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A NOTE ON THE SILVER HESSIAN STRUCTURE

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Abstract. *In this paper, we study locally decomposable Silver-Hessian manifolds and holomorphic Silver Norden Hessian manifolds. We investigate that if the smooth function* $f: M \rightarrow \mathbb{R}$ *is decomposable, then the triplet* $(M, \Theta, \nabla^2 f)$ *is a locally decomposable Silver Hessian manifold, where* Θ *is a Silver structure and* ∇ *represents the Levi-Civita Connection of* g*. We obtain some conditions under which the manifold* M *is associated with Hessian metric* h *and complex Silver structure. Moreover, we investigate the twin Norden Silver Hessian metric for a Kaehler-Norden Silver Hessian manifold.*

Keywords: *Silver structure; pure tensor; Hessian metric; decomposable and holomorphic tensors*

1. INTRODUCTION

Gezer et al. [1] investigated the integrability condition and curvature properties of Golden Riemannian structures with ϕ-operator. Gezer also studied some fascinating properties of twin Golden Riemannian metrics and obtained some examples. In [2], Gezer et al. studied the properties of Riemannian manifolds equipped with the Hessian metric h and complex Golden structure. Primo et al. [3], discussed some algebraic and geometric characterizations of the Silver ratio. In [4], Salimov et al. obtained some properties of Riemannian curvature tensors of paraholomorphic B-manifolds. In [5], Salimov and Gezer proved that if the function f is holomorphic then Kaehler manifolds admit a Norden-Hessian metric and gave conditions of para-Kaehler structures for Norden-Hessian metrics. In [6], Crasmareanu *et al*. studied the geometry of the Golden structure. Motivated by the Golden ratio, Crasmareanu deduced the Golden structure and studied various properties of the Riemannian manifold associated with the Golden structure in [7].

Shima introduced the notion of the Hessian metrics in [8]. Udriste et al. [9], defined the Hessian metric on a pseudo-Riemannian manifold and is given by $h = \nabla^2 f$, where ∇ is the Levi-Civita connection of g and $f: M \rightarrow \mathbb{R}$ is a smooth function. In [10-12], Bercu et al. studied Hessian metrics and some of its applications. He also deduced the Partial differential equations systems which are produced by two connections, one determined by a pseudo-Riemannian metric g and other determined by pseudo-Riemannian Hessian metrics h. In [13], Shima investigated the geometry of the Euclidean space $\mathbb{R}^n$ associated with the Hessian Riemannian metrics. Moreover, the hessian metrics have been studied by several authors [14-15].

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In [18], Iscan et al. investigated the geometry of Kaehler-Norden manifolds. Iscan and Salimov used the Tachibana operators and studied the properties of curvature scalars and Riemannian curvature tensors of Kaehler-Norden manifolds. In [20], Kumar defined the n^{th} order Hessian structures on manifolds and studied in detail for third order $(n = 3)$. He deduced a one-to-one correspondence between a certain class of connections and third-order Hessian structures on the second-order tangent bundle of a manifold. Moreover, Kumar gave a geometric characterization of symmetric third-order Hessian structures.

The notion of the polynomial structure on a manifold was introduced by Goldberg and Yano [22] in 1970. Suppose M be a C^{∞}-differentiable real manifold. A (1,1)- tensor field P is said to be a polynomial structure of degree n, if P satisfies the following algebraic equation

$$Q(P) = P^n + a_n P^{n-1} + \cdots + a_2 P + a_1 I = 0$$

and P has constant rank on M. Here I denote identity tensor field of type $(1,1)$ and $a_1, a_2, \ldots, a_n$ are real numbers.

Inspired by the research of the above authors, we have studied locally decomposable Silver-Hessian manifolds and holomorphic Silver Norden Hessian manifolds. We have shown that if the function f is decomposable, then the triplet $(M, \Theta, \nabla^2 f)$ is a locally decomposable Silver Hessian manifold and obtains some conditions under which the manifold M associated with Hessian metric h and complex Silver structure.

2. PRELIMINARIES

In this section, we have some results and definitions for the later use.

ϕ_θ-Operator: Suppose θ be a tensor field of type $(1, 1)$ and M be an n-dimensional manifold. A tensor field t of type (r, s) satisfying the following condition

$$\begin{aligned} t(\theta U_1, U_2, \ldots, U_s; \xi_1, \xi_2, \ldots, \xi_r) &= t(U_1, \theta U_2, \ldots, U_s; \xi_1, \xi_2, \ldots, \xi_r) \\ &\vdots \\ &= t(U_1, U_2, \ldots, \theta U_s; \xi_1, \xi_2, \ldots, \xi_r) \\ &= t(U_1, U_2, \ldots, U_s; {}'\theta \xi_1, \xi_2, \ldots, \xi_r) \\ &\vdots \\ &= t(U_1, U_2, \ldots, U_s; \xi_1, \xi_2, \ldots, {}'\theta \xi_r) \end{aligned}$$

for any $U_1, U_2, \ldots, U_s \in \mathfrak{I}_0^1(M)$ and $\xi_1, \xi_2, \ldots, \xi_r \in \mathfrak{I}_1^0(M)$ is said to be a pure tensor field associated with θ, where ${}'\theta$ is the adjoint operator of θ and it is defined by

$$({}'\theta \xi)(U) = \xi(\theta U) = (\xi \circ \theta)(U)$$

Now, consider an operator

$$\phi_\theta : \mathfrak{I}_s^0(M) \to \mathfrak{I}_{s+1}^0(M)$$

operated on the pure tensor field t of type $(0, s)$ associated with θ by

$$(\phi_\theta t)(U, V_1, V_2, \ldots, V_s) = (\theta U)\big(t(V_1, V_2, \ldots, V_s)\big)$$

$$-U\big(t(\theta V_1, V_2, \dots, V_s)\big)$$
$$+t\Big(\big(L_{V_1}\big)U, V_2, \dots, V_s\Big)$$
$$\cdots$$
$$+t\big(V_1, V_2, \dots, \big(L_{V_1}\big)U\big)$$

for any $U, V_1, V_2, \dots, V_s \in \mathfrak{I}_0^1(M)$, where L_V is the Lie differentiation with respect to V. The pure tensor t is said to be ϕ-tensor if $\phi_\theta t = 0$. If θ is a product structure, then a ϕ-tensor is said to be a decomposable tensor. A ϕ-tensor is called an analytic tensor if θ is a complex structure [16, 23].

3. LOCALLY DECOMPOSABLE SILVER-HESSIAN STRUCTURE

Let M is an n-dimensional differentiable manifold and Θ be a tensor field of type $(1, 1)$ on M. The structure Θ is said to be a Silver structure if it satisfies the condition

$$\Theta^2 - 2\Theta - \mathrm{I} = 0, \tag{3.1}$$

and the pair (M, Θ) is called a Silver manifold [19].

Suppose g be a Riemannian metric on a differentiable manifold M and (M, g) is a Riemannian manifold associated with the Silver structure Θ such that

$$\mathrm{G}(\Theta U, V) = g(U, \Theta V) \tag{3.2}$$

Let Riemannian metric g and differentiable manifold M associated with a tensor field F of type (1,1), satisfying the following conditions

$$g(FU, V) = g(U, FV)$$

and

$$F^2 = I, \qquad \nabla F = 0,$$

where ∇ represents the Levi-Civita connection of g. This type of manifold is said to be a locally decomposable Riemannian manifold.

Let almost product structure F induces a Silver structure on the manifold M as follows

$$\Theta = (I \mp \sqrt{2}F) \tag{3.3}$$

Conversely, the Silver structure yields an almost product structure as

$$F = \mp \frac{1}{\sqrt{2}}(\Theta - I) \tag{3.4}$$

Theorem 3.1. *Suppose* (M, Θ, g) *is a Silver Riemannian manifold. If* $\varphi_\Theta g = 0$, *then* Θ *is said to be integrable.*

Proof: From (3.2) and $\nabla g = 0$, it follows that

$$g(U,(\nabla_W\Theta)V) = g\big((\nabla_W\Theta)U,V\big) \tag{3.5}$$

for any vector fields U, V and W on M.

Now making use of (3.5) and $[U,V] = \nabla_U V - \nabla_V U$, we can transform (2.1) as given below:

$$(\phi_\Theta g)(U,V,W) = -g\big((\nabla_U\Theta)V,W\big) + g\big((\nabla_V\Theta)U,W\big) + g(V,(\nabla w\Theta)U) \tag{3.6}$$

Similarly, we have

$$(\phi_\Theta g)(W,V,U) = -g\big((\nabla_W\Theta)V,U\big) + g\big((\nabla_V\Theta)W,U\big) + g(V,(\nabla_U\Theta)W) \tag{3.7}$$

Adding (3.6) and (3.7), we get

$$(\phi_\Theta g)(U,V,W) + (\phi_\Theta g)(W,V,U) = 2g(U,(\nabla_V\Theta)W) \tag{3.8}$$

Now, putting $\phi_\Theta g = 0$ in (3.8), we get $\nabla\Theta = 0$.

Corollary 3.1. Suppose $(\boldsymbol{M},\boldsymbol{\Theta},\boldsymbol{g})$ is a Silver Riemannian manifold and $\boldsymbol{\nabla}$ be the Levi-Civita connection of $\boldsymbol{g}$. The condition $\boldsymbol{\phi_\Theta g = 0}$ is equivalent to $\boldsymbol{\nabla\Theta = 0}$.

If a Riemannian metric g is pure endowed with Silver structure, then the Riemannian metric g is pure associated with the almost product structure F. A direct computation via (3.4) gives

$$\phi_F g = \frac{1}{\sqrt{2}}\phi_\Theta g$$

For an almost product Riemannian manifold, Salimov et al. [4] shown that the almost product structure F is integrable, if $\phi_F g = 0$. Consequently, from the Theorem 3.1 and equation (3.9), we have

Proposition 3.1. Let F be an almost product structure of a Silver Riemannian manifold (M,Θ,g). If $\phi_F g = 0$, then Silver structure Θ is integrable.

A Silver Riemannian manifold (M,Θ,g) with an integrable Silver structure Θ is known as a locally Silver Riemannian manifold. Then, the manifold M is said to be a locally decomposable Silver Riemannian manifold if the metric g of the local Silver Riemannian manifold has the following form

$$ds^2 = g_{xy}(\xi^z)d\xi^x d\xi^y + g_{\bar{x}\bar{y}}(\xi^{\bar{z}})d\xi^{\bar{x}} d\xi^{\bar{y}},$$

where $x,y,z = 1,2,\dots,m,\ \ \bar{x},\bar{y},\bar{z} = m+, m+2,\dots,n,$ and g_{xy} are functions of ξ^z only, $g_{x\bar{y}} = 0$ and $g_{\bar{x}\bar{y}}$ are the functions of $\xi^{\bar{z}}$ only. On the other side, we say that F is covariantly constant associated with the Levi-Civita connection of g if and only if the local product Riemannian manifold with respect to almost all product structure F is locally decomposable [17].

Salimov et al. [4] proved that the condition $\phi_F g = 0$ is equivalent to $\nabla F = 0$, then we have the following statement:

Proposition 3.2. Let F be an almost product structure of a Silver Riemannian manifold $(\mathrm{M}, \Theta, \mathrm{g})$. The condition $\phi_{\mathrm{F}}\mathrm{g} = 0$ if and only if the manifold M is a locally decomposable Silver Riemannian manifold.

Theorem 3.2. Suppose f is a smooth function such that the Hessian $\nabla^2\mathrm{f}$ is non-degenerate and $(\mathrm{M}, \mathrm{F}, \mathrm{g})$ denote the locally decomposable Riemannian manifold, where ∇ be the Levi-Civita connection of g. The triplet $(\mathrm{M}, \Theta, \mathrm{h} = \nabla^2\mathrm{f})$ is a locally decomposable Silver Hessian manifold, if the smooth function f is decomposable, i.e. $\phi_{\mathrm{F}}(\mathrm{df}) = 0$, where Θ be the Silver structure related to the almost product structure.

Proof: Suppose f is a decomposable function on (M, F, g) that is $\phi_F(df) = 0$.

Then, from (2.1) and (3.3), we have

$$\begin{aligned}
&0 = (\phi_F(df))(U,V)\\
&(FU)\big((df)(V)\big) - U\big((df)(FV)\big) + (df)\big((L_V F)(X)\big)\\
&= \frac{1}{\sqrt{2}}[\Theta(U)((df)((V))) - U((df)(\Theta V)) + (df)([V,\Theta U]) - (df)\Theta(([V,U])]\\
&= \frac{1}{\sqrt{2}}\big[\Theta(U)((df)(V)) - U((df)(\Theta V)) + (df)\big(\nabla_V \Theta U - \nabla_{\Theta U} V - \Theta(\nabla_V U - \nabla_U\\
&(df)(-\nabla_U \Theta V + \nabla_U \Theta V)]\\
&= \frac{1}{\sqrt{2}}[(\nabla_{\Theta U} df)(V) + (\nabla_U df)(\Theta V) - (df)(\nabla\Theta)(V,U) + (df)(\nabla\Theta)(U,V)]
\end{aligned} \tag{3.10}$$

From (3.4), it follows that $\nabla F = 0$ is an equivalent to $\nabla\Theta = 0$. Then, the equation (3.10) becomes

$$(\nabla^2 f)(V,\Theta U) = (\nabla^2 f)(\Theta V, U). \tag{3.11}$$

Then, the triplet $(M, \Theta, h = \nabla^2 f)$ is said to be a Silver Hessian manifold. Now applying ϕ_Θ-operator to the Hessian metric h, we obtain

$$\begin{aligned}
(\phi_\Theta h)(U,V,W)&\\
&= \Theta(U)(h(V,W)) - U(h(\Theta V,W)) - h(\nabla_{\Theta U}V,W)\\
&+ h((\nabla\Theta)(U,V)W) + h(V,(\nabla\Theta)(U,W)) - h(V,\nabla_{\Theta U}W)\\
&+ h(\Theta(\nabla_U V),W) + h(\Theta V,\nabla_U W)
\end{aligned} \tag{3.12}$$

$$= (\nabla_{\Theta U}h)(V,W) - (\nabla_U h)(\Theta V,W) + h((\nabla\Theta)(U,V),W) + h\big(V,(\nabla\Theta)(U,W)\big)$$

Pulting $h(V,W) = (\nabla_V\nabla_W f)$ and $\nabla\Theta = 0$ in (3.12), we get

$$\begin{aligned}
(\phi_\Theta h)(U,V,W) &= (\phi_\Theta h)(U,W,V)\\
&= \big(\nabla_{\Theta U}(\nabla^2 f)\big)(W,V) - \big(\nabla_U(\nabla^2 f)\big)(W,\Theta V)\\
&= (\nabla^3 f)(W,V,\Theta U) - (\nabla^3 f)(W,\Theta V,U)
\end{aligned} \tag{3.13}$$

From the equation (3.11), we get

$$(\nabla^3 f)(\Theta U, V, W) = (\nabla^3 f)(U, \Theta V, W) \tag{3.14}$$

Now using Ricci identity, then from (3.14) we obtain

$$\begin{aligned}(\nabla^3 f)(\Theta U, V, W) &= \nabla_W\big(\nabla_V(\nabla_{\Theta U} f)\big) \\ &= \nabla_V\big(\nabla_W(\nabla_{\Theta U} f)\big) - (df)(R(W,V)\Theta U) \\ &= (\nabla^3 f)(\Theta U, W, V) - (df)(R(W,V)\Theta U)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}(\nabla^3 f)(U, \Theta V, W) &= \nabla_W\big(\nabla_{\Theta V}(\nabla_U f)\big) \\ &= \nabla_{\Theta V}\big(\nabla_W(\nabla_U f)\big) - (df)(R(W,\Theta V)U) \\ &= (\nabla^3 f)(U, W, \Theta V) - (df)(R(W,\Theta V)U)\end{aligned} \tag{3.16}$$

In (M, F, g), the curvature tensor R of g is totally pure and Hessian metric h is symmetric. So that curvature tensor R of g is totally pure endowed with the Silver structure Θ. Thus, from (3.15) and (3.16) it follows that

$$(\nabla^3 f)(W, \Theta U, V) = (\nabla^3 f)(W, U, \Theta V), \tag{3.17}$$

that is, the tensor field $\nabla^3 f$ is totally pure with Silver structure Θ. Thus we have a higher-order Hessian structure $\nabla^3 f$ on manifold M as given in the [20]. From (3.13) and (3.17), we get $\phi_\Theta h = 0$.

Analogous to [2], we have the following example.

Example 3.1. Suppose g be a Euclidean metric, that is

$$g = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \tag{3.18}$$

where $i, j = 1,2,\dots,k, \bar{i}, \bar{j} = k+1, k+2, \dots, n$, endowed with the positive orthant $\mathbb{R}_+^n$, *i.e.*

$$\mathbb{R}_+^n = \{u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n : u_1 > 0, \dots, u_n > 0\}$$

$$F = \begin{pmatrix} 0 & \delta_{\bar{j}}^i \\ \delta_j^{\bar{i}} & 0 \end{pmatrix} \tag{3.19}$$

where $i, j = 1,2,\dots,k, \bar{i}, \bar{j} = k+1, k+2, \dots, n$.

Then, the triplet $(\mathbb{R}_+^n, F, g)$ is known as locally decomposable Euclidean space. The Silver structure Θ on $\mathbb{R}_+^n$ obtained from canonical product structure F is as follows:

$$\Theta = \begin{pmatrix} \delta_j^i & \sqrt{2}\delta_{\bar{j}}^i \\ \sqrt{2}\delta_j^{\bar{i}} & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \tag{3.20}$$

where $i, j = 1,2,\dots,k, \bar{i}, \bar{j} = k+1, k+2, \dots, n$.

Also, the triplet $(\mathbb{R}_+^n, \Theta, g)$ is a locally decomposable Silver Euclidean space. Since, the condition $\phi_F(df) = 0$ is locally represented as follows:

$$(\Theta_F df)_{ij} = F_i^m \, \partial_m \, \partial_j f - \partial_i \left(F_j^m \, \partial_m f \right) + \left(\partial_j F_i^m \right) \partial_m f = 0. \tag{3.21}$$

Now, the function $f: \mathbb{R}_+^n \to \mathbb{R}$ of the following form

$$f(u_1, u_2, \dots, u_n) = f_1(u_1) + f_2(u_2) + \cdots + f_n(u_n), \tag{3.22}$$

where the differentiable function $f_i: \mathbb{R}_+ \to \mathbb{R}$ is a decomposable function and it satisfies $f_1'' > 0, i = 1,2, \dots, n$. Now, by putting (3.19) and (3.22) into equation (3.21). Then, equation (3.21) is satisfied. Therefore, the positive definite Hessian is given by

$$\nabla_g^2 f(u_1, u_2, \dots, u_n) = \begin{pmatrix} \dfrac{\partial^2 f_1(u_1)}{\partial u_1 \, \partial u_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\partial^2 f_n(u_n)}{\partial u_n \, \partial u_n} \end{pmatrix}. \tag{3.23}$$

Therefore, $\left(\mathbb{R}_+^n, \nabla_g^2 f\right)$ is said to be Riemannian manifold and $\nabla_g^2 f$ is pure endowed with the Silver structure Θ. The components of $\left(\phi_\Theta \nabla_g^2 f\right)$ can be represented as

$$\begin{aligned} \left(\phi_\Theta \nabla_g^2 f\right)_{kij} = \Theta_k^m \partial_m \left(\nabla_g^2 f\right)_{ij} - \Theta_i^m \partial_k \left(\nabla_g^2 f\right)_{mj} + \left(\nabla_g^2 f\right)_{mj} (\partial_i \Theta_k^m - \partial_k \Theta_i^m) \\ + \left(\nabla_g^2 f\right)_{im} \partial_j \Theta_k^m \end{aligned} \tag{3.24}$$

Now, from (3.20), (3.23) and (3.24), we have, $\phi_\Theta \nabla_g^2 f = 0$. Hence, $\left(\mathbb{R}_+^n, \Theta, \nabla_g^2 f\right)$ is known as locally decomposable Silver Hessian Euclidean spuce.

Some particular cases:

Case 1. Consider the function $f: \mathbb{R}_+^n \to \mathbb{R}$ of the form

$$f(u_1, u_2, \dots, u_n) = e^{u_1} + e^{u_2} + \cdots + e^{u_n}.$$

Then, the following Hessian structure is a Riemannian metric.

$$\nabla_g^2 f(u_1, u_2, \dots, u_n) = \begin{pmatrix} e^{u_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{u_n} \end{pmatrix}. \tag{3.25}$$

Case 2. Now choosing the Shanon entropy [24] function $f: \mathbb{R}_+^n \to \mathbb{R}$ and is given by

$$f(u_1, u_2, \dots, u_n) = -\frac{1}{k^2} (\ln k^2 \, u_1 + \ln k^2 \, u_2 + \cdots + \ln k^2 \, u_n)$$

Then, the Riemannian metric is given by

$$\nabla_g^2 f(u_1, u_2, \dots, u_n) = \begin{pmatrix} \dfrac{1}{k^2 (u_1)^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{1}{k^2 (u_2)^2} \end{pmatrix} \tag{3.26}$$

Therefore, $\left(\mathbb{R}_{+}^{n}, \Theta, \nabla_{g}^{2} f\right)$ are locally decomposable Silver Hessian Euclidean spaces for above two cases.

4. HOLOMORPHIC SILVER NORDEN HESSIAN MANIFOLD

Suppose M is an $2n$-dimensional manifold and Θ_c be a tensor field of type (1,1) on M^{2n}. If Θ_c satisfying the condition

$$\Theta_c{}^2 - 2\Theta_c + 3I = 0,$$

then Θ_c is said to be almost complex Silver structure and the pair (M^{2n}, Θ_c) is said to be an almost complex Silver manifold [19].

Since, it is well known that the integrability of almost complex Silver structure Θ_c is equivalent to the vanishing of the Nijenhuis tensor N_{Θ_c}. The structure Θ_c is said to be integrable if and only if it is possible to establish a torsion-free affine connection ∇ associated with the structure tensor Θ_c is covariantly constant.

Suppose θ is an almost complex structure on M^{2n}, then

$$\Theta_c^2 = (\mathrm{I} \pm \sqrt{2}\Theta) \tag{4.1}$$

is known as an almost complex Silver structure on M^{2n}. In fact,

$$\Theta_c^2 = \left(\mathrm{I} \mp 2\sqrt{2}\Theta - 2\mathrm{I}\right) = -\mathrm{I} \mp 2\sqrt{2}\left(\frac{\Theta_c - I}{\mp\sqrt{2}}\right)$$
$$= 2\Theta_c - 3\mathrm{I}$$

Conversely, let Θ_c be an almost complex Silver structure on M^{2n}, then the structure $\Theta = \mp\frac{1}{\sqrt{2}}(\Theta_c - \mathrm{I})$ is an almost comprx structure.

$$\Theta^2 = \frac{(\Theta_c^2 - 2\Theta_c) + I}{2} = \frac{-3I + I}{2} = -I \tag{4.2}$$

Let (M^{2n}, g) denotes the pseudo-Riemannian manifold associated with an almost complex structure θ. If the pseudo-Riemannian metric g is pure associated with almost complex structure θ, *i.e.*,

$$g(\theta U, V) = g(U, \theta V),$$

for any vector fields U and V on M. The pair (M^{2n}, g) is called an almost complex Norden manifold and M is known as Norden manifold if θ is integrable.

Suppose Θ_c be the almost complex Silver structure associated with pseudo-Riemannian manifold (M^{2n}, g) satisfies the following condition

$$g\,(\Theta_c U, V) = g\,(U, \Theta_c V) \tag{4.3}$$

where U and V are vector fields on M, then the triplet (M^{2n}, Θ_c, g) is said to be almost complex Norden Silver manifold.

Thus, for almost complex Norden Silver manifold (M^{2n}, Θ_c, g), we obtain

$$g(\Theta_c U, \Theta_c V) = 2g(\Theta_c U, V) - 3g(U, V) \tag{4.4}$$

In [18], it is already shown that g is holomorphic, *i.e.*, $\phi_\theta g = 0$ if and only if the almost complex Norden manifold is a holomorphic Norden, i.e., $\nabla\theta = 0$, where ∇ denote the Levi-Civita connection of g. Then, (M^{2n}, Θ_c, g) is said to be holomorphic Norden manifold if (M^{2n}, Θ_c, g) is a Norden manifold associated with holomorphic Norden metric g.

In the study of almost complex structure, the ϕ-operator method can be used for almost complex Silver structures because the almost complex structure θ is interrelated to the almost complex Silver structure Θ_c. Therefore, for the integrability of Θ_c on pseudo-Riemannian manifolds, another possible condition is introduced by the following theorem given below:

Theorem 4.1. Suppose (M^{2n}, Θ_c, g) is an almost complex Silver Norden manifold and ∇ denote the Levi-Civita connection of g. Then
(i)if $\phi_{\Theta_c} g = 0$, then Θ_c is integrable,
(ii) the condition $\phi_{\Theta_c} g = 0$ is equivalent to $\nabla\, \Theta_c = 0$.

Proof: Proof of the theorem is similar to the theorem 3.1.

If g (pseudo-Riemannian metric) is pure with regard to an almost complex Silver structure Θ_c, then g is pure along with the almost complex structureθ. From (4,1), we have

$$\phi_{\Theta_c} g = \sqrt{2}\, \phi_\theta g \tag{4.5}$$

As a consequence of the above equality and Theorem 4.1, we have the following.

Proposition 4.1. Let θ be the almost complex structure of almost complex Silver Norden manifold (M^{2n}, Θ_c, g). If $\phi_\theta g = 0$, then the almost complex Silver structure Θ_c is integrable.

Since $\phi_\theta g = 0$ is equivalent to $\nabla\theta = 0$, we have

Proposition 4.2. Suppose (M^{2n}, Θ_c, g) is an almost complex Silver Norden manifold and θ be its corresponding almost complex structure. The term $\phi_\theta g = 0$ if and only if the triplet (M^{2n}, Θ_c, g) is a holomorphic Silver Norden manifold.

In holomorphic Norden manifold (M^{2n}, θ, g), f is a holomorphic function, *i.e.*, $\phi_\theta(df) = 0$. For this instance, repeating the procedure of Theorem 3.2 for the almost complex Silver structure Θ_c, by the above proposition and equation (4.5), we have

Proposition 4.3. Suppose f is a smooth function with non-degenerate Hessian $\nabla^2 f$, where ∇ denote the Levi-Civita connection of g and (M^{2n}, θ, g) be a holomorphic Norden manifold. Suppose the structure Θ_c related to the almost complex structure θ be an almost complex Silver structure. Then, the triplet $(M^{2n}, \Theta_c, \nabla^2 f = h)$ is said to be a holomorphic Silver Norden Hessian manifold, if the smooth function f is holomorphic.

5. KAEHLER-NORDEN SILVER HESSIAN MANIFOLDS

Let $\tilde{g}$ be a twin Norden Silver metric for an almost complex Norden Silver manifold (M^{2n}, Θ_c, g), given by

$$\tilde{g}(U,V) = (g \circ \Theta_c)(U,V) = g(\Theta_c U, V), \tag{5.1}$$

Obviously $\tilde{g}(\Theta_c U, V) = \tilde{g}(U, \Theta_c V)$, where U, V are the vector fields on M. It should be noted that both the metrics g and $\tilde{g}$ are necessary of signature (n, n). Moreover, by equation (4.4), we can write

$$\tilde{g}(U, \Theta_c V) = 2\tilde{g}(U,V) - 3g(U,V) \tag{5.2}$$

Suppose ∇ and $\widetilde{\nabla}$ are the Levi-Civita connections of pseudo-Riemannian metrics g and $\tilde{g}$, respectively. Then the potential tensor $Q \in \Im_2^1(M)$ of $\widetilde{\nabla}$ associated with ∇ is given by

$$Q(U,V) = \widetilde{\nabla}_U V - \nabla_U V \tag{5.3}$$

The potential tensor Q is symmetric because both the connections are torsion-free, i.e.,

$$Q(U,V) = Q(V,U)$$

for any vector fields $U, V \in M$. Now the corresponding (0,3)-tensor field associated with g is defined as

$$Q(U,V,W) = g(Q(U,V),W)$$

Now, for an almost complex Norden Silver manifold (M^{2n}, Θ_c, g), the tensor field F of type $(0,3)$ is defined by

$$F(U,V,W) = (\nabla_U \tilde{g})(V,W) = g\big((\nabla_U \Theta_c)V, W\big) \tag{5.4}$$

which satisfies the following properties

$$\begin{cases} F(U,V,W) = F(U,W,V) \\ F(U,\Theta_c V, \Theta_c W) = -F(U,V,\Theta_c W) + 3F(U,V,W) \end{cases} \tag{5.5}$$

Evidently $(M^{2\mathrm{n}}, \Theta_c, g)$ be a Kaehler-Norden Silver manifold if and only if $F(U,V,W) = 0$ or equivalently $\nabla \Theta_c = 0$. Let U, V and W be any vector fields on (M^{2n}, Θ_c, g), then making use of equation (5.3), we get

$$(\nabla_U \tilde{g})(V,W) = (\widetilde{\nabla}_U \tilde{g})(V,W) + \tilde{g}(Q(U,V),W) + \tilde{g}(Q(U,W),V)$$

Now, using equation (5.4) and $\widetilde{\nabla}$ be a Levi-Civita connection of $\tilde{g}$, we get

$$F(U,V,W) = \tilde{g}(Q(U,V),W) + \tilde{g}(Q(U,W),V)$$

Further using equation (5.5), it follows that

$$\tilde{g}(Q(U,V),W) = \frac{1}{2}[F(U,V,W) + F(V,W,U) - F(W,U,V)] \quad (5.6)$$

Replacing W by $\Theta_c W$ and then making use of equations (5.2) and (5.6), we obtain the relation between Q and F as given below:

$$\begin{aligned} Q(U,V,W) \quad &= \frac{1}{3}F(U,V,W) - \frac{1}{6}F(U,V,\Theta_c W) + \frac{1}{3}F(V,W,U) \\ &- \frac{1}{6}F(V,\Theta_c W,U) - \frac{1}{3}F(W,U,V) + \frac{1}{6}F(\Theta_c W,U,V) \end{aligned} \quad (5.7)$$

Remark 5.1. Suppose (M^{2n},Θ_c,g) is a Kaehler-Norden Silver Manifold then the tensor F vanishes and from equation (5.7), then potential tensor Q of $\widetilde{\nabla}$ with ∇ also vanishes. Therefore, the Levi-Civita connections of g and $\tilde{g}$ coincide, *i.e.*, $\widetilde{\nabla}= \nabla$ if (M^{2n},Θ_c,g) is a Kaehler-Norden Silver manifold.

Suppose g be a metric tensor associated with Riemannian manifold (M^{2n},g). The gradient $\mathrm{grad}(f)$ of a function $f\colon M^{2\mathrm{n}} \to \mathbb{R}$ is the vector field metrically equivalent to the differential df. Thus

$$g(\mathrm{grad}(f),V) = g(df,V) = df(V) = Vf \quad (5.8)$$

The Hessian of the smooth function f on M is its second covariant differential $h = \nabla(\nabla f) = \nabla^2 f$, associated with the Levi-Civita connection ∇ of [7]. Since,

$$\nabla_V f = \nabla f(V) = df(V) = Vf \quad (5.9)$$

Then,

$$\begin{aligned} h(V,U) &= (\nabla(\nabla f))(V,U) = (\nabla(df))(V,U) = (\nabla_U(df))(V) \\ &= U((df)V) - (df)(\nabla_U V) \\ &= U(Vf) - (\nabla_U V)f \end{aligned} \quad (5.10)$$

Since, h is a symmetric tensor field. From (5.8), we obtain

$$(\nabla_U V)f = g(\mathrm{grad}(f),\nabla_U V)$$

Then, from the equation (5.10), we have

$$h(V,U) = g(\nabla_U \mathrm{grad}(f),V) \quad (5.11)$$

The metric defined by $h=\nabla^2 f$ is said to be pseudo-Riemannian Hessian metric if the Hessian $\nabla^2 f$ of a smooth function f associated with the metric g is non-degenerate of constant index [9].

Let $\tilde{g}$ denote the twin Norden Silver metric as given in (5.1) and $(M^{2\mathrm{n}},\Theta_c,g)$ denote the almost complex Norden Silver manifold. Suppose ∇ and $\widetilde{\nabla}$ represent the Levi-Civita connections of pseudo-Riemannian metrics g and $\tilde{g}$, respectively. Let $f\colon(M^{2n},\Theta_c,g)\to\mathbb{R}$ be a smooth function on M^{2n} then, we represent the Hessian of f by h and $\tilde{h}$, respectively associated with the ∇ and $\tilde{\nabla}$ of g and $\tilde{g}$. Then by equation (5.8), pseudo-Riemannian Hessian metrics on (M^{2n},Θ_c,g) are given by

$$h(U,V)=g(\nabla_U grad(f),V) \tag{5.12}$$

and

$$\tilde{h}(U,V)=\tilde{g}\big(\widetilde{\nabla}_U \mathrm{grad}(f),V\big) \tag{5.13}$$

From (5.2) and (5.12), we have

$$h(U,V)=\frac{2}{3}\tilde{g}(\nabla_U grad(f),V)-\frac{1}{3}\tilde{g}(\nabla_U grad(f),\Theta_c V)$$

Now, using (5.3) and (5.13) in above equation, we get

$$h(U,V)=\frac{2}{3}\tilde{h}(U,V)-\frac{2}{3}\tilde{g}(Q(U,grad(f)),V)-\frac{1}{3}\tilde{g}(\nabla_U grad(f),\Theta_c V) \tag{5.14}$$

From (5.4), $F(U,V,W)=(\nabla_U\tilde{g})(V,W)$ this implies that

$$\begin{aligned}\tilde{g}(\nabla_U grad(f),\Theta_c V) &= U\tilde{g}(grad(f),\Theta_c V)-\tilde{g}(grad(f),\nabla_U\Theta_c V)\\ &\quad -F(U,grad(f),\Theta_c V)\end{aligned} \tag{5.15}$$

Putting above equation in (5.14), we get

$$\begin{aligned}h(U,V)= {} & \frac{2}{3}\tilde{h}(U,V)-\frac{2}{3}\tilde{g}(Q(U,grad(f)),V)-\frac{1}{3}[U\tilde{g}(grad(f),\Theta_c V)\\ & -\tilde{g}\big(grad(f),\tilde{\nabla}_U\Theta_c V\big)-\tilde{g}\big(grad(f),Q(U,\Theta_c V)\big)\\ & -F(U,grad(f),\Theta_c V)]\end{aligned}$$

Moreover, using (5.13) and $\tilde{\nabla}\tilde{g}=0$ in above equation, we get

$$\begin{aligned}h(U,V) &= \frac{2}{3}\tilde{h}(U,V)-\frac{2}{3}\tilde{g}(Q(U,\mathrm{gradf}),V)-\frac{1}{3}\big[\tilde{h}(U,\Theta_c V)\\ &\quad -\tilde{g}\big(\mathrm{grad}(f),Q(U,\Theta_c V)\big)-F(U,\mathrm{grad}(f),\Theta_c V)\big]\end{aligned} \tag{5.16}$$

The tensors F and Q vanishes for a Kachler-Norden Silver manifold (see remark (5.1)) then from (5.16), we have

Theorem 5.1. Let h and $\tilde{h}$ be pseudo-Riemannian Hessian metrics and $(M^{2\mathrm{n}},\Theta_c,g)$ be an almost complex Norden Silver manifold. If (M^{2n},Θ_c,g) is Kaehler-Norden Silver manifold then

$$\tilde{h}(U, \Theta_c V) = 2\tilde{h}(U, V) - 3h(U, V) \tag{5.17}$$

and

$$\tilde{h}(U, V) = h(U, \Theta_c V) \tag{5.18}$$

Remark 5.2. From the proposition (4.3), if the smooth function f is holomorphic then $(M^{2n}, \Theta_c, h = \nabla^2 f)$ is Kaehler-Norden Silver Hessian manifold, *i.e.*, $h(\Theta_c U, V) = h(U, \Theta_c V)$. So that, from the equation (5.18), we have

$$\tilde{h}(U, V) = h(\Theta_c U, V) = (h \circ \Theta_c)(U, V) \tag{5.19}$$

Therefore, equations (5.19) and (5.17) are equivalent to (5.1) and (5.2), respectively. Hence, the Hessian metric $\tilde{h}$ is a twin Norden Silver Hessian metric for Kaehler-Norden Silver Hessian manifold $(M^{2n}, \Theta_c, h = \nabla^2 f)$.

CONCLUSION

In this paper, we have investigated some conditions for locally decomposable Silver-Hessian manifold and an integrability condition for Complex Structure. We gave an example of locally decomposable Silver-Hessian Euclidean space. We obtained some results on Holomorphic Silver Norden Hessian Manifold. Finally, we studied some reults of twin Norden Silver Hessian metric for a Kaehler-Norden Silver Hessian manifold.

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