach and Bellman [1], while Popoviciu's inequality was formulated by Tiberiu Popoviciu in 1965. See [9].

2. ON RADÓ'S INEQUALITY

We provide the following:

Theorem 1. Let $a_1, a_2, \dots, a_n$ be non-negative numbers, with $n \geq 2$ and $2 \leq r \leq n$.

Then the following inequality holds true:

$$\begin{aligned} &n(A_n[a_1,a_2,\dots,a_n] - G_n[a_1,a_2,\dots,a_n]) \\ &\qquad \geq r \cdot \max_{1\leq i_1<i_2<\dots<i_r\leq n}\left(A_r\left[a_{i_1},a_{i_2},\dots,a_{i_r}\right] - G_r\left[a_{i_1},a_{i_2},\dots,a_{i_r}\right]\right). \end{aligned} \tag{1}$$

Proof: We have:

$$\begin{aligned} &A_n[a_1,a_2,\dots,a_n] - \frac{r}{n}\left(A_r\left[a_{i_1},a_{i_2},\dots,a_{i_r}\right] - G_r\left[a_{i_1},a_{i_2},\dots,a_{i_r}\right]\right) \\ &= \frac{1}{n}\left[\sum_{k=1}^{n} a_k - \left(\sum_{1\leq i_1<i_2<\dots<i_r\leq n} a_k - r\cdot\sqrt[r]{a_{i_1}a_{i_2}\dots a_{i_r}}\right)\right] \\ &= \frac{1}{n}\left[\left(\sum_{k=1}^{n} a_k - \sum_{1\leq i_1<i_2<\dots<i_r\leq n} a_k\right) + r\cdot\sqrt[r]{a_{i_1}a_{i_2}\dots a_{i_r}}\right] \\ &= \frac{1}{n}\left[\sum_{k\neq i_1,i_2,\dots,i_r} a_k + r\cdot\sqrt[r]{a_{i_1}a_{i_2}\dots a_{i_r}}\right]. \end{aligned}$$

Now, by AM-GM inequality, we deduce that

$$\begin{aligned} &\frac{1}{n}\left[\sum_{k\neq i_1,i_2,\dots,i_r} a_k + r\cdot\sqrt[r]{a_{i_1}a_{i_2}\dots a_{i_r}}\right] \geq \sqrt[n]{\left(\prod_{k\neq i_1,i_2,\dots,i_r} a_k\right)\left(\sqrt[r]{a_{i_1}a_{i_2}\dots a_{i_r}}\right)^r} \\ &= \sqrt[n]{\left(\prod_{k\neq i_1,i_2,\dots,i_r} a_k\right)a_{i_1}a_{i_2}\dots a_{i_r}} = \sqrt[n]{a_1a_2\dots a_n} = G_n[a_1,a_2,\dots,a_n]. \end{aligned}$$

Hence

$$A_n[a_1, a_2, \dots, a_n] - G_n[a_1, a_2, \dots, a_n] \geq \frac{r}{n}\left(A_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right] - G_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]\right),$$

so

$$n(A_n[a_1, a_2, \dots, a_n] - G_n[a_1, a_2, \dots, a_n]) \geq r \cdot \left(A_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right] - G_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]\right). \tag{2}$$

The conclusion follows by taking the maximum with respect to all sequences $i_1 < i_2 < \cdots < i_r$. Other proof can be given by successively passing from n to $n-1$, then from $n-1$ to $n-2$ variables and so on in Radó's inequality.

Remark that if we take $r = n-1$, then the following inequality of Radó's type is obtained:

$$n(A_n[a_1, a_2, \dots, a_n] - G_n[a_1, a_2, \dots, a_n]) \geq (n-1) \cdot \max_{1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n}\left(A_{n-1}\left[a_{i_1}, a_{i_2}, \dots, a_{i_{n-1}}\right] - G_{n-1}\left[a_{i_1}, a_{i_2}, \dots, a_{i_{n-1}}\right]\right)$$

We now give two corollaries (see [4], [6]).

Corollary 1. Let $a_1, a_2, \dots, a_n$ be non-negative, where $n \geq 2$. Then the following inequalities hold true:

$$\frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{1}{n} \cdot \max_{1 \leq i < j \leq n}\left(\sqrt{a_i} - \sqrt{a_j}\right)^2$$

and

$$\frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{1}{n} \cdot \max_{1 \leq i < j < k \leq n}\left(a_i + a_j + a_k - 3 \cdot \sqrt[3]{a_i a_j a_k}\right).$$

The proof follows by taking $r = 2$ and $r = 3$ in (1). By summation inequalities (2) with respect to all subsets $\{i_1 < i_2 < \dots < i_r\} \subset \{1, 2, \dots, n\}$ (with C_n^r terms), we obtain the following:

Corollary 2. Let $a_1, a_2, \dots, a_n$ be non-negative numbers. Then for every $\{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, 2, \dots, n\}$, the following inequality holds true:

$$n(A_n[a_1, a_2, \dots, a_n] - G_n[a_1, a_2, \dots, a_n]) \geq \frac{r}{C_n^r} \cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n}\left(A_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right] - G_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]\right).$$

3. ON POPOVICIU'S INEQUALITY

We provide the following extension of Popoviciu's inequality:

Theorem 2. Let $a_1, a_2, \ldots, a_n$ be non-negative numbers, with $n \geq 2$ and $2 \leq r \leq n$. Then the following inequality holds true:

$$\left(\frac{A_n[a_1, a_2, \ldots, a_n]}{G_n[a_1, a_2, \ldots, a_n]}\right)^n \geq \max_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}\right)^r. \tag{3}$$

Proof: We have, by using the AM-GM inequality:

$$A_n[a_1, a_2, \ldots, a_n] = \frac{1}{n} \cdot \sum_{k=1}^{n} a_k = \frac{1}{n}\left(\sum_{k \neq i_1, i_2, \ldots, i_r} a_k + r \cdot \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r}}{r}\right)$$

$$\geq \sqrt[n]{\left(\prod_{k \neq i_1, i_2, \ldots, i_r} a_k\right)\left(\frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r}}{r}\right)^r} = \sqrt[n]{\left(\prod_{k=1}^{n} a_k\right)\left(\frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r}}{r \cdot \sqrt[r]{a_{i_1} a_{i_2} \ldots a_{i_r}}}\right)^r}$$

$$= \sqrt[n]{\prod_{k=1}^{n} a_k} \cdot \left(\frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r}}{r \cdot \sqrt[r]{a_{i_1} a_{i_2} \ldots a_{i_r}}}\right)^{\frac{r}{n}} = G_n[a_1, a_2, \ldots, a_n] \cdot \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}\right)^{\frac{r}{n}}.$$

We obtained:

$$\frac{A_n[a_1, a_2, \ldots, a_n]}{G_n[a_1, a_2, \ldots, a_n]} \geq \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}\right)^{\frac{r}{n}}.$$

By raising to the nth power, we deduce that

$$\left(\frac{A_n[a_1, a_2, \ldots, a_n]}{G_n[a_1, a_2, \ldots, a_n]}\right)^n \geq \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \ldots, a_{i_r}\right]}\right)^r, \tag{4}$$

The conclusion follows by taking the maximum with respect to all sequences $i_1 < i_2 < \cdots < i_r$.

Other proof can be given by successively passing from n to $n-1$, then from $n-1$ to $n-2$ variables and so on in Popoviciu's inequality. Remark that if we take $r = n-1$, then the following extended form of Popoviciu's inequality is obtained:

$$\left(\frac{A_n[a_1, a_2, \ldots, a_n]}{G_n[a_1, a_2, \ldots, a_n]}\right)^n \geq \max_{1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n} \left(\frac{A_{n-1}\left[a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\right]}{G_{n-1}\left[a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\right]}\right)^{n-1}.$$

By summation inequalities (3) with respect to all subsets $\{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n\}$ (with C_n^r terms), we obtain the following result (see [5], [6]):

Corollary 3. Let $a_1, a_2, \dots, a_n$ be positive numbers, with $n \geq 2$. Then the following inequalities hold true:

$$\frac{1}{n} \cdot \left(\sum_{k=1}^{n} a_k\right) / \left(\sqrt[n]{\prod_{k=1}^{n} a_k}\right) \geq \max_{1 \leq i < j \leq n} \left(\frac{a_i + a_j}{2\sqrt{a_i a_j}}\right)^{\frac{2}{n}}$$

and

$$\frac{1}{n} \cdot \left(\sum_{k=1}^{n} a_k\right) / \left(\sqrt[n]{\prod_{k=1}^{n} a_k}\right) \geq \max_{1 \leq i < j < k \leq n} \left(\frac{a_i + a_j + a_k}{3\sqrt[3]{a_i a_j a_k}}\right)^{\frac{3}{n}}.$$

The proof follows by taking $r = 2$, respective $r = 3$ in inequalities (3).

Corollary 4. Let $a_1, a_2, \dots, a_n$ be positive numbers. Then for every $\{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, 2, \dots, n\}$, the following inequalities hold true:

$$\left(\frac{A_n[a_1, a_2, \dots, a_n]}{G_n[a_1, a_2, \dots, a_n]}\right)^n \geq \frac{1}{C_n^r} \cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]}\right)^r$$

and

$$\left(\frac{A_n[a_1, a_2, \dots, a_n]}{G_n[a_1, a_2, \dots, a_n]}\right)^{nC_n^r} \geq \prod_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \left(\frac{A_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]}{G_r\left[a_{i_1}, a_{i_2}, \dots, a_{i_r}\right]}\right)^r.$$

The proof follows by summation, respective by multiplication of the inequalities (4), with respect to all combinations $\{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, 2, \dots, n\}$.

4. FURTHER RESEARCH DIRECTIONS

We are convinced that our ideas in this paper will inspire other researchers to give new extensions and refinements of Radó and Popoviciu inequalities. To be more precisely, we mention the results obtained by Mortici [8], who proved that Radó and Popoviciu inequalities remain true by replacing the arithmetic and geometric means by other means, such as the harmonic mean

$$H_n[a_1, a_2, \dots, a_n] := \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

or

$$W_n[a_1, a_2, \dots, a_n] := \exp\left[\left(\frac{\ln^p a_1 + \ln^p a_2 + \cdots + \ln^p a_n}{n}\right)^{\frac{1}{p}}\right],$$

and

$$V_n[a_1, a_2, \dots, a_n] \coloneqq \exp\left[(\ln a_1 \cdot \ldots \cdot \ln a_n)^{\frac{1}{n}}\right].$$

REFERENCES

[1] Beckenbach, E. F., Bellman, R., *Inequalities*, Springer Verlag, 1961.
[2] Bullen, P. S., *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
[3] Marghidanu, D., Bencze, M., *OCTOGON Mathematical Magazine*, **12**, 532, 2004.
[4] Marghidanu, D., *Proposed Problem in Mathematical Inequalities*, **11**, 2022.
[5] Marghidanu, D., *Proposed Problem in arHivMath*, **11**, 2022.
[6] Marghidanu, D., *Romanian Mathematical Magazine*, **2022**(11**)**, 2022.
[7] Mitrinović, D. S., Vasić, P. M., *Analytic Inequalities*, Springer Verlag, Berlin, 1970.
[8] Mortici, C., *ANZIAM J.*, **50**, 137, 2008.
[9] Popoviciu, T., *Annals of the "Alexandru Ioan Cuza" University of Iaşi*, **11**, 155, 1965.