ORIGINAL PAPER

BINOMIAL SUMS WITH HARMONIC AND FIBONACCI NUMBERS

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Abstract. *In this paper, we define new sequence $S_n(a,b)$ with parameters a and b with the help of the generalized harmonic numbers. Also, we get some new sums involving harmonic, Fibonacci and Lucas numbers.*

Keywords: *Fibonacci numbers; harmonic numbers; generating function.*

1. INTRODUCTION

The harmonic numbers, denoted as H_n are defined by

$$H_0 = 0, \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \geq 1, \tag{1}$$

and the generating function of these numbers is

$$\sum_{n=0}^{\infty} H_n t^n = -\frac{\ln(1-t)}{1-t}. \tag{2}$$

Alternating or skew-harmonic numbers, denoted as H_n^-, are defined by

$$H_0^- = 0, \text{ and } H_n^- = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \text{ for } n \geq 1, \tag{3}$$

and the generating function of these numbers is

$$\sum_{n=0}^{\infty} H_n^- t^n = \frac{\ln(1+t)}{1-t}.$$

Recently, these numbers have been generalized by several authors. There are a lot of works involving harmonic numbers and generalized of them [1-4].

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Guo and Chu [4] combined numbers in (1) and (3) by

$$H_0(\sigma) = 0, \text{ and } H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \text{ for } n \geq 1,$$

where σ is a variable, with the generating function

$$\sum_{n=0}^{\infty} H_n(\sigma) t^n = -\frac{\ln(1-\sigma t)}{1-t}. \tag{4}$$

When $\sigma = 1/\alpha$, for $\alpha \in \mathbb{R}^+$, we will take $H_n(1/\alpha) = h_n(\alpha)$ and here $h_n(\alpha)$ are called the generalized harmonic numbers by Genčev [5]. Throughout this paper, we will take $\sigma \in \mathbb{R}\backslash\{0\}$.

By using Euler's transform for power series, some authors work various binomial identities with harmonic numbers [6-11].

Boyadzhiev [10] studied binomial sums with harmonic numbers using the Euler transform. He proved the following identity valid for positive integer n:

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = ((a+b)^n - b^n) H_n - a \sum_{k=0}^{n-1} (a+b)^k b^{n-k-1} H_{n-k-1}.$$

This formula allowed to derive the following identity:

$$\sum_{k=1}^{n} 2^{k-1} H_{n-k} = 2^n H_n(2) - H_n$$

which can be used as a new defining equation for harmonic numbers. He gave the following expansion in a neighborhood of zero,

$$\frac{-\ln(1-ct)}{1-dt} = \sum_{n=1}^{\infty} \left(c d^{n-1} + \frac{1}{2} c^2 d^{n-2} + \cdots + \frac{1}{n} c^n \right) t^n, \tag{5}$$

where c, d are appropriate parameters. Also, it is known that

$$\frac{-\ln(1-bt)}{1-(a+b)t} = \sum_{n=1}^{\infty} \left(a \sum_{k=0}^{n-1} (a+b)^{n-k-1} b^k H_k + b^n H_n \right) t^n. \tag{6}$$

In [11], Frontczak proved a new expression for binomial sums with harmonic numbers. His derivation was based on an alternative argument for the Euler transform of these sums. The findings complemented a result of Boyadzhiev. He discovered some known identities for harmonic numbers and derived some new identities involving harmonic numbers, and Fibonacci and Lucas numbers. For example, for positive integer n,

$$\sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k}H_k^- = ((a-b)^n - b^n)H_n^- + b^n H_n + 2b\sum_{k=0}^{n-1}(a+b)^k(a-b)^{n-k-1}H_{n-k-1}^-$$
$$+a\sum_{k=0}^{n-1}(a+b)^k b^{n-k-1}H_{n-k-1}.$$

For $f(t)=\sum_{n=0}^{\infty} f_n t^n$, the Euler transform can be defined as

$$\frac{1}{1-t}f\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}f_k\, t^n.$$

More generally,

$$\frac{1}{1-bt}f\left(\frac{at}{1-bt}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}f_k\, t^n,$$

where a and b are real parameters.

For non-negative integer n, Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ are given by recurrence relations

$$F_{n+2}=F_{n+1}+F_n \text{ and } L_{n+2}=L_{n+1}+L_n$$

with initial conditions $F_0=0, F_1=1$ and $L_0=2, L_1=1$, respectively. The Binet formulas for these sequences are

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } L_n=\alpha^n+\beta^n,$$

where $\alpha=(1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2$. In here, F_n and L_n are called Fibonacci numbers and Lucas numbers, respectively. Recently, there are many works and interesting results including these numbers (see [12-14]).

2. MAIN RESULTS

In this section, for parameters a and b with $a\neq 0$, let $S_n(a,b)$ be defined as

$$S_n(a,b)=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}H_k(1/a). \tag{7}$$

Now, we have the following theorem involving $S_n(a,b)$ and harmonic numbers.

Theorem 2.1. For positive integer n, it holds that

$$S_n(a,b)=((b+1)^n-b^n)H_n$$
$$+\sum_{k=0}^{n-1}\Big((a-1)(b+1)^k-ab^k\Big)(a+b)^{n-k-1}H_k. \tag{8}$$

Proof. Let $S(t)$ be the ordinary generating function of $S_n(a,b)$, namely $S(t)=\sum_{n=0}^{\infty} S_n(a,b)t^n$ and

$$A_\sigma(t) := \frac{-\ln(1-\sigma t)}{1-t}.$$

(4) and Euler transform yield that

$$S(t) = \frac{1}{1-bt}A_{1/a}\left(\frac{at}{1-bt}\right) = \frac{-\ln(1-(b+1)t)}{1-(a+b)t} + \frac{\ln(1-bt)}{1-(a+b)t}.$$

By (2), geometric series, production of two formal power series and

$$\frac{\ln(1-bt)}{1-(a+b)t} = \frac{-at}{1-(a+b)t}\frac{-\ln(1-bt)}{1-bt} + \frac{-\ln(1-bt)}{1-bt},$$

we have,

$$S(t) = \sum_{n=1}^{\infty}\left(((b+1)^n - b^n)H_n + \sum_{k=0}^{n-1}\Big((a-1)(b+1)^k - ab^k\Big)(a+b)^{n-k-1}H_k\right)t^n.$$

Since $S_0(a,b)=0$ and equality of two formal power series, we have (8). So, proof is complete.

We can get some sums for some special numbers to Theorem 2.1. For example, taking $(a,b)=(1,-1)$ and $(-1,1)$ in (8), respectively, we have that for positive integer n

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}H_k = \frac{1}{n} \text{ and } \sum_{k=1}^{n}\binom{n}{k}(-1)^k H_k^- = \frac{1-2^n}{n}.$$

Lemma 2.2. For positive integers n and r, we have

$$\sum_{k=1}^{r}\frac{F_{n-k}}{k} = \frac{\alpha^n H_r(1/\alpha) - \beta^n H_r(1/\beta)}{\sqrt{5}}, \qquad \sum_{k=1}^{r}\frac{L_{n-k}}{k} = \alpha^n H_r(1/\alpha) + \beta^n H_r(1/\beta),$$

$$\sum_{k=1}^{n}\frac{(-1)^k F_k}{k} = \frac{H_n(1/\beta) - H_n(1/\alpha)}{\sqrt{5}}, \qquad \sum_{k=1}^{n}\frac{(-1)^k L_k}{k} = H_n(1/\alpha) + H_n(1/\beta),$$

$$\sum_{k=1}^{r}\frac{(-1)^k F_{n-k}}{k} = \frac{\alpha^n H_r(\beta) - \beta^n H_r(\alpha)}{\sqrt{5}}, \text{ and } \sum_{k=1}^{r}\frac{(-1)^k L_{n-k}}{k} = \alpha^n H_r(\beta) + \beta^n H_r(\alpha).$$

Proof. The proof can be given by Binet formulas of $\{F_n\}$ and $\{L_n\}$.

Now we will give some sums involving Fibonacci and Lucas numbers by using Theorem 2.1.

Corollary 2.3. For positive integer n, we have sums

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{k+i}F_i}{i} = \frac{F_{2n}}{n}, \tag{9}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{k+i}L_i}{i} = \frac{L_{2n}-2}{n}. \tag{10}$$

Proof. Let us take $(a,b)=(\alpha,-\alpha)$ in (8). Since $\alpha+\beta=1$ and $H_n-H_{n-1}=1/n$ for positive integer n, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}\alpha^n H_k(1/\alpha)\\
&\qquad= ((1-\alpha)^n-(-\alpha)^n)H_n+\big((\alpha-1)(1-\alpha)^{n-1}-\alpha(-\alpha)^{n-1}\big)H_{n-1}\\
&\qquad= (\beta^n-(-\alpha)^n)H_n+(-\beta^n+(-\alpha)^n)H_{n-1}\\
&\qquad= \frac{\beta^n-(-\alpha)^n}{n}.
\end{aligned}$$

Multipling both sides with $(-\alpha^{-1})^n$ and using $-\beta/\alpha=\beta^2$, we get

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k H_k(1/\alpha) = \frac{\beta^{2n}-1}{n}. \tag{11}$$

Similarly,

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k H_k(1/\beta) = \frac{\alpha^{2n}-1}{n}. \tag{12}$$

Combining (11) and (12), we can write

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(H_k(1/\beta)-H_k(1/\alpha)) = \frac{\alpha^{2n}-\beta^{2n}}{n}.$$

Dividing both sides with $\sqrt{5}$ and by Lemma 2.2, we have the identity (9). In similar way, if we addition with (11) and (12), we find (10). So, proof is complete.

Corollary 2.4. For positive integer n, we have sums

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{n-k}F_{k-i}}{i} = \sum_{k=0}^{n-1}(-1)^{k+1}F_{k-n+2}H_k, \tag{13}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{n-k}L_{k-i}}{i} = \sum_{k=0}^{n-1}(-1)^{k+1}L_{k-n+2}H_k + 2(-1)^{n+1}H_n. \tag{14}$$

Proof. Taking $(a,b)=(\alpha,-1)$ and $(a,b)=(\beta,-1)$ in (8), and calculating $S_n(\alpha,-1)\pm S_n(\beta,-1)$, we have (13) and (14) immediately.

Corollary 2.5. For positive integer n, we have sums

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_{n-k}H_k^- = (F_n-F_{2n})H_n+\sum_{k=0}^{n-1}(2F_{3k-n+1}-F_{2k-n+1})H_k,$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{n-k}H_k^- = (L_n-L_{2n})H_n+\sum_{k=0}^{n-1}(2L_{3k-n+1}-L_{2k-n+1})H_k.$$

Proof. Taking $(a,b)=(-1,\alpha)$ and $(a,b)=(-1,\beta)$ in (8), by $H_n(-1)=-H_n^-$, and calculating $S_n(-1,\alpha)\pm S_n(-1,\beta)$, we have proof.

Corollary 2.6. For positive integer n, we have

$$\sum_{k=0}^{n}\binom{n}{k}F_{n-k}H_k=(F_{2n}-F_n)H_n-\sum_{k=0}^{n-1}F_{2n-k+2}H_k, \tag{15}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}L_{n-k}H_k=(L_{2n}-L_n)H_n-\sum_{k=0}^{n-1}L_{2n-k+2}H_k. \tag{16}$$

Proof. Taking $(a,b)=(1,\alpha)$ and $(a,b)=(1,\beta)$ in (8), respectively, and calculating $S_n(1,\alpha)\pm S_n(1,\beta)$, we have (15) and (16) immediately.

Corollary 2.7. For positive integer n, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{i+1}F_i}{i} = F_nH_n+\sum_{k=0}^{n-1}2^{n-k-1}F_{k-2}H_k,$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{i}L_i}{i} = (L_n-2)H_n+\sum_{k=0}^{n-1}2^{n-k-1}(L_{k-2}-2)H_k.$$

Proof. Taking $(a,b)=(\alpha,\alpha)$ and $(a,b)=(\beta,\beta)$ in (8), and calculating $S_n(\alpha,\alpha)\pm S_n(\beta,\beta)$, by Lemma 2.2, we have proof.

Corollary 2.8. For positive integer n, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{F_{k-i}}{i} = \sum_{k=0}^{n-1}(2^kF_{2n-2k-3}-F_{2n-2k-1})H_k, \tag{17}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{L_{k-i}}{i} = (2^{n+1}-2)H_n+\sum_{k=0}^{n-1}(2^kL_{2n-2k-3}-L_{2n-2k-1})H_k.$$

Proof. We will firstly prove (17). Taking $(a,b)=(\alpha,1)$ and $(a,b)=(\beta,1)$ in (8), respectively, by Lemma 2.2, we can write

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{F_{k-i}}{i}=\frac{S_n(\alpha,1)-S_n(\beta,1)}{\sqrt{5}}. \tag{18}$$

On the other hand, from $\alpha\beta=-1$, and (8), we get

$$\begin{aligned}S_n(\alpha,1)-S_n(\beta,1)&=\sum_{k=0}^{n-1}(-\beta 2^k-\alpha)\alpha^{2n-2k-2}H_k-\sum_{k=0}^{n-1}(-\alpha 2^k-\beta)\beta^{2n-2k-2}H_k\\&=\sum_{k=0}^{n-1}(\alpha^{-1}2^k-\alpha)\alpha^{2n-2k-2}H_k-\sum_{k=0}^{n-1}(\beta^{-1}2^k-\beta)\beta^{2n-2k-2}H_k\\&=\sum_{k=0}^{n-1}(2^k(\alpha^{2n-2k-3}-\beta^{2n-2k-3})-(\alpha^{2n-2k-1}-\beta^{2n-2k-1}))H_k\\&=\sqrt{5}\sum_{k=0}^{n-1}(2^kF_{2n-2k-3}-F_{2n-2k-1})H_k.\end{aligned}$$

Thus, we have proof of (17). Similarly, second identity can be found by calculating of $S_n(\alpha,1)+S_n(\beta,1)$.

Lemma 2.9. [15] For non-negative integer n and any real number x, we have

$$\sum_{k=0}^{n}\binom{n}{k}\frac{x^{k+1}}{k+1}=\frac{(1+x)^{n+1}-1}{n+1}.$$

Theorem 2.10. For positive integer n, we have

$$\sum_{k=1}^{n}\binom{n}{k}H_k(\sigma)=2^nH_n\left(\frac{1+\sigma}{2}\right)-2^nH_n\left(\frac{1}{2}\right). \tag{19}$$

Proof. We will use inductive method for proof. For $n=1$, the identity is true. Assume that (19) is true for n. For $n+1$, next step is

$$\begin{aligned}\sum_{k=1}^{n+1}\binom{n+1}{k}H_k(\sigma)&=\sum_{k=1}^{n}\binom{n+1}{k}H_k(\sigma)+H_{n+1}(\sigma)\\&=\sum_{k=1}^{n}\binom{n}{k}H_k(\sigma)+\sum_{k=1}^{n}\binom{n}{k-1}H_k(\sigma)+H_{n+1}(\sigma)\\&=\sum_{k=1}^{n}\binom{n}{k}H_k(\sigma)+\sum_{k=0}^{n-1}\binom{n}{k}H_{k+1}(\sigma)+H_{n+1}(\sigma)\\&=\sum_{k=1}^{n}\binom{n}{k}H_k(\sigma)+\sum_{k=0}^{n-1}\binom{n}{k}\left(H_k(\sigma)+\frac{\sigma^{k+1}}{k+1}\right)+H_n(\sigma)+\frac{\sigma^{n+1}}{n+1}\end{aligned}$$

$$= 2\sum_{k=1}^{n}\binom{n}{k}H_k(\sigma)+\sum_{k=0}^{n}\binom{n}{k}\frac{\sigma^{k+1}}{k+1}.$$

From the inductive hypothesis and Lemma 2.9, we get

$$\sum_{k=1}^{n+1}\binom{n+1}{k}H_k(\sigma) = 2^{n+1}\sum_{k=1}^{n}\frac{(1+\sigma)^k-1}{2^k k}+\frac{(1+\sigma)^{n+1}-1}{n+1} = 2^{n+1}\sum_{k=1}^{n+1}\frac{(1+\sigma)^k-1}{2^k k}.$$

So, we have the proof.

For example, when $\sigma = 1$, we have result in [14] as follows:

$$\sum_{k=1}^{n}\binom{n}{k}H_k = 2^n\big(H_n - H_n(1/2)\big).$$

Corollary 2.11. For positive integer n, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{i+1}F_i}{i} = 2^n\sum_{k=1}^{n}\frac{F_k}{2^k k},$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{n}{k}\frac{(-1)^{i}L_i}{i} = 2^n\sum_{k=1}^{n}\frac{L_k}{2^k k} - 2^{n+1}H_n(1/2).$$

Proof. Taking $\sigma = \alpha$ and $\sigma = \beta$ in (19), respectively, we have proof clearly.

Now, we define $S_n(a,b;m)$ as the following:

$$S_n(a,b;m) = \sum_{k=1}^{n}\binom{n}{k}k^m a^{n-k}b^k H_{n-k}\left(\frac{1}{a}\right).$$

Theorem 2.12. For positive integer n, we have

$$S_n(a,b;m) = H_n\sum_{k=0}^{n-1}\binom{n}{k}k^m b^k + \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}a^{k-i}$$
$$\times\left((a-1)\sum_{j=0}^{n-k-1}(i+j)^m\binom{n-k-1}{j}b^{i+j} - (n+i-k-1)^m ab^{n+i-k-1}\right)H_{n-k-1}. \quad (20)$$

Proof. We have $S_n(a,b) = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^kH_{n-k}(1/a)$. From Theorem 2.1, we can write

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^kH_{n-k}(1/a)$$
$$= \sum_{k=0}^{n-1}(a+b)^k\big((a-1)(b+1)^{n-k-1} - ab^{n-k-1}\big)H_{n-k-1} + ((b+1)^n - b^n)H_n$$

$$= \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i\left((a-1)(b+1)^{n-k-1}-ab^{n-k-1}\right)H_{n-k-1}+((b+1)^n-b^n)H_n$$
$$= \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i\left((a-1)\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}b^j-ab^{n-k-1}\right)H_{n-k-1}+\sum_{k=0}^{n-1}\binom{n}{k}b^k H_n.$$

Using the operator $\left(b\frac{d}{db}\right)^m$ both sides, we have the proof.

Corollary 2.13. For positive integers n and m, we have

$$\sum_{k=1}^{n}\binom{n}{k}k^m F_k H_{n-k} = H_n\sum_{k=0}^{n-1}\binom{n}{k}k^m F_k - \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}(n+i-k-1)^m F_{n+i-k-1}H_{n-k-1},$$

and

$$\sum_{k=1}^{n}\binom{n}{k}k^m L_k H_{n-k} = H_n\sum_{k=0}^{n-1}\binom{n}{k}k^m L_k - \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}(n+i-k-1)^m L_{n+i-k-1}H_{n-k-1}.$$

Proof. Taking $(a,b;m)=(1,\alpha;m)$ and $(a,b;m)=(1,\beta;m)$ in (20), respectively, and calculating $S_n(1,\alpha;m)\pm S_n(1,\beta;m)$, we have the proof.

Corollary 2.14. For positive integers n and m, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{n-k}\binom{n}{k}k^m\frac{F_{n-k-i}}{i}$$
$$=\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}\left(\sum_{j=0}^{n-k-1}(i+j)^m\binom{n-k-1}{j}F_{k-i-1}-(n+i-k-1)^m F_{k-i-1}\right)H_{n-k-1}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{n-k}\binom{n}{k}k^m\frac{L_{n-k-i}}{i} = 2H_n\sum_{k=0}^{n-1}\binom{n}{k}k^m$$
$$+\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}\left(\sum_{j=0}^{n-k-1}(i+j)^m\binom{n-k-1}{j}L_{k-i-1}-(n+i-k-1)^m L_{k-i-1}\right)H_{n-k-1}.$$

Proof. Taking $(a,b;m)=(\alpha,1;m)$ and $(a,b;m)=(\beta,1;m)$ in (20), respectively, and calculating $S_n(\alpha,1;m)\pm S_n(\beta,1;m)$, by Lemma 2.2, we have the proof.

Corollary 2.15. For positive integers n and m, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{n-k}\binom{n}{k}k^m\frac{F_{n-i}}{i} = H_n\sum_{k=0}^{n-1}\binom{n}{k}k^mF_k$$
$$+\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}\left(\sum_{j=0}^{n-k-1}(i+j)^m\binom{n-k-1}{j}F_{k+j-1}-(n+i-k-1)^mF_n\right)H_{n-k-1}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{n-k}\binom{n}{k}k^m\frac{L_{n-i}}{i} = H_n\sum_{k=0}^{n-1}\binom{n}{k}k^mL_k$$
$$+\sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i}\left(\sum_{j=0}^{n-k-1}(i+j)^m\binom{n-k-1}{j}L_{k+j-1}-(n+i-k-1)^mL_n\right)H_{n-k-1}.$$

Proof. Taking $(a,b;m)=(\alpha,\alpha;m)$ and $(a,b;m)=(\beta,\beta;m)$ in (20), respectively, and calculating $S_n(\alpha,\alpha;m)\pm S_n(\beta,\alpha;m)$, by Lemma 2.2, we have the proof.

3. CONCLUSION

In this paper, we define $S_n(a,b)$ and $S_n(a,b;m)$ with the help of a generalization of harmonic numbers. Using these, we obtained interesting identities and sums among Fibonacci numbers, Lucas numbers, and harmonic numbers. We foresee that the obtained results will support future studies in number theory and combinatorics.

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