ORIGINAL PAPER

ON THE GEODESIC CURVES OF CONSTANT BREADTH ACCORDING TO THE EXTENDED DARBOUX FRAME

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Abstract. *In this study, we investigate the geodesic curves of constant breadth on an oriented hypersurface in Euclidean 4-space according to the extended Darboux frame and give the general differential equation characterizing these curves.*
Keywords: *constant breadth curve; extended Darboux frame; geodesic curve.*

1. INTRODUCTION

Constant breadth curves defined in the plane by Euler in 1778 [1] have since been used in different fields such as kinematics, mechanical engineering, and computer design [2]. Many researchers have become interested in these curves and have investigated the properties of planar constant breadth curves. By taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve and for which the distance $d(P,Q)$ is constant, Fujiwara [3] defined constant breadth curves and also obtained these curves on a surface of constant breadth. The properties of plane curves of constant breadth were investigated in [4-8]. Plane curves of constant breadth were studied in Euclidean three and four spaces in [9,10]. The curves of constant breadth were studied in Euclidean n−space in [11]. Altunkaya and Aksoyak [12] have investigated constant breadth curves on a surface according to the Darboux frame and have given some characterizations of these curves. Some characterizations of the constant breadth curves according to the Bishop frame and type-2 Bishop frame have been presented in [13,14].

There are also some studies in which such curves are discussed in different spaces such as Minkowski spaces and Galilean spaces. In Minkowski 3-space, the differential equations characterizing the timelike and spacelike constant breadth curves have been given in [15] and some characterizations of the timelike curves of constant breadth have been presented in [16]. Kazaz et al. [17] have studied spacelike constant breadth curves in Minkowski 4-space. Altunkaya and Aksoyak [18] have studied null constant breadth curves in Minkowski 4-space. Yoon [19] has investigated the properties of curves of constant breadth in 3-dimensional Galilean space and Ünlütürk et al. [20] have studied constant breadth curves in pseudo-Galilean space. The curves of constant breadth according to the Darboux frame in the 3-dimensional Galilean space are investigated by Bozok in [21].

In this paper, we consider the curves of constant breadth on an oriented hypersurface according to the extended Darboux frame in Euclidean 4-space and give the general differential equation characterizing the geodesic curves of constant breadth according to the extended Darboux frame in Euclidean 4-space.

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2. PRELIMINARIES

In this part, let us give some fundamental definitions and theorems about the curves of constant breadth and the extended Darboux frame in Euclidean 4-space E^4.

Definition 2.1. Let α be a unit speed closed curve in E^3. If the curve α has parallel tangents in opposite directions at the opposite points and the distance between these opposite points is always constant, then α is called a curve of constant breadth, [3]. Moreover, if there is a pair of curves α and α^* where the tangents at the corresponding points are parallel and in opposite directions and the distance between the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ is always constant, then (α, α^*) is called a pair of curves of constant breadth, where s and s^* are the arc length parameters of the curves α and α^*, respectively.

Definition 2.2. Let $\mathcal{M}$ be an oriented hypersurface with the unit normal vector field $\mathcal{N}$ in E^4 and α be a Frenet curve lying on $\mathcal{M}$ with its tangent vector field T. Also, let N be the hypersurface unit normal vector field restricted to the curve. Then the extended Darboux frame field or shortly the ED-frame field along the Frenet curve α is given as follows, [22]:

Case 1. If the set $\{N, T, \alpha''\}$ is linearly independent, then the orthonormal set $\{N, T, E\}$ is valid, where $E = \frac{\alpha'' - \langle \alpha'', N\rangle N}{\|\alpha'' - \langle \alpha'', N\rangle N\|}$.

Case 2. If the set $\{N, T, \alpha''\}$ is linearly dependent, then the orthonormal set $\{N, T, E\}$ is valid, where $E = \frac{\alpha''' - \langle \alpha''', N\rangle N - \langle \alpha''', T\rangle T}{\|\alpha''' - \langle \alpha''', N\rangle N - \langle \alpha''', T\rangle T\|}$.

In both cases, the vector field D is defined as $D = N \otimes T \otimes E$. So the vector fields $T, E, D,$ and N which are mutually orthogonal at each point of $\alpha(s)$ are obtained. Thus, the orthonormal frame field $\{T, E, D, N\}$ along the curve α is been constructed. These frame fields are called the ED-frame field of the first kind in Case 1 and the ED-frame field of the second kind in Case 2. The derivative equations related to these frame fields are given as

Case 1.

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{1}$$

and

Case 2.

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{2}$$

where κ_n is the normal curvature of the hypersurface in the direction of the tangent vector T, κ_g^i and τ_g^i are the geodesic curvatures and the geodesic torsions of order i (i=1,2), respectively. So, we have the following equalities:

$$\kappa_n = \langle T', N\rangle,\ \kappa_g^1 = \langle T', E\rangle,\ \kappa_g^2 = \langle E', D\rangle,\ \tau_g^1 = \langle E', N\rangle,\ \tau_g^2 = \langle D', N\rangle.$$

Theorem 2.3. Let α be a unit speed geodesic curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. Then, Case 2 is valid and

$$\kappa_n = k_1,$$

$$\kappa_g^1 = 0,$$

$$\kappa_g^2 = k_3,$$

$$\tau_g^1 = -k_2,$$

$$\tau_g^2 = 0$$

where k_i(i= 1,2,3) denotes the i-th curvature functions of α, [22].

3. GEODESIC CURVES OF CONSTANT BREADTH ACCORDING TO THE ED −FRAME

In this section, curves of constant breadth on an oriented hypersurface in E^4 according to the extended Darboux frame will be considered. Later, by giving the characterization of geodesic curves of constant breadth these curves will be determined.

Let α and α^* be a pair of unit speed curves of constant breadth in E^4 with the arc length parameters s and s^*, respectively. Then α and α^* have parallel tangents T and T^* in opposite directions at the points $\alpha(s)$ and $\alpha^*(s^*)$. Also, the distance between these points is always constant, i.e. $d^2 = \|\alpha^*(s^*) - \alpha(s)\|^2 = constant$. If we suppose that the curve α lies on an oriented hypersurface $\mathcal{M}$ in E^4, then it has the ED-frame $\{T, E, D, N\}$ in addition to the Frenet frame. So, we may write the following equation:

$$\alpha^*(s) = \alpha(s) + m_1(s)T(s) + m_2(s)E(s) + m_3(s)D(s) + m_4(s)N(s), \quad (3)$$

where $m_i(s)$, $1 \le i \le 4$, is the differentiable functions of s. If we differentiate this equation with respect to s and use (1), we obtain

$$\frac{d\alpha^*}{ds} = T^*\frac{ds^*}{ds} = \left(1 + m_1' - m_2\kappa_g^1 - m_4\kappa_n\right)T + \left(m_2' + m_1\kappa_g^1 - m_3\kappa_g^2 - m_4\tau_g^1\right)E + \left(m_3' + \kappa_g^2 - m_4\tau_g^2\right)D + (m_4' + m_1\kappa_n + m_2\tau_g^1 + m_3\tau_g^2)N. \quad (4)$$

Since $T^* = -T$ at the corresponding points of α and α*, from (4), we have

$$\begin{cases} 1+m_1'-m_2\kappa_g^1-m_4\kappa_n=-\dfrac{ds^*}{ds} \\ m_2'+m_1\kappa_g^1-m_3\kappa_g^2-m_4\tau_g^1=0 \\ m_3'+m_2\kappa_g^2-m_4\tau_g^2=0 \\ m_4'+m_1\kappa_n+m_2\tau_g^1+m_3\tau_g^2=0. \end{cases} \tag{5}$$

Let θ be the angle between the tangent of curve α and a certain fixed direction at point α(s). If the angle between the tangent vectors at the points α(s) and α(s+Δs) is represented by Δθ and the vector T(s+Δs)-T(s) by ΔT, the equation $\lim_{\Delta s\to 0}\frac{\Delta T}{\Delta s}=\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s}=\frac{d\theta}{ds}=k_1$ can be written. Then, we can rewrite (5) as follows:

$$\begin{cases} \dot{m}_1=\rho\left(m_2\kappa_g^1+m_4\kappa_n\right)-f(\theta) \\ \dot{m}_2=\rho\left(-m_1\kappa_g^1+m_3\kappa_g^2+m_4\tau_g^1\right) \\ \dot{m}_3=\rho\left(-m_2\kappa_g^2+m_4\tau_g^2\right) \\ \dot{m}_4=\rho\left(-m_1\kappa_n-m_2\tau_g^1-m_3\tau_g^2\right) \end{cases}, \tag{6}$$

where $(\dot{})$ denotes the differentiation with respect to θ, $\rho=\frac{1}{k_1}$, $\rho^*=\frac{1}{{k_1}^*}$, and $f(\theta)=\rho+\rho^*$. Let α be a unit speed geodesic curve on $\mathcal{M}$. Then using Theorem 2.3., from the equation system (6), we obtain

$$\begin{cases} \dot{m}_1=m_4-f(\theta) \\ \dot{m}_2=\rho(k_3m_3-k_2m_4) \\ \dot{m}_3=-\rho k_3m_2 \\ \dot{m}_4=\rho k_2m_2-m_1 \end{cases} \tag{7}$$

Differentiating both sides of equation $m_4=\dot{m}_1+f(\theta)$ with respect to θ yields

$$\dot{m}_4=\ddot{m}_1+\dot{f}=\rho k_2m_2-m_1. \tag{8}$$

If m_2 is obtained from (8) and its derivative is found, we have

$$\dot{m}_2=\frac{d}{d\theta}\left[\frac{k_1}{k_2}\left(\ddot{m}_1+m_1+\dot{f}\right)\right]=\rho(k_3m_3-k_2m_4). \tag{9}$$

Similarly, if m_3 is obtained from (9) and its derivative is found, we get

$$\dot{m}_3=\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left[\frac{k_1}{k_2}\left(\ddot{m}_1+m_1+\dot{f}\right)\right]+\frac{k_2}{k_3}(\dot{m}_1+f)\right\}=-\frac{k_3}{k_2}\left(\ddot{m}_1+m_1+\dot{f}\right).$$

Rewriting this last equation gives

$$\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left[\frac{k_1}{k_2}\left(\ddot{m}_1+m_1+\dot{f}\right)\right]+\frac{k_2}{k_3}(\dot{m}_1+f)\right\}+\frac{k_3}{k_2}\left(\ddot{m}_1+m_1+\dot{f}\right)=0$$

or

$$\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left[\frac{k_1}{k_2}(\ddot{m}_1+m_1)\right]+\frac{k_2}{k_3}\dot{m}_1\right\}+\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left[\frac{k_1}{k_2}\dot{f}\right]+\frac{k_2}{k_3}f\right\}+\frac{k_3}{k_2}(\ddot{m}_1+m_1)+ \tag{10}$$

$$\frac{k_3}{k_2}\dot{f} = 0.$$

Then, the following theorem can be given.

Theorem 3.1. The general differential equation characterizing the geodesic curves of constant breadth in E^4 is given by (10).

Now let us take into consideration the equation system (7). Since the distance d between the opposite points $\alpha(s)$ and $\alpha^*(s^*)$ is constant, i.e.

$$d^2 = \|\alpha^* - \alpha\|^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 = constant,$$

we get

$$m_1\dot{m}_1 + m_2\dot{m}_2 + m_3\dot{m}_3 + m_4\dot{m}_4 = 0.$$

Using the equation system (7) in this last equation, we obtain

$$m_1 f(\theta) = 0.$$

Then, we have two cases: $m_1 = 0$ or $f(\theta) = 0$. So, we get the following equations for each case from (7):

$$m_1 = 0 \Rightarrow \begin{cases} m_4 = f(\theta) \\ \dot{m}_2 = \rho(k_3 m_3 - k_2 m_4) \\ \dot{m}_3 = -\rho k_3 m_2 \\ \dot{m}_4 = \rho k_2 m_2 \end{cases} \tag{11}$$

or

$$f(\theta) = 0 \Rightarrow \begin{cases} \dot{m}_1 = m_4 \\ \dot{m}_2 = \rho(k_3 m_3 - k_2 m_4) \\ \dot{m}_3 = -\rho k_3 m_2 \\ \dot{m}_4 = \rho k_2 m_2 - m_1 \end{cases}. \tag{12}$$

First of all, let us investigate the case $f(\theta) = 0$.

Case $\boldsymbol{f(\theta) = 0}$.

First of all let us consider m_1 as a constant in the equation system (12). Then, we get

$$\begin{cases} \dot{m}_1 = 0 \\ \dot{m}_2 = \rho k_3 m_3 \\ \dot{m}_3 = -\rho k_3 m_2 \\ \dot{m}_4 = 0 \end{cases}. \tag{13}$$

Therefore, we have $m_1 = \rho k_2 m_2$ and $m_3 = \frac{\dot{m}_2}{\rho k_3}$. Besides, we obtain

$$m_2 = -\frac{\dot{m}_3}{\rho k_3} = -\frac{\left(\frac{\dot{m}_2}{\rho k_3}\right)^{\cdot}}{\rho k_3} = -\frac{\ddot{m}_2 \rho k_3 - \dot{m}_2(\rho k_3)^{\cdot}}{(\rho k_3)^3}$$

or

$$(\rho k_3)^3 m_2 + \dddot{m}_2 \rho k_3 - \ddot{m}_2 (\rho k_3)^{\cdot} = 0. \tag{14}$$

Similarly, we find

$$m_3 = \frac{\dot{m}_2}{\rho k_3} = \frac{\left(-\frac{\ddot{m}_3}{\rho k_3}\right)^{\cdot}}{\rho k_3} = -\frac{\dddot{m}_3 \rho k_3 - \ddot{m}_3 (\rho k_3)^{\cdot}}{(\rho k_3)^3}$$

or

$$(\rho k_3)^3 m_3 + \dddot{m}_3 \rho k_3 - \ddot{m}_3 (\rho k_3)^{\cdot} = 0. \tag{15}$$

So, (14) and (15) gives the following equation system:

$$\begin{cases} \rho k_3 \dddot{m}_2 - (\rho k_3)^{\cdot} \ddot{m}_2 + (\rho k_3)^3 m_2 = 0 \\ \rho k_3 \dddot{m}_3 - (\rho k_3)^{\cdot} \ddot{m}_3 + (\rho k_3)^3 m_3 = 0 \end{cases}, \rho k_3 \neq 0. \tag{16}$$

Let us change the variable θ of the form $\eta = \int_0^\theta \rho(t) k_3(t) dt$. Therefore, we have

$$\dot{m}_2 = \frac{dm_2}{d\eta} \rho k_3, \qquad \ddot{m}_2 = \frac{d^2 m_2}{d\eta^2} (\rho k_3)^2 + \frac{dm_2}{d\eta} (\rho k_3)^{\cdot}$$

and

$$\dot{m}_3 = \frac{dm_3}{d\eta} \rho k_3, \qquad \ddot{m}_3 = \frac{d^2 m_3}{d\eta^2} (\rho k_3)^2 + \frac{dm_3}{d\eta} (\rho k_3)^{\cdot}.$$

Then we can transform the equation system (16) into the following differential equations with constant coefficients as

$$\frac{d^2 m_2}{d\eta^2} + m_2 = 0 \tag{17}$$

and

$$\frac{d^2 m_3}{d\eta^2} + m_3 = 0, \tag{18}$$

respectively. If we solve (17) and (18), we obtain

$$m_2 = A \cos\left(\int_0^\theta \rho k_3 dt\right) + B \sin\left(\int_0^\theta \rho k_3 dt\right),$$

and

$$m_3 = C \cos\left(\int_0^\theta \rho k_3 dt\right) + D \sin\left(\int_0^\theta \rho k_3 dt\right),$$

where A, B, C, D are constants. Moreover, we get

$$\dot{m}_2 = \rho k_3 m_3 = \rho k_3 \left[C \cos\left(\int_0^\theta \rho k_3 dt\right) + D \sin\left(\int_0^\theta \rho k_3 dt\right)\right]$$

or

$$\dot{m}_2 = \rho k_3 \left[-A \sin\left(\int_0^\theta \rho k_3 dt\right) + B \cos\left(\int_0^\theta \rho k_3 dt\right)\right].$$

So, we have $D = -A$ and $C = B$. Thus, the solution of the system (13) can be found as

$$\begin{cases} m_1 = c = constant \\ m_2 = A \cos\left(\int_0^\theta \rho k_3 dt\right) + B \sin\left(\int_0^\theta \rho k_3 dt\right) \\ m_3 = B \cos\left(\int_0^\theta \rho k_3 dt\right) - A \sin\left(\int_0^\theta \rho k_3 dt\right) \\ m_4 = 0 \end{cases}. \tag{19}$$

Then, we can give the following theorem.

Theorem 3.2. Let $f(\theta) = 0$ and $m_1 = constant$ for the geodesic curves of constant breadth in E^4. Then the differentiable functions $m_i(s)$, $1 \le i \le 4$, in (3) are as given in (19). Thus, the geodesic curve of constant breadth is determined.

Corollary 3.3. If the equation system (19) is substituted in (3), we obtain $d^2 = \|\alpha^* - \alpha\|^2 = constant$. Therefore, the breadth of the geodesic curve α is found as $d^2 = c^2 + A^2 + B^2$.

Now, let us consider m_1 as not a constant in the equation system (12). From this system, we get

$$\ddot{m}_1 = \dot{m}_4 = \rho k_2 m_2 - m_1.$$

Then, we have

$$\dot{m}_2 = \left(\frac{1}{\rho k_2}\right)^{\cdot} (\ddot{m}_1 + m_1) + \frac{1}{\rho k_2}(\dddot{m}_1 + \dot{m}_1) = \rho(k_3 m_3 - k_2 m_4).$$

From this equation, we obtain

$$m_3 = \frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot} m_1 + \left(\frac{1}{\rho^2 k_2 k_3} + \frac{k_2}{k_3}\right)\dot{m}_1 + \frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot} \ddot{m}_1 + \frac{1}{\rho^2 k_2 k_3}\dddot{m}_1 .$$

Differentiating this last equation gives

$$\begin{aligned} &\left[\frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot\cdot} + \left(\frac{1}{\rho k_3}\right)^{\cdot}\left(\frac{1}{\rho k_2}\right)^{\cdot} + \frac{k_3}{k_2}\right] m_1 + \left[\frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot} + \left(\frac{1}{\rho^2 k_2 k_3}\right)^{\cdot} + \left(\frac{k_2}{k_3}\right)^{\cdot}\right]\dot{m}_1 + \\ &\left[\frac{1}{\rho^2 k_2 k_3} + \frac{k_2}{k_3} + \frac{k_3}{k_2} + \left(\frac{1}{\rho k_3}\right)^{\cdot}\left(\frac{1}{\rho k_2}\right)^{\cdot} + \frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot\cdot}\right]\ddot{m}_1 + \\ &\left[\frac{1}{\rho k_3}\left(\frac{1}{\rho k_2}\right)^{\cdot} + \left(\frac{1}{\rho^2 k_2 k_3}\right)^{\cdot}\right]\dddot{m}_1 + \frac{1}{\rho^2 k_2 k_3} m_1^{(iv)} = 0. \end{aligned} \tag{20}$$

Moreover, if we substitute $f(\theta) = 0$ into (10), we find

$$\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left[\frac{k_1}{k_2}(\ddot{m}_1 + m_1)\right] + \frac{k_2}{k_3}\dot{m}_1\right\} + \frac{k_3}{k_2}(\ddot{m}_1 + m_1) = 0. \tag{21}$$

Then, the following theorem can be given.

Theorem 3.4. Let $f(\theta) = 0$ and $m_1 \neq constant$ for the geodesic curves of constant breadth in E^4. Then the equation (20) is valid for m_1 and its derivatives and the general differential equation characterizing the geodesic curves of constant breadth is given by (21).

Secondly, let us investigate the case $m_1 = 0$.

Case $m_1 = 0$:

Let us consider the equation system (11). If we change the variable θ of the form $u = \int_0^\theta \rho(t)k_3(t)dt$, we obtain

$$\frac{dm_2}{du} = m_3 - \frac{k_2}{k_3}m_4 \tag{22}$$

and

$$\frac{dm_3}{du} = -m_2. \tag{23}$$

From (22) and (23), we get

$$\frac{d^2m_2}{du^2} + m_2 = -\frac{d}{du}\left(\frac{k_2}{k_3}m_4\right). \tag{24}$$

Then, the general solution of (24) is

$$m_2 = F_1(\theta) - \int_0^\theta cos[u(\theta) - u(t)]\rho k_2 f(t)dt, \tag{25}$$

where

$$F_1(\theta) = A_1 cos\int_0^\theta \rho k_3 dt + B_1\, sin\int_0^\theta \rho k_3 dt$$

and A_1, B_1 are real constants.

From (23), we have

$$\frac{d^2m_3}{du^2} + m_3 = \frac{k_2}{k_3}m_4. \tag{26}$$

Then, we obtain the general solution of (26) as

$$m_3 = F_2(\theta) + \int_0^\theta sin[u(\theta) - u(t)]\rho k_2 f(t)dt \tag{27}$$

where

$$F_2(\theta) = B_1 cos\int_0^\theta \rho k_3 dt - A_1\, sin\int_0^\theta \rho k_3 dt.$$

Therefore, the general solution of the system (11) is found as

$$\begin{cases} m_1 = 0 \\ m_2 = A_1 cos \int_0^{\theta} \rho k_3 dt + B_1 \, sin \int_0^{\theta} \rho k_3 dt - \int_0^{\theta} cos[u(\theta) - u(t)] \rho k_2 f(t) dt \\ m_3 = B_1 cos \int_0^{\theta} \rho k_3 dt - A_1 \, sin \int_0^{\theta} \rho k_3 dt + \int_0^{\theta} sin[u(\theta) - u(t)] \rho k_2 f(t) dt \\ m_4 = f(\theta) \end{cases}. \tag{28}$$

Besides, substituting $m_1 = 0$ into (10) gives

$$\frac{d}{d\theta}\left\{\frac{k_1}{k_3}\frac{d}{d\theta}\left(\frac{k_1}{k_2}\dot{f}\right) + \frac{k_2}{k_3}f\right\} + \frac{k_2}{k_3}\dot{f} = 0. \tag{29}$$

Then, we can give the following theorem.

Theorem 3.5. Let $m_1 = 0$ for the geodesic curves of constant breadth in E^4. Then the differentiable functions $m_i(s)$, $1 \leq i \leq 4$, in (3) are as given in (28) and the general differential equation characterizing the geodesic curves of constant breadth is given by (29).

4. CONCLUSION

In this paper, the geodesic curves of constant breadth on an oriented hypersurface according to the ED-frame in Euclidean 4-space are examined and the differential equation characterizing these curves is given. Similar studies can be done for the asymptotic lines of constant breadth and the principal lines of constant breadth on an oriented hypersurface according to the ED-frame in Euclidean 4-space in the future.

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