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QUADRA FIBONA-PELL HYBRID NUMBERS AND HYBRINOMIALS

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Abstract. *In this paper, we define quadra Fibona-Pell hybrid numbers by using the definition of hybrid numbers. After that, we introduce quadra Fibona-Pell hybrinomials and investigate some properties of quadra Fibona-Pell hybrid numbers and hybrinomials. Finally, we show the matrix representation of quadra Fibona-Pell hybrinomials.*
Keywords: *Quadra Fibona-Pell sequence; hybrid numbers; generating function.*

1. INTRODUCTION

The integer sequences and the polynomials of these sequences were studied by many mathematicians. Some of these are Fibonacci polynomials [1] and Pell polynomials [2] which are the base of this work.

In [3], Tasci defined quadrapell numbers with the fourth-order recurrence relation

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4}, n \geq 4,$$

where $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$ are the initial values. After that, some properties and matrix sequences of these numbers were given in [4].

Inspiring the definition of quadrapell numbers in [5], Özkoç introduced quadra Fibona-Pell numbers recursively by

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}$$

for $n \geq 4$, where $W_0 = W_1 = 0, W_2 = 1$ and $W_3 = 3$. The most important property of this sequence is that the characteristic equation of the sequence consists of the roots of the characteristic equations of the Fibonacci and Pell sequences.

For $n \geq 0$,

$$W_n = P_n - F_n, \tag{1}$$

where P_n and F_n are the n −th Pell and Fibonacci numbers, respectively [5].

After that, in [6] quadra Fibona-Pell polynomials introduced by

$$W_n(x) = 3xW_{n-1}(x) - (2x^2 - 2)W_{n-2}(x) - 3xW_{n-3}(x) - W_{n-4}(x),$$

with the initial conditions $W_0(x) = W_1(x) = 0, W_2(x) = x$ and $W_3(x) = 3x^2$.

For $n \geq 0$,

$$W_n(x) = P_n(x) - F_n(x), \tag{2}$$

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where $P_n(x)$ and $F_n(x)$ are Pell and Fibonacci polynomials, respectively [6].

The hybrid numbers were defined as a generalization of complex, hyperbolic and dual numbers [7]. These numbers are the elements of the set

$$K = a + bi + \boldsymbol{c}\epsilon + \boldsymbol{d}h: a, b, c, d \in \mathbb{R}.$$

Let $Z_1 = a_1 + b_1 i + c_1 \epsilon + d_1 h$ and $Z_2 = a_2 + b_2 i + c_2 \epsilon + d_2 h$ be any two hybrid numbers. Then, the main operations on hybrid numbers are defined as follows:

- $Z_1 = Z_2$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_{2,} d_1 = d_2$,
- $Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)\epsilon + (d_1 + d_2)h$,
- $Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)\epsilon + (d_1 - d_2)h$,
- $sZ_1 = sa_1 + sb_1 i + sc_1 \epsilon + sd_1 h$, where $s \in \mathbb{R}$.

By using Table 1 any two hybrid numbers can be multiplied:

Table 1. Multiplication Table.

•	1	i	ϵ	h
1	1	i	ϵ	h
i	i	-1	$1 - \mathrm{h}$	$\epsilon + i$
ϵ	ϵ	$h + 1$	0	$-\epsilon$
h	h	$-\epsilon - i$	ϵ	1

After this work, Szynal-Liana introduced Fibonacci hybrid numbers by using the definition of hybrid numbers [8]. Also, Catarino defined k −Pell hybrid numbers in [9]. Recently, some properties between Mersenne, Jacobsthal and Jacobsthal-Lucas hybrid number were given in [10]. Moreover, hybrinomials which are the polynomials of hybrid number sequences were defined by many authors. Firstly, Szynal-Liana and Wloch defined Fibonacci and Lucas hybrinomials in [11]. Then, Pell hybrinomials were defined in [12]. Lastly, the generalized Lucas hybrinomials with two variables were given [13].

2. QUADRA FIBONA-PELL HYBRID NUMBERS

The quadra Fibona-Pell hybrid number is recursively defined by

$$WH_n = W_n + W_{n+1} i + W_{n+2} \epsilon + W_{n+3} h, n \geq 0, \tag{3}$$

where W_n is the n −th quadra Fibona-Pell number. By using the equation (3), the first few elements of quadra Fibona-Pell hybrid numbers can be obtained as in Table 2:

Table 2. The first few elements of quadra Fibona-Pell hybrid numbers.

n	WH_n
0	$\epsilon + 3h$
1	$1 + 3\epsilon + 9h$
2	$1 + 3\mathrm{i} + 9\epsilon + 24h$
3	$3 + 9\mathrm{i} + 24\epsilon + 62h$
4	$9 + 24\mathrm{i} + 62\epsilon + 156h$
5	$24 + 62\mathrm{i} + 156\epsilon + 387h$
6	$62 + 156\mathrm{i} + 387\epsilon + 951h$

Note that, for $n \geq 4$, the sequence of quadra Fibona-Pell hybrid numbers satisfies the following fourth order recurrence relation

$$WH_n = 3WH_{n-1} - 3WH_{n-3} - WH_{n-4}$$

with the initial conditions

$$WH_0 = \epsilon + 3h,$$
$$WH_1 = 1 + 3\epsilon + 9h,$$
$$WH_2 = 1 + 3\mathrm{i} + 9\epsilon + 24h,$$
$$WH_3 = 3 + 9\mathrm{i} + 24\epsilon + 62h.$$

Lemma 2.1. For $n \geq 0$, we have

$$WH_n = PH_n - FH_n,$$

where PH_n is the $n-$th Pell hybrid number and FH_n is the $n-$th Fibonacci hybrid number.

Proof: By substituting equation (1) in (3), we have

$$\begin{aligned} WH_n &= W_n + W_{n+1}i + W_{n+2}\epsilon + W_{n+3}h \\ &= P_n - F_n + (P_{n+1} - F_{n+1})i + (P_{n+2} - F_{n+2})\epsilon + (P_{n+3} - F_{n+3})h \\ &= P_n + P_{n+1}i + P_{n+2}\epsilon + P_{n+3}h - (F_n + F_{n+1}i + F_{n+2}\epsilon + F_{n+3}h) \\ &= PH_n - FH_n. \end{aligned}$$

■

Theorem 2.2. The generating function for the quadra Fibona-Pell hybrid numbers WH_n is

$$\sum_{n=0}^{\infty} WH_n x^n = \frac{\epsilon + 3h + ix + (1-3h)x^2 - hx^3}{1-3x + 3x^3 + x^4}.$$

Proof: The formal power series expansion of the generating function for WH_n at $x = 0$ is

$$G(x) = \sum_{n=0}^{\infty} WH_n x^n = WH_0 + WH_1 x + WH_2 x^2 + \cdots + WH_n x^n + \cdots \quad (4)$$

Then, by multiplying the equation (4) by $-3x, 3x^3$ and x^4, respectively, we have

$$\begin{aligned} -3xG(x) &= -3WH_0 x - 3WH_1 x^2 - 3WH_2 x^3 - \cdots - 3WH_n x^{n+1} + \cdots, \\ 3x^3 G(x) &= 3WH_0 x^3 + 3WH_1 x^4 + 3WH_2 x^5 + \cdots + 3WH_n x^{n+3} + \cdots, \\ x^4 G(x) &= WH_0 x^4 + WH_1 x^5 + WH_2 x^6 + \cdots + WH_n x^{n+4} + \cdots. \end{aligned}$$

By using the above equations, we get

$$(1-3x + 3x^3 + x^4)G(x) = \epsilon + 3h + ix + (1-3h)x^2 - hx^3.$$

The above equation yields the result.

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Notice that, by using the partial fractions decomposition of the generating function for the quadra Fibona-Pell hybrid numbers we can write

$$\begin{aligned} \sum_{n=0}^{\infty} WH_n x^n &= \frac{(1 + \epsilon + 2h)x + i + 2\epsilon + 5h}{1-2x-x^2} - \frac{(1 + \epsilon + h)x + i + \epsilon + 2h}{1-x-x^2} \\ &= \sum_{n=0}^{\infty} PH_n x^n - \sum_{n=0}^{\infty} FH_n x^n. \end{aligned}$$

Theorem 2.3. The Binet formula for the quadra Fibona-Pell hybrid numbers WH_n is

$$WH_n = \frac{\Phi^n\widehat{\Phi} - \Psi^n\widehat{\Psi}}{\Phi - \Psi} - \frac{\alpha^n\hat{\alpha} - \beta^n\hat{\beta}}{\alpha - \beta},$$

where $\widehat{\Phi}, \widehat{\Psi}, \hat{\alpha}$ and $\hat{\beta}$ are defined by

$$\begin{aligned} \widehat{\Phi} &= 1 + i\Phi + \epsilon\Phi^2 + h\Phi^3, \\ \widehat{\Psi} &= 1 + i\Psi + \epsilon\Psi^2 + h\Psi^3, \\ \hat{\alpha} &= 1 + i\alpha + \epsilon\alpha^2 + h\alpha^3, \\ \hat{\beta} &= 1 + i\beta + \epsilon\beta^2 + h\beta^3. \end{aligned}$$

Proof: By using Lemma 2.1 and the Binet formulas of Pell hybrid numbers and Fibonacci hybrid numbers the theorem can be proved easily.

■

Theorem 2.4. The sum of the first n terms of the quadra Fibona-Pell hybrid numbers WH_n is

$$\sum_{i=0}^{n} WH_i = \frac{WH_{n-3} + 4WH_{n-2} + 4WH_{n-1} + WH_n + (1 + i - \epsilon - 7h)}{2}.$$

Proof: By modifying the recurrence relation of quadra Fibona-Pell hybrid numbers, we get

$$WH_{n-3} + WH_{n-4} = 3WH_{n-1} - 2WH_{n-3} - WH_n.$$

By the help of the above equation, we get

$$\begin{aligned} WH_1 + WH_0 &= 3WH_3 - 2WH_1 - WH_4, \\ WH_2 + WH_1 &= 3WH_4 - 2WH_2 - WH_5, \\ WH_3 + WH_2 &= 3WH_5 - 2WH_3 - WH_6, \\ &\vdots \\ WH_{n-4} + WH_{n-5} &= 3WH_{n-2} - 2WH_{n-4} - WH_{n-1}, \\ WH_{n-3} + WH_{n-4} &= 3WH_{n-1} - 2WH_{n-3} - WH_n. \end{aligned} \tag{5}$$

Then, by taking the summation of both sides of (5), we obtain

$$\begin{aligned} &WH_{n-3} + WH_0 + 2(WH_1 + \cdots + WH_{n-4}) \\ &\quad = 3(WH_3 + \cdots + WH_{n-1}) - 2(WH_1 + \cdots + WH_{n-3}) - (WH_4 + \cdots + WH_n. \end{aligned}$$

From the above equation, we get

$$\begin{aligned} &2(WH_1 + \cdots + WH_{n-3}) \\ &\quad = -WH_{n-3} + 2WH_{n-2} + 2WH_{n-1} - WH_n - WH_0 - 2WH_1 - 2WH_2 + WH_3. \end{aligned}$$

Finally, we obtain the desired result

$$\sum_{i=0}^{n} WH_i = \frac{WH_{n-3} + 4WH_{n-2} + 4WH_{n-1} + WH_n + (1 + i - \epsilon - 7h)}{2}.$$

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3. QUADRA FIBONA-PELL HYBRINOMIALS

The quadra Fibona-Pell hybrinomials are defined by

$$WH_n(x) = W_n(x) + W_{n+1}(x)i + W_{n+2}(x)\epsilon + W_{n+3}(x)h, n \geq 0, \quad (6)$$

where W_n(x) is the n −th quadra Fibona-Pell polynomial.

Theorem 3.1. For $n \geq 4$ the quadra Fibona-Pell hybrinomials $WH_n(x)$ provides the recurrence relation

$$WH_n(x) = 3xWH_{n-1}(x) - (2x^2 - 2)WH_{n-2}(x) - 3xWH_{n-3}(x) - WH_{n-4}(x),$$

with the initial conditions

$$WH_0(\mathrm{x}) = \mathrm{x}\epsilon + 3x^2h,$$
$$WH_1(\mathrm{x}) = \mathrm{xi} + 3x^2\epsilon + (7x^3 + 2\mathrm{x})h,$$
$$WH_2(\mathrm{x}) = \mathrm{x} + 3x^2\mathrm{i} + (7x^3 + 2\mathrm{x})\epsilon + (15x^4 + 9x^2)h,$$
$$WH_3(\mathrm{x}) = 3x + (7x^3 + 2\mathrm{x})\mathrm{i} + (15x^4 + 9x^2)\epsilon + (31x^5 + 28x^3 + 3\mathrm{x})h.$$

Proof: By using the definition of quadra Fibona-Pell polynomial, we get

$$\begin{aligned}
WH_n(x) &= W_n(x) + W_{n+1}(x)i + W_{n+2}(x)\epsilon + W_{n+3}(x)h \\
&= 3xW_{n-1}(x) - (2x^2 - 2)W_{n-2}(x) - 3xW_{n-3}(x) - W_{n-4}(x) \\
&+ (3xW_n(x) - (2x^2 - 2)W_{n-1}(x) - 3xW_{n-2}(x) - W_{n-3}(x))i \\
&+ (3xW_{n+1}(x) - (2x^2 - 2)W_n(x) - 3xW_{n-1}(x) - W_{n-2}(x))\epsilon \\
&+ (3xW_{n+2}(x) - (2x^2 - 2)W_{n+1}(x) - 3xW_n(x) - W_{n-1}(x))h \\
&= 3x(W_{n-1}(x) + W_n(x)i + W_{n+1}(x)\epsilon + W_{n+2}(x)h) \\
&- (2x^2 - 2)(W_{n-2}(x) + W_{n-1}(x)i + W_n(x)\epsilon + W_{n+1}(x)h) \\
&- 3x(W_{n-3}(x) + W_{n-2}(x)i + W_{n-1}(x)\epsilon + W_n(x)h) \\
&- (W_{n-4}(x) + W_{n-3}(x)i + W_{n-2}(x)\epsilon + W_{n-1}(x)h) \\
&= 3xWH_{n-1}(x) - (2x^2 - 2)WH_{n-2}(x) - 3xWH_{n-3}(x) - WH_{n-4}(x)
\end{aligned}$$

■

Notice that for $x = 1$, $WH_n(1)$ gives the quadra Fibona-Pell hybrid numbers.

Theorem 3.2. For $n \geq 0$, we have

$$WH_n(x) = PH_n(x) - FH_n(x), \quad (7)$$

where $PH_n(x)$ is the n −th Pell hybrinomial and $FH_n(x)$ is the n −th Fibonacci hybrinomial.

Proof: By substituting equation (2) in (6), we have

$$\begin{aligned}
WH_n(x) &= W_n(x) + W_{n+1}(x)\, i + W_{n+2}(\mathrm{x})\, \epsilon + W_{n+3}(x)\, h \\
&= P_n(\mathrm{x}) - F_n(x) + (P_{n+1}(\mathrm{x}) - F_{n+1}(x))i + (P_{n+2}(x) - F_{n+2}(x))\epsilon + (P_{n+3}(x) \\
&- F_{n+3}(x))h \\
&= P_n(x) + P_{n+1}(x)\, i + P_{n+2}(x)\, \epsilon + P_{n+3}(x)\, h \\
&- (F_n(x) + F_{n+1}(x)\, i + F_{n+2}(x)\, \epsilon + F_{n+3}(x)\, h) \\
&= PH_n(x) - FH_n(x).
\end{aligned}$$

■

Theorem 3.3. The Binet formula for the quadra Fibona-Pell hybrinomials $WH_n(x)$ is

$$WH_n(x) = \frac{\Phi^n(x)\widehat{\Phi}(x) - \Psi^n(x)\widehat{\Psi}(x)}{\Phi(x) - \Psi(x)} - \frac{\alpha^n(x)\hat{\alpha}(x) - \beta^n(x)\hat{\beta}(x)}{\alpha(x) - \beta(x)},$$

where $\widehat{\Phi}(x), \widehat{\Psi}(x), \hat{\alpha}(x)$ and $\hat{\beta}(x)$ are defined by

$$\begin{aligned}
\widehat{\Phi}(x) &= 1 + i\Phi(x) + \epsilon\Phi^2(x) + h\Phi^3(x),\\
\widehat{\Psi}(x) &= 1 + i\Psi(x) + \epsilon\Psi^2(x) + h\Psi^3(x),\\
\hat{\alpha}(x) &= 1 + i\alpha(x) + \epsilon\alpha^2(x) + h\alpha^3(x),\\
\hat{\beta}(x) &= 1 + i\beta(x) + \epsilon\beta^2(x) + h\beta^3(x).
\end{aligned}$$

Proof: By using Theorem 3.2 and the Binet formulas of Pell hybrinomials and Fibonacci hybrinomails the theorem can be proved easily.

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Theorem 3.4. The generating function for the quadra Fibona-Pell hybrinomials $WH_n(x)$ is

$$\sum_{n=0}^{\infty} WH_n(x)\mathrm{t}^n = \frac{\epsilon x + h(3x^2) + (ix + h(-2x^3 + 2x))t + (x - h(3x^2))t^2 - hxt^3}{1 - 3xt + (2x^2 - 2)t^2 + 3xt^3 + t^4}.$$

Proof: The formal power series expansion of the generating function for $WH_n(x)$ at $x = 0$ is

$$\begin{aligned}
G(\mathrm{t}) &= \sum_{n=0}^{\infty} WH_n(x)\mathrm{t}^n \\
&= WH_0(x) + WH_1(x)\mathrm{t} + WH_2(x)\mathrm{t}^2 + \cdots + WH_n(x)\mathrm{t}^n + \cdots
\end{aligned} \tag{8}$$

Then, by multiplying the equation (8) by $-3x\mathrm{t}, \left(2x^2 - 2\right)t^2, 3xt^3$ and t^4, respectively, we have

$$\begin{aligned}
-3x\mathrm{t}G(\mathrm{t}) &= -3xWH_0(x)\mathrm{t} - 3xWH_1(x)\mathrm{t}^2 - 3xWH_2(x)\mathrm{t}^3 - \cdots - 3xWH_n(x)\mathrm{t}^{n+1} + \cdots,\\
(2x^2 - 2)t^2G(\mathrm{t}) &= (2x^2 - 2)WH_0(x)\mathrm{t}^2 + (2x^2 - 2)WH_1(x)\mathrm{t}^3 + (2x^2 - 2)WH_2(x)\mathrm{t}^4\\
&\quad + \cdots + (2x^2 - 2)WH_n(x)\mathrm{t}^{n+2} + \cdots,\\
3xt^3G(\mathrm{t}) &= 3xWH_0(x)\mathrm{t}^3 + 3xWH_1(x)\mathrm{t}^4 + 3xWH_2(x)\mathrm{t}^5 + \cdots + 3xWH_n(x)\mathrm{t}^{n+3} + \cdots,\\
t^4G(\mathrm{t}) &= WH_0(x)\mathrm{t}^4 + WH_1(x)\mathrm{t}^5 + WH_2(x)\mathrm{t}^6 + \cdots + WH_n(x)\mathrm{t}^{n+4} + \cdots.
\end{aligned}$$

By using the above equations, we get

$$\begin{aligned}
&(1 - 3xt + (2x^2 - 2)t^2 + 3xt^3 + t^4)G(t)\\
&= \epsilon x + h(3x^2) + (ix + h(-2x^3 + 2x))t + (x - h(3x^2))t^2 - hxt^3.
\end{aligned}$$

The above equation yields the result.

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Notice that, by using the partial fractions decomposition of the generating function for the quadra Fibona-Pell hybrinomials we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} WH_n(x)t^n\\
&= \frac{i + \epsilon(2x) + h(4x^2 + 1) + (1 + \epsilon + h(2x))t}{1 - 2xt - t^2} - \frac{i + \epsilon x + h(x^2 + 1) + (1 + \epsilon + hx)t}{1 - xt - t^2}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} PH_n(x)t^n - \sum_{n=0}^{\infty} FH_n(x)t^n .$$

Theorem 3.5. For any nonnegative integer n, we have

$$\begin{bmatrix} WH_{n+6}(x) & WH_{n+5}(x) & WH_{n+4}(x) & WH_{n+3}(x) \\ WH_{n+5}(x) & WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) \\ WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) \\ WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) & WH_{n}(x) \end{bmatrix} = \begin{bmatrix} WH_{6}(x) & WH_{5}(x) & WH_{4}(x) & WH_{3}(x) \\ WH_{5}(x) & WH_{4}(x) & WH_{3}(x) & WH_{2}(x) \\ WH_{4}(x) & WH_{3}(x) & WH_{2}(x) & WH_{1}(x) \\ WH_{3}(x) & WH_{2}(x) & WH_{1}(x) & WH_{0}(x) \end{bmatrix} M^n,$$

where $M = \begin{bmatrix} 3x & 1 & 0 & 0 \\ -(2x^2-2) & 0 & 1 & 0 \\ -3x & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

Proof: We prove this theorem by using induction on n. For $n = 0$, since the zero power of a matrix gives identity matrix the result is obvious. Assume that the theorem holds for any integer $n - 1 > 0$. We will show that the theroem holds for n.

By using the assumption in the hypothesis of induction, we get

$$\begin{aligned} &\begin{bmatrix} WH_{6}(x) & WH_{5}(x) & WH_{4}(x) & WH_{3}(x) \\ WH_{5}(x) & WH_{4}(x) & WH_{3}(x) & WH_{2}(x) \\ WH_{4}(x) & WH_{3}(x) & WH_{2}(x) & WH_{1}(x) \\ WH_{3}(x) & WH_{2}(x) & WH_{1}(x) & WH_{0}(x) \end{bmatrix} M^n \\ &= \begin{bmatrix} WH_{6}(x) & WH_{5}(x) & WH_{4}(x) & WH_{3}(x) \\ WH_{5}(x) & WH_{4}(x) & WH_{3}(x) & WH_{2}(x) \\ WH_{4}(x) & WH_{3}(x) & WH_{2}(x) & WH_{1}(x) \\ WH_{3}(x) & WH_{2}(x) & WH_{1}(x) & WH_{0}(x) \end{bmatrix} M^{n-1} M \\ &= \begin{bmatrix} WH_{n+5}(x) & WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) \\ WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) \\ WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) & WH_{n}(x) \\ WH_{n+2}(x) & WH_{n+1}(x) & WH_{n}(x) & WH_{n-1}(x) \end{bmatrix} M \\ &= \begin{bmatrix} WH_{n+6}(x) & WH_{n+5}(x) & WH_{n+4}(x) & WH_{n+3}(x) \\ WH_{n+5}(x) & WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) \\ WH_{n+4}(x) & WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) \\ WH_{n+3}(x) & WH_{n+2}(x) & WH_{n+1}(x) & WH_{n}(x) \end{bmatrix}, \end{aligned}$$

which completes the proof.

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Corollary 3.6. Let $n \geq 0$ be an integer and $x = 1$. Then, since $WH_n(1) = WH_n$ we have

$$\begin{bmatrix} WH_{n+6} & WH_{n+5} & WH_{n+4} & WH_{n+3} \\ WH_{n+5} & WH_{n+4} & WH_{n+3} & WH_{n+2} \\ WH_{n+4} & WH_{n+3} & WH_{n+2} & WH_{n+1} \\ WH_{n+3} & WH_{n+2} & WH_{n+1} & WH_{n} \end{bmatrix} = \begin{bmatrix} WH_{6} & WH_{5} & WH_{4} & WH_{3} \\ WH_{5} & WH_{4} & WH_{3} & WH_{2} \\ WH_{4} & WH_{3} & WH_{2} & WH_{1} \\ WH_{3} & WH_{2} & WH_{1} & WH_{0} \end{bmatrix} M^n.$$

Theorem 3.7. For any nonnegative integer n, we have

$$\begin{bmatrix} WH_{n+2}(x) & WH_{n+1}(x) \\ WH_{n+1}(x) & WH_{n}(x) \end{bmatrix} = \begin{bmatrix} PH_{2}(x) & PH_{1}(x) \\ PH_{1}(x) & PH_{0}(x) \end{bmatrix} \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n - \begin{bmatrix} FH_{2}(x) & FH_{1}(x) \\ FH_{1}(x) & FH_{0}(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof: By using the equation (7), we deduced the desired result as follows;

$$\begin{aligned} \begin{bmatrix} WH_{n+2}(x) & WH_{n+1}(x) \\ WH_{n+1}(x) & WH_{n}(x) \end{bmatrix} &= \begin{bmatrix} PH_{n+2}(x)-FH_{n+2}(x) & PH_{n+1}(x)-FH_{n+1}(x) \\ PH_{n+1}(x)-FH_{n+1}(x) & PH_{n}(x)-FH_{n}(x) \end{bmatrix} \\ &= \begin{bmatrix} PH_{n+2}(x) & PH_{n+1}(x) \\ PH_{n+1}(x) & PH_{n}(x) \end{bmatrix} - \begin{bmatrix} FH_{n+2}(x) & FH_{n+1}(x) \\ FH_{n+1}(x) & FH_{n}(x) \end{bmatrix} \\ &= \begin{bmatrix} PH_{2}(x) & PH_{1}(x) \\ PH_{1}(x) & PH_{0}(x) \end{bmatrix} \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n - \begin{bmatrix} FH_{2}(x) & FH_{1}(x) \\ FH_{1}(x) & FH_{0}(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n. \end{aligned}$$

4. CONCLUSION

In this study, we define quadra Fibona-Pell hybrid numbers by using the definition of hybrid numbers. After that, we introduce quadra Fibona-Pell hybrinomials and investigate some properties of quadra Fibona-Pell hybrid numbers and hybrinomials. Finally, we show the matrix representation of quadra Fibona-Pell hybrinomials.

REFERENCES

[1] Koshy, T., *Fibonacci and Lucas numbers with applications*, John Wiley & Sons, New York, 2001.
[2] Horadam, A., Mahon, J., *Fibonacci Quarterly*, **23**(1), 7, 1985.
[3] Tasci, D., *Hacettepe Journal of Mathematics and Statistics*, **38**(3), 265, 2009.
[4] Kızılateş, C., Tuglu, N., Çekim, B., *Journal of Science and Arts*, **1**(38), 69, 2017.
[5] Ozkoc, A., *Advances in Difference Equations*, **2015**, 1, 2015.
[6] Sevgi, E., *Notes on Number Theory and Discrete Mathematics*, **30**(3), 2024.
[7] Ozdemir, M., *Advances in Applied Clifford Algebras*, **28**(11), 2018.
[8] Szynal-Liana, A., *Utilitas Mathematica*, **110**, 1428, 2019.
[9] Catarino, P., *Journal of Discrete Mathematical Sciences and Cryptography*, **22**(1), 83, 2019.
[10] Tasci, D., Sevgi, E., *Chaos, Solitons and Fractals*, **146**, 110862, 2021.
[11] Szynal-Liana, A., Wloch, I., *Complex Variables and Elliptic Equations*, **65**(10), 1736, 2020.
[12] Liana, M., Szynal-Liana, A., WLoch, I., *Miskolc Mathematical Notes*, **20**(2), 1051, 2019.
[13] Sevgi, E., *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*, **70**(2), 622, 2021.