ORIGINAL PAPER

IMPLICIT FUNCTIONAL TRANSFORMATION METHOD FOR PERTURBED AND UNPERTURBED DIFFERENTIAL EQUATIONS

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Abstract. *Implicit functional transformations in the context of ordinary differential equations are discussed. The transformations can be applied to both perturbed and unperturbed equations. The link between the general transformations and various perturbation methods is exploited. The transformations can be used basically to eliminate the secularities in the solutions, which may lead to unphysical results. For unperturbed equations, the transformations can be employed to simplify the equations, allowing for the construction of exact analytical solutions, if they exist. Numerous worked examples are provided to illustrate the applications.*

Keywords: *Ordinary differential equation; exact solution; approximate solution; implicit function; perturbation.*

1. INTRODUCTION

Differential equations are one of the most common tools for expressing physical and engineering problems. The solutions can be categorized into various qualitative forms: Exact analytical, approximate analytical, and numerical. Determining the exact solutions is the first and best choice. However, equations possessing exact analytical solutions are rare, and one usually resorts to approximate analytical solutions as the next best option. Occasionally, approximate analytical solutions may have limitations, such as validity for small parameters in the case of perturbed equations and convergence limitations for the series solutions of unperturbed equations. Eventually, numerical solutions may be inevitable for constructing solutions in a wider parameter space.

In the past, perturbation methods proved to be effective in producing approximate analytical solutions. Many techniques and their variants were developed over time to eliminate the shortcomings of the so-called regular expansions [1,2]. A fundamental characteristic limitation of perturbation solutions is the validity of solutions for small parameters. There are also attempts to validate the solutions for large perturbation parameters by modifying the existing methods [3-6] or by incorporating iterations over the perturbation expansions [7-9].

For constructing exact analytical solutions, Lie Group techniques proved to be the most systematic and effective way of producing such solutions [10]. The theory unites the solutions determined by ad hoc transformations and various analytical techniques. However, the Lie Group transformations may also lead to differential equations that cannot be analytically solved. For collections of a wide range of analytical and numerical techniques in producing exact solutions, see [11,12].

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In this work, implicit functional transformations are employed in search of approximate and exact analytical solutions. For equations with small perturbation parameters, it is shown that the implicit functional transformations are more general compared to the well-known Lindstedt-Poincaré method [1,2], the method of multiple scales [1,2], and the newly proposed shift-perturbation method [13]. The mentioned classical methods are special cases of the implicit functional transformation method (IFTM). It is also pointed out in [13] that more advanced techniques of perturbations, such as the Lindstedt-Poincaré method or the method of multiple scales, can be interpreted as functional shifts in both the vertical and horizontal axes, whereas regular expansions only consider vertical functional shifts. It is shown in this work that the implicit functions can be used as tools to eliminate the secularities that spoil the uniformity of the solutions. The expansions allow vertical and horizontal shifts in both directions. In the more general proposed form presented here, the implicit functions may not be in the linear functional form as in the cases of Lindstedt-Poincaré and multiple scales methods, but may be nonlinear functions.

For differential equations without a small parameter, the implicit transformations are more general in some sense than the functional substitution method proposed in [14]. The goal is to transform the differential equation to a simpler form via transformations that can easily be solved. For both perturbed and unperturbed differential equations, a number of problems are considered as worked examples to exploit the applicability of the algorithm.

For the sake of completeness, there appeared several papers in the literature citing the so-called "*implicit function method*", which are algorithmically different than the algorithm presented here but which have a common base of using the implicit functions in determining analytical solutions. The implicit function method has been applied for determining the stability boundaries of Hill's equation [15]. For an application of the method to the nonlinear Schrödinger equation, see [16]. The method has been applied to the dynamic modeling of a slider-crank mechanism [17]. For a theoretical basis of the method, see [18-20].

2. PERTURBED EQUATIONS

Consider the k'th order nonlinear ordinary differential equation with a small parameter ε

$$F\left(t,u,\dot{u},\ddot{u},\dots u^{(k)};\varepsilon\right)=0 \tag{1}$$

The solution to the problem is $u=u(t)$. An approximate series solution in the form below

$$u(t,\delta_i(t,\varepsilon);\varepsilon)=u_0\left(t,\delta_i(t,\varepsilon)\right)+\varepsilon u_1\left(t,\delta_i(t,\varepsilon)\right)+\cdots, \tag{2}$$

may be assumed in which $\delta_i(t,\varepsilon)$ are determined by the elimination of secularities appearing at each order of approximation or by selecting a form that simplifies the equation. In the conventional method of multiple scales [1,2], $\delta_i=\varepsilon^i t, i=1,2,\dots$, represent the slow time scales, which are linear in time. However, δ_i need not be linear functions, and the expansion (2) with arbitrary nonlinear δ_i functions are more general than the method of multiple scales. Similar linear functions also appear in the Lindstedt-Poincaré method [1,2] and the newly proposed shift-perturbation method [13]. Note that $\delta_i\sim O(\varepsilon)$ or smaller in the expansion, therefore that they do not appear explicitly at the very first level of approximation.

2.1. FIRST ORDER EQUATIONS

A sample problem is treated in this section.

Example 1. Consider the nonlinear variable coefficient first-order differential equation with a small parameter ε

$$\dot{u}+\varepsilon\dot{f}(t)u^2=0 \qquad u(0)=1, \tag{3}$$

with the function having the property $f(0)=0$. Assuming an expansion given in (2) with only one implicit function $\delta(t;\varepsilon)$, and substituting into (3) yields

$$\frac{\partial u_0}{\partial t}+\frac{\partial u_0}{\partial \delta}\dot{\delta}+\varepsilon\dot{f}(t)u_0^2+\cdots=0. \tag{4}$$

Since $\delta\sim O(\varepsilon)$, separation with respect to orders yields

$$\frac{\partial u_0}{\partial t}=0 \tag{5}$$

$$\frac{\partial u_0}{\partial \delta}\dot{\delta}+\varepsilon\dot{f}(t)u_0^2=0 \tag{6}$$

From (5), $u_0=u_0(\delta)$. Now selecting $\dot{\delta}=\varepsilon\dot{f}(t)$, or $\delta=\varepsilon f(t)$ simplifies (6) with the remaining equation being

$$\frac{\partial u_0}{\partial \delta}+u_0^2=0 \tag{7}$$

whose solution is $u_0=\frac{1}{c+\delta}$. Inserting $\delta=\varepsilon f(t)$ and applying the initial condition, the final solution is

$$u_0=\frac{1}{1+\varepsilon f(t)}. \tag{8}$$

Although approximate methods are used, the above solution is indeed the exact solution. It can be verified by separation of variables or by using the integrability conditions and integration factor methods given in [21]. For direct integration, the variation of the integral of the differential equation should be identically equal to zero, i.e.

$$\delta\int F(u,\dot{u})dt\equiv 0. \tag{9}$$

or for $F=\dot{u}+\varepsilon\dot{f}(t)u^2$, the Euler equation $\frac{\partial F}{\partial u}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right)\equiv 0$ yields

$$2\varepsilon\dot{f}u\neq 0, \tag{10}$$

which implies that there is a need for an integration factor. Assuming an integration factor $\mu=\mu(u)$, $F=\mu\dot{u}+\varepsilon\mu\dot{f}(t)u^2$, the Euler equation yields

$$\varepsilon\dot{f}u(\mu' u+2\mu)\equiv 0, \tag{11}$$

from which $\mu = \frac{1}{u^2}$. Multiplying the whole equation by the integration factor, the equation can be expressed in a direct integrable form as

$$\frac{d}{dt}\left(-\frac{1}{u} + \varepsilon f(t)\right) \equiv 0, \tag{12}$$

which yields the exact solution given in (8).

2.2. SECOND ORDER EQUATIONS

Two sample problems are treated.

Example 2. Consider the lightly damped linear oscillator with a small parameter ε

$$\ddot{u} + 2\varepsilon\dot{u} + u = 0, \tag{13}$$

Inserting the expansion given in (2) with only one implicit function $\delta(t;\varepsilon)$ into (13) yields

$$\frac{\partial^2 u_0}{\partial t^2} + 2\frac{\partial^2 u_0}{\partial t \partial \delta}\dot{\delta} + \frac{\partial^2 u_0}{\partial \delta^2}\dot{\delta}^2 + \frac{\partial u_0}{\partial \delta}\ddot{\delta} + 2\varepsilon\left(\frac{\partial u_0}{\partial t} + \frac{\partial u_0}{\partial \delta}\dot{\delta}\right) + u_0 + \cdots = 0. \tag{14}$$

To simplify the equation, one may choose $\frac{\partial u_0}{\partial t} = 0$, yielding

$$\frac{\partial^2 u_0}{\partial \delta^2}\dot{\delta}^2 + \frac{\partial u_0}{\partial \delta}\ddot{\delta} + 2\varepsilon\frac{\partial u_0}{\partial \delta}\dot{\delta} + u_0 + \cdots = 0. \tag{15}$$

Choosing $u_0 = e^{\delta}$, the above equation further simplifies

$$\ddot{\delta} + \dot{\delta}^2 + 2\varepsilon\dot{\delta} + 1 = 0. \tag{16}$$

For the reduction of order, choose $\dot{\delta} = p$

$$\dot{p} + p^2 + 2\varepsilon p + 1 = 0 \tag{17}$$

For constant solutions

$$p = -\varepsilon \mp i\sqrt{1-\varepsilon^2} \tag{18}$$

and

$$\delta = \left(-\varepsilon \mp i\sqrt{1-\varepsilon^2}\right)t. \tag{19}$$

The solution is then

$$u_0 = c_1 e^{\left(-\varepsilon + i\sqrt{1-\varepsilon^2}\right)t} + c_2 e^{\left(-\varepsilon - i\sqrt{1-\varepsilon^2}\right)t}, \tag{20}$$

or in the alternative form

$$u_0 = a e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\,t + \beta\right), \tag{21}$$

which is the exact solution for a lightly damped linear oscillator. Using the method of multiple scales and taking two slow time scales, the solution turns out to be [1]

$$u_0 = ae^{-\varepsilon t}\cos\left(\left(1-\frac{1}{2}\varepsilon^2\right)t+\beta\right), \tag{22}$$

in which the amplitude decay is the same with the exact solution, but the damped frequency is an approximation of the exact one.

Example 3. Consider the Duffing equation with variable coefficient nonlinearity

$$\begin{gathered}\ddot{u}+u+\varepsilon\dot{f}(t)u^3=0,\\ u(0)=1,\dot{u}(0)=0\end{gathered} \tag{23}$$

with the function having the property $f(0)=0$. Inserting the expansion given in (2) with only one implicit function $\delta(t;\varepsilon)$ into (23) yields

$$\begin{gathered}\frac{\partial^2 u_0}{\partial t^2}+2\frac{\partial^2 u_0}{\partial t\partial\delta}\dot{\delta}+\frac{\partial^2 u_0}{\partial\delta^2}\dot{\delta}^2+\frac{\partial u_0}{\partial\delta}\ddot{\delta}+\varepsilon\left(\frac{\partial^2 u_1}{\partial t^2}+2\frac{\partial^2 u_1}{\partial t\partial\delta}\dot{\delta}+\frac{\partial^2 u_1}{\partial\delta^2}\dot{\delta}^2+\frac{\partial u_1}{\partial\delta}\ddot{\delta}\right)\\ +u_0+\varepsilon u_1+\varepsilon\dot{f}(t)u_0^3+\cdots=0.\end{gathered} \tag{24}$$

Assuming $\delta\sim O(\varepsilon)$, the equation separates into orders

$$\frac{\partial^2 u_0}{\partial t^2}+u_0=0 \tag{25}$$

$$\varepsilon\left(\frac{\partial^2 u_1}{\partial t^2}+u_1\right)=-2\frac{\partial^2 u_0}{\partial t\partial\delta}\dot{\delta}-\frac{\partial u_0}{\partial\delta}\ddot{\delta}-\varepsilon\dot{f}(t)u_0^3 \tag{26}$$

where $O(\varepsilon^2)$ terms are neglected. The solution of the first-order equation is

$$u_0=a(\delta)\cos\big(t+\beta(\delta)\big). \tag{27}$$

Substituting this solution into the next level of approximation yields

$$\begin{gathered}\varepsilon\left(\frac{\partial^2 u_1}{\partial t^2}+u_1\right)=\left(2a'\dot{\delta}+a\beta'\ddot{\delta}\right)\sin(t+\beta)+\Big(2a\beta'\dot{\delta}-a'\ddot{\delta}-\\ \varepsilon\dot{f}(t)\frac{3}{4}a^3\Big)\cos(t+\beta)+NST.\end{gathered} \tag{28}$$

where NST stands for non-secular terms. The coefficients of the secular terms are annihilated next for uniformly valid expansions

$$2a'\dot{\delta}+a\beta'\ddot{\delta}=0 \tag{29}$$

$$2a\beta'\dot{\delta}-a'\ddot{\delta}-\varepsilon\dot{f}(t)\frac{3}{4}a^3=0. \tag{30}$$

From (29), if $a'=0$, then $a=a_0$ and $\ddot{\delta}=0$ as a consequence. Since $\beta'\dot{\delta}=\dot{\beta}$, from (30)

$$\dot{\beta}=\frac{3}{8}\varepsilon\dot{f}(t)a_0^2, \tag{31}$$

or integrating

$$\beta=\frac{3}{8}\varepsilon f(t)a_0^2+\beta_0. \tag{32}$$

The solution is then

$$u_0 = a_0 \cos(t + \frac{3}{8}\varepsilon f(t) a_0^2 + \beta_0) \tag{33}$$

The initial conditions given in (23) require $a_0 = 1$ and $\beta_0 = 0$ which yields the approximate solution as follows

$$u = \cos\left(t + \frac{3}{8}\varepsilon f(t)\right) + \cdots \tag{34}$$

This approximate solution is contrasted with the numerical solution for $f(t) = \sin t$ (Fig. 1), $f(t) = 1 - e^{-t}$ (Fig. 2), $f(t) = \ln(1+t)$ (Fig. 3). To distinguish the curves from each other and yet maintain reasonable agreement, the perturbation parameter ε is taken as its limiting value.

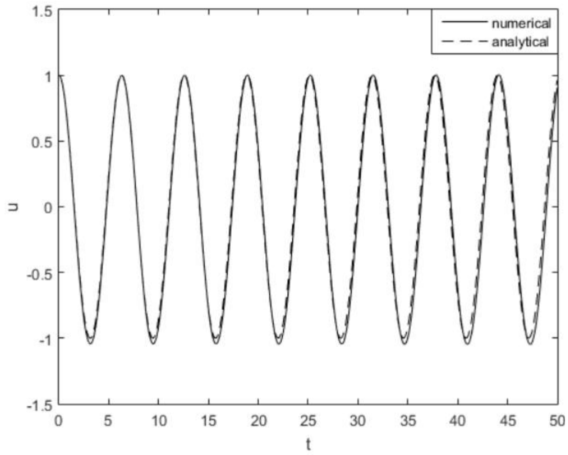

Figure 1. Comparison of the perturbation and numerical solutions for $f(t) = \sin t$ ($\varepsilon = 0.1$).

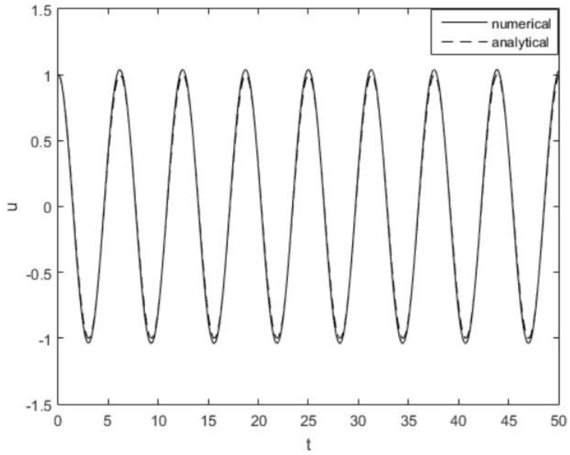

Figure 2. Comparison of the perturbation and numerical solutions for $f(t) = 1 - e^{-t}$ ($\varepsilon = 0.3$).

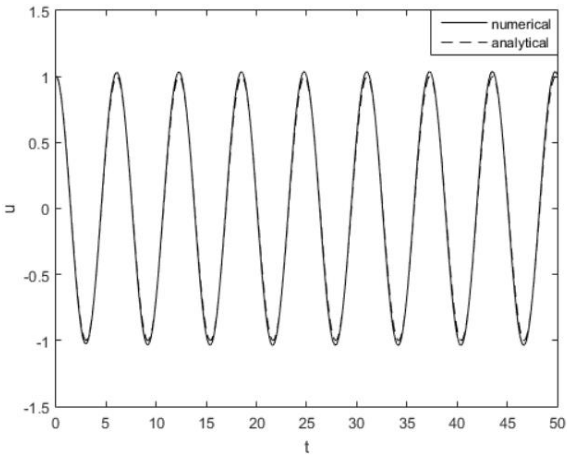

Figure 3. Comparison of the perturbation and numerical solutions for $f(t) = \ln(1+t)$ ($\varepsilon = 0.3$).

For oscillatory functions such as the sine function, the limiting value is as low as 0.1. The precision of the solutions increases for non-oscillatory functions such as the exponential and natural logarithm, as the limiting values for reasonable agreement can be as high as 0.3. It is well-known that perturbative solutions can be constructed for arbitrarily large values of the perturbation parameters for some nonlinear dynamical systems if variants of the existing methods can be employed. For a variant of the Lindstedt-Poincaré method, see [3, 4], and for a variant of the multiple scales method, see [5, 6]. A variant of the current technique can also be developed, which makes it possible to have admissible solutions for arbitrarily large perturbation parameters.

3. UNPERTURBED EQUATIONS

In this section, the equations treated will not possess small parameters, and the goal will be to determine the exact solutions. The role of the implicit transfer function would not be to eliminate secularities, but rather to simplify the equation.

3.1. FIRST ORDER EQUATIONS

The general solution of the Bernoulli equation is given in this section. In [12], the transformation leading to the solution without the general solution is given.

Example 4. Consider the Bernoulli equation

$$\frac{dy}{dx}+a(x)y=b(x)y^{\alpha} \tag{35}$$

The transformation

$$y=y(x,\delta(x)) \tag{36}$$

with its derivative

$$\frac{dy}{dx}=\frac{\partial y}{\partial x}+\frac{\partial y}{\partial \delta}\frac{d\delta}{dx} \tag{37}$$

leads to

$$\frac{\partial y}{\partial x}+\frac{\partial y}{\partial \delta}\frac{d\delta}{dx}+a(x)y=b(x)y^{\alpha}. \tag{38}$$

Assuming the more restricted case $y=y(\delta)$ and a solution $y=\delta^{\beta}$

$$\frac{\partial \delta}{\partial x}+\frac{a}{\beta}\delta=\frac{b}{\beta}\delta^{\beta\alpha+1-\beta}. \tag{39}$$

To simplify the equation, choose $\beta\alpha+1-\beta=0$, or

$$\beta=\frac{1}{1-\alpha}, \tag{40}$$

which yields the solution

$$y=\delta^{1/(1-\alpha)}. \tag{41}$$

To determine δ, substitute (40) into (39)

$$\frac{\partial \delta}{\partial x}+a(x)(1-\alpha)\delta=b(x)(1-\alpha), \tag{42}$$

which is a variable coefficient linear first order equation which can be solved by a number of techniques including the integrating factor method

$$\delta = e^{-(1-\alpha)\int a(x)dx}\left((1-\alpha)\int b(x)e^{(1-\alpha)\int a(x)dx}dx + c\right). \tag{43}$$

The general solution of the Bernoulli equation is found by inserting (43) into (41)

$$y(x) = e^{-\int a(x)dx}\left((1-\alpha)\int b(x)e^{(1-\alpha)\int a(x)dx}dx + c\right)^{1/(1-\alpha)}. \tag{44}$$

As a specific example, consider the Bernoulli equation with the condition

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^2}y^3 \ , y(1) = 1. \tag{45}$$

Defining $a(x) = -\frac{1}{x}$, $b(x) = \frac{1}{x^2}$ and $\alpha = 3$, and substituting all into the general solution given in (44) and performing the integrals, applying the condition, the final result is

$$y = \frac{x}{\sqrt{3-2x}}. \tag{46}$$

3.2. SECOND ORDER EQUATIONS

Variable coefficient linear differential equations are treated in this section.

Example 5. Consider the equation

$$\ddot{u} - \frac{1}{t}\dot{u} + 4t^2u = 0,$$
$$u(0) = 1, \dot{u}(0) = 0. \tag{47}$$

The transformation

$$u = u\big(t, \delta(t)\big), \tag{48}$$

with its derivatives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \delta}\dot{\delta} \ , \frac{d^2u}{dt^2} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial \delta}\dot{\delta} + \frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 \tag{49}$$

lead to

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial \delta}\dot{\delta} + \frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 - \frac{1}{t}\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \delta}\dot{\delta}\right) + 4t^2u = 0. \tag{50}$$

One may assume $u = u(\delta)$. If inconsistencies arise, then one has to resort to the more general form given in (48). Under this assumption, equation (50) reads

$$\frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 - \frac{1}{t}\frac{\partial u}{\partial \delta}\dot{\delta} + 4t^2u = 0 \ . \tag{51}$$

There are two options: i) Assume a form for $u = u(\delta)$ and solve for δ; ii) Assume a form for δ and solve for $u = u(\delta)$. If the assumptions are correct, then an exact solution will be retrieved.

i) Assume $u = cos\delta$

Substituting this form into (51)

$$cos\delta\left(-\dot{\delta}^2 + 4t^2\right) + sin\delta(-\ddot{\delta} + \frac{1}{t}\dot{\delta}) = 0. \tag{52}$$

To satisfy the equation, the coefficients should vanish, leading to $\delta = t^2$. Hence, the exact solution for the problem is

$$u = \cos(t^2), \tag{53}$$

which satisfies the initial conditions given.

ii) Assume $\delta = t^2$

Substituting this form into (51) and simplifying

$$\frac{\partial^2 u}{\partial \delta^2} + u = 0, \tag{54}$$

which yields $u = cos\delta$ or $u = \cos(t^2)$ for the given initial conditions.

Example 6. Consider the equation

$$\ddot{u} - \left(\frac{2}{t} + 1\right)\dot{u} + \left(\frac{2}{t^2} + \frac{1}{t} + e^{2t}\right)u = 0, \quad u(0) = 0, \dot{u}(0) = 0. \tag{55}$$

Applying the transformation given in (48) and (49) to the equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial \delta}\dot{\delta} + \frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 - \left(\frac{2}{t} + 1\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \delta}\dot{\delta}\right) + \left(\frac{2}{t^2} + \frac{1}{t} + e^{2t}\right)u = 0. \tag{56}$$

For simplification, one may assume a form $u = tsin\delta$. Substituting into (56)

$$tcos\delta\left(\ddot{\delta} - \dot{\delta}\right) + tsin\delta(-\dot{\delta}^2 + e^{2t}) = 0. \tag{57}$$

To satisfy the equation, each coefficient should vanish. If the assumed form is incorrect, then both coefficients cannot be eliminated. Requiring

$$\ddot{\delta} - \dot{\delta} = 0, -\dot{\delta}^2 + e^{2t} = 0, \tag{58}$$

leads to

$$\delta = e^t + c. \tag{59}$$

Hence, the solution satisfying the initial conditions is

$$u = tsin(e^t - 1) \tag{60}$$

Before closing the section, the following theorem is proved with the use of the implicit function transformation method.

Theorem 1. The general variable coefficient second-order homogeneous differential equation can be reduced to the Riccati equation from which the solution can be retrieved □

Proof: Consider the general equation

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0 \tag{61}$$

Inserting (48) and (49)

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial \delta}\dot{\delta} + \frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 + p\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \delta}\dot{\delta}\right) + qu = 0. \tag{62}$$

Assume $u = u(\delta)$. Then (62) reads

$$\frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 + p\frac{\partial u}{\partial \delta}\dot{\delta} + qu = 0 \tag{63}$$

To simplify the equation, choose $u = e^{\delta}$,

$$\ddot{\delta} + \dot{\delta}^2 + p\dot{\delta} + q = 0 \tag{64}$$

Define $\mu = \dot{\delta}$. Hence the equation

$$\dot{\mu} + \mu^2 + p\mu + q = 0 \tag{65}$$

is the well-known Riccati equation. For details of solution methods and properties of the Riccati equation, see [12, 22]. If $\mu_1(t)$ and $\mu_2(t)$ are two independent solutions of the Riccati equation, then $\delta_1(t) = \int \mu_1(t)dt$ and $\delta_2(t) = \int \mu_2(t)dt$. Since $u = e^{\delta}$, the general solution of the original equation is

$$u = c_1 e^{\int \mu_1(t)dt} + c_2 e^{\int \mu_2(t)dt} \tag{66}$$

in terms of the Riccati solutions.

Example 7. Consider the equation

$$\ddot{u} + \frac{4}{t}\dot{u} - \frac{10}{t^2}u = 0 \tag{67}$$

Define $p(t) = \frac{4}{t}$ and $q(t) = -\frac{10}{t^2}$. Equation (65) therefore reduces to

$$\dot{\mu} + \mu^2 + \frac{4}{t}\mu - \frac{10}{t^2} = 0. \tag{68}$$

The two independent solutions are $\mu_1(t) = \frac{2}{t}$, $\mu_2(t) = -\frac{5}{t}$. Substituting into (66), the solution of the second-order linear equation is

$$u = c_1 t^2 + c_2 t^{-5} \tag{69}$$

An equation without an exact solution cannot be solved by this method. This can be identified by the unsolvable Riccati equation for general second-order variable-coefficient equations. A worked example is given below.

Example 8. Consider the equation

$$\ddot{u} + t\dot{u} + (sint)u = 0. \quad (70)$$

Define $p(t) = t$ and $q(t) = sint$. Following the general steps given in the above proof, equation (70) reduces to

$$\dot{\mu} + \mu^2 + t\mu + sint = 0. \quad (71)$$

The Riccati equation above does not possess an exact solution. Therefore, the original equation also does not have an exact analytical solution. The method fails when no analytical solution exists for the original equation. For variable coefficient second-order equations in the form of (61), one can say that if the corresponding Riccati equation (65) does have an analytical solution, then the original equation also possesses an analytical solution.

For general nonlinear problems, however, no such general statements can be written, and when one encounters an unsolvable transformed equation, one may understand that the exact solution does not exist. The next example discusses the issue.

Example 9. Reconsider the Duffing equation in Example 3 with variable coefficient nonlinearity, but the nonlinearity does not possess a small parameter coefficient.

$$\ddot{u} + u + \dot{f}(t)u^3 = 0. \quad (72)$$

The implicit function transformation

$$u = u(t, \delta(t)), \quad (73)$$

leads to the transformed equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial \delta}\dot{\delta} + \frac{\partial u}{\partial \delta}\ddot{\delta} + \frac{\partial^2 u}{\partial \delta^2}\dot{\delta}^2 + u + \dot{f}(t)u^3 = 0. \quad (74)$$

which is highly nonlinear and unsolvable even if simplified forms are assumed for the functions. The success in finding approximate analytical solutions in the case of Example 3 highly depends on the small parameter, which disintegrates the equations with respect to the orders of small parameters and produces simpler, solvable forms. Since the original equation (72) does not possess an exact analytical solution, the method is incapable to produce such a solution from the transformed equation (74).

4. CONCLUSIONS

The implicit functional transformation method (IFTM) is outlined in this paper with possible applications for perturbed and unperturbed ordinary differential equations. In the case of perturbed equations, the method is more straightforward and mechanistic, with the employment of the implicit functions to eliminate secularities or introduce simplifications in the expansions. Contrary to the well-established methods, which employ linear implicit functions in time, depending on the nature of the problem, nonlinear functions may be required, which are given in worked examples. The application of the method for unperturbed equations is not as straightforward as in the case of perturbed equations, since the role of the implicit functions is to simplify the equation for solvability. This requires some intuition and ad hoc transformations that highly depend on the nature of the differential equation. The method will definitely fail if there is no exact solution available for the problem. Single implicit functions are employed in the expansions. Some problems may require more than one implicit function to eliminate the inconsistencies.

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