ORIGINAL PAPER

SURFACE FAMILIES WITH COMMON MANNHEIM-B CURVATURE LINE

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Abstract. *In this study, we obtain necessary and sufficient conditions for the Mannheim B curve pair of the curve given by the Bishop frame in to be both parametric and curvature lines on the surface given by its parametric equation. Then, the same situation was expressed in the case of a ruled surface, which is a special and important type of surface, and the conditions for developable it among these surfaces were examined. Finally, examples supporting the study were given through the Maple15 program.*

Keywords: *Bishop Frame; Curvature line; Mannheim B pair; Parametric surface; Ruled surface.*

1. INTRODUCTION

The theory of curves and surfaces is one of the basic and important topics included in almost every differential geometry book [1-6]. Curves are called connected curves or pairs of curves; They are curves where one of the Frenet vectors of one curve and one of the Frenet vectors of the other curve are equivalent at opposite points. The best-known examples of these curves are Bertrand curves, evolute-involute curves, and Mannheim curves. Various studies on these curves and their characterization have survived to this day [7-21]. The first study on Mannheim curves was made by Mannheim in 1878 and has a special place in the theory of curves [22]. The principal normal vector at each point of the Mannheim curve is the binormal vector of another curve called the Mannheim curve pair [23]. The roof that can be built on a curve is not only the Frenet roof, but L. Bishop defined the Bishop frame, which we call an alternative parallel frame, in 1975 [24]. The Bishop framework is defined even if the curvature of the curve is zero and is more advantageous than the Frenet framework. Additionally, it has become possible to make calculations about the DNA helix with the help of a curve defined by the Bishop frame [25]. In addition, the Bishop frame has made significant contributions to the application by providing a new control method for controlling virtual cameras in computer animations [26]. Masal and Azak expressed the characterization of these curves by defining the Mannheim B curve pair of the curve given according to the Bishop frame. They also gave theorems and results regarding the relationship between the Bishop framework and the Frenet frame [27]. The concept of a family of surfaces passing through a given curve in 3-dimensional Euclidean space and accepting this curve as a special curve was first expressed by Wang et al. [28]. Kasap et al. generalized Wang's work by introducing new types of deviation functions [29]. Liu et al. examined the problem of the surface family by taking the curve mentioned in Wang's study as a curvature line [30].

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Additionally, there are various studies on the problem of finding a surface family that passes through a given curve and accepts this curve as a special curve [31-37].

In this study, by obtaining the necessary and sufficient conditions for the Mannheim B pair of a curve whose curvature is different from zero on the surface given by the parametric equation in 3-dimensional Euclidean space to be both parametric and curvature line on this surface, the problem of surface families with common Mannheim B curvature line is solved. It is discussed. In addition, in case the surface is a ruled surface, the specified conditions are given and the conditions for opening these surfaces are also given. Finally, various examples that support the study are given.

2. MATERIALS AND METHODS

Let E^3 be a 3-dimensional Euclidean space provided with the metric given by

$$\langle\, , \rangle = dx_1^{\ 2} + dx_2^{\ 2} + dx_3^{\ 2}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. Let $\alpha = \alpha(s): I \subset IR \to E^3$ is an arbitrary curve of arc-length parameter s. The curve α is called a unit speed curve if the velocity vector α' of a satisfies $\|\alpha'\| = 1$. Let $\{\mathrm{T(s), N(s), B(s)}\}$ be the moving Frenet frame along α, Frenet formulas is given by

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where the function $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s) \rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [2].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well-defined even when the curve has a vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [2]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as.

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}, \tag{2}$$

Here, we shall call the set $\{T, N_1, N_2\}$ as Bishop Frame and k_1 and k_2 as Bishop curvatures.

Let $\{T, N, B, \kappa, \tau\}$ and $\{T, N_1, N_2, k_1, k_2\}$ be the Frenet and Bishop apparatus of regular curve α with the arc-lenght parameter s respectively. The relations between Frenet and Bishop frames are given as follows:

$$\begin{cases} T = \alpha', \\ N = \cos\theta N_1 + \sin\theta N_2, \\ B = -\sin\theta N_1 + \cos\theta N_2, \end{cases} \tag{3}$$

and

$$\tau(s) = -\theta'(s), \quad \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)} \tag{4}$$

where $\theta(s) = \arctan\left(\dfrac{k_2(s)}{k_1(s)}\right), \quad N_2 = T \times N_1.$

Furthermore, the relations

$$\begin{cases} k_1(s) = \kappa(s)\cos\theta(s) \\ k_2(s) = \kappa(s)\sin\theta(s) \end{cases} \tag{5}$$

can be written for the Bishop curvatures of the curve α .

A regular curve α in $M \subset IR^3$ is a line of curvature provided its velocity α' always points in a principal direction [4].

Theorem 1. A surface curve is a line of curvature if and only if the surface normals along the curve forms a developable surface [6].

An isoparametric curve α(s) is a curve on a surface $\varphi = \varphi(s, v)$ is that has a constant s or v-parameter value. In other words, there exist a parameter s_0 or v_0 such that $\alpha(s) = \varphi(s, v_0)$ or $\alpha(v) = \varphi(s_0, v)$ [1].

Now, the Mannheim B-curves and some characterizations of these curves will be introduced:

Definition 2. Let C and C^* be unit speed curves with the arc-length parameters of s and s^* respectively. Denote the Bishop apparatus of C and C^* by $\{T, N_1, N_2, k_1, k_2\}$ and $\{T^*, N_1^*, N_2^*, k_1^*, k_2^*\}$ respectively. If the Bishop vector N_1 coincides with the Bishop vector N_2^* at the corresponding points of the curves C and C^* then the curve C is said to be a Mannheim partner B-curve of C^* or a (C, C^*) curve couple is called Mannheim B-pair, see Fig. 1 [27].

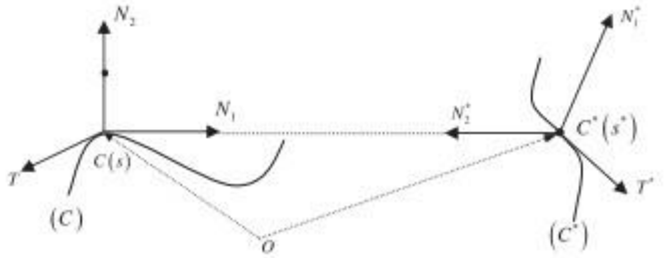

Figure 1. Mannheim B Curves.

Let us consider that the pair (C, C^*) is a Mannheim B-pair. Then we can write

$$\mathrm{C(s)} = \mathrm{C}^*(\mathrm{s}^*) + \lambda(\mathrm{s}^*)\mathrm{N}_2{}^*(\mathrm{s}^*)$$

where λ is a function, see Fig. 1.

Let γ be the angle between the tangents T and T^* of (C, C^*) Mannheim B-pair. Thus from the definition of the Mannheim B-pair, the following matrix representation can be written

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ 0 & 0 & 1 \\ -\sin\gamma & \cos\gamma & 0 \end{bmatrix} \begin{bmatrix} T^* \\ N_1^* \\ N_2^* \end{bmatrix} \tag{6}$$

Theorem 3. The distance between the corresponding points of the Mannheim B-curves is constant E^3 [27].

Theorem 4. Let (C, C*) be a Mannheim B-pair in E^3. Then the relationships between the Bishop vectors of C and C* are given as follows $T = \mu T^*, \quad N_1 = N_2^*, \quad N_2 = \mu N_1^*$ such that

$$\mu = \begin{cases} 1, & for\ \gamma = 0 \\ -1, & for\ \gamma = \pi \end{cases}$$

where γ is the angle between the tangent vectors of C and C* [27].

3. RESULTS AND DISCUSSION

3.1. SURFACES FAMILY WITH A COMMON MANNHEIM B CURVATURE LINE

Suppose we are given a 3-dimensional parametric curve $\alpha(s)$, $L_1 \le s \le L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0$. Let $\overline{\alpha}(s)$, $L_1 \le s \le L_2$, be Mannheim partner B- curve of the given curve $\alpha(s)$.

Surface family that interpolates $\overline{\alpha}(s)$ as a common curve is given in the parametric form as

$$\phi(s,v)=\overline{\alpha}(s)+[x(s,v)\,\overline{T}(s)+y(s,v)\,\overline{N}_1(s)+z(s,v)\,\overline{N}_2(s)],\ L_1\le s\le L_2,\ K_1\le v\le K_2, \tag{7}$$

where $x(s,v), y(s,v)$ and $z(s,v)$ are C^1 functions and are called marching-scale functions such that: $x(s,v), y(s,v)$ and $z(s,v)$ indicate, respectively; the extension-like, flexion-like, and retortion-like effects, by the point unit through the time v, starting from $\overline{\alpha}(s)$ and $\left\{\overline{T}(s),\ \overline{N}_1(s),\ \overline{N}_2(s)\right\}$ is the Bishop frame associated with the curve $\overline{\alpha}(s)$.

$\overline{\alpha}(s)$ be Mannheim partner B-curve of the given curve $\alpha(s)$ .Thus $\overline{\alpha}(s)$ is given by

$$\overline{\alpha}(s)=\alpha(s)+\lambda N_1(s), \tag{8}$$

where λ is a non-zero constant.

Using Eqn. (8) and theorem 4, firstly for $\gamma=0$, we obtain

$$\phi(s,v)\ =\alpha(s)+x(s,v)T(s)\ +(\lambda+z(s,v))N_1(s)+\ y(s,v)N_2(s) \tag{9}$$

Remark 5. Observe that choosing different marching-scale functions yields different surfaces possessing $\overline{\alpha}(s)$ as a common curve.

Our goal is to find the necessary and sufficient conditions for which the Mannheim partner B-curve of the given curve $\alpha(s)$ is isoparametric and curvature line on the surface $\varphi(s,v)$. Firstly, as $\overline{\alpha}(s)$ is an isoparametric curve on the surface $\varphi(s,v)$, there exists a parameter $v_0\in[K_1,K_2]$ such that

$$x(s,v_0)=y(s,v_0)\equiv 0,\ z(s,v_0)=-\lambda,\ \ L_1\le s\le L_2,\ K_1\le v_0\le K_2. \tag{10}$$

The normal vector of $\phi=\phi(s,v)$ can be written as follows:

$$n(s,v)=\frac{\partial\phi(s,v)}{\partial s}\times\frac{\partial\phi(s,v)}{\partial v}$$

Along the $\overline{\alpha}(s)$ curve, the normal vector can be expressed as, if the necessary calculations and arrangements are made:

$$\begin{aligned}n(s,v)=&\left[\frac{\partial y(s,v)}{\partial v}\left(-x(s,v)k_1+\frac{\partial z(s,v)}{\partial s}\right)-\frac{\partial z(s,v)}{\partial v}\left(x(s,v)k_2+\frac{\partial y(s,v)}{\partial s}\right)\right]T(s)\ +\\&\left[\frac{\partial x(s,v)}{\partial v}\left(x(s,v)k_2+\frac{\partial y(s,v)}{\partial s}\right)-\frac{\partial y(s,v)}{\partial v}\left(1+\frac{\partial x(s,v)}{\partial s}-k_1\left(z(s,v)+\lambda\right)-k_2y(s,v)\right)\right]N_1(s)+\\&\left[\frac{\partial z(s,v)}{\partial v}\left(1+\frac{\partial x(s,v)}{\partial s}-k_1\left(z(s,v)+\lambda\right)-k_2y(s,v)\right)-\frac{\partial x(s,v)}{\partial v}\left(x(s,v)k_1+\frac{\partial z(s,v)}{\partial s}\right)\right]N_2(s)\end{aligned}$$

Thus,

$$n(s,v_0)=\varsigma_1(s,v_0)T(s)+\varsigma_2(s,v_0)N_1(s)+\varsigma_3(s,v_0)N_2(s)$$

where

$$\begin{cases} \varsigma_1(s,v_0)=0, \\ \varsigma_2(s,v_0)=-\dfrac{\partial y(s,v_0)}{\partial v}, \\ \varsigma_3(s,v_0)=\dfrac{\partial z(s,v_0)}{\partial v}. \end{cases}$$

Also, we obtain

$$n(s,v_0)=-\frac{\partial y(s,v_0)}{\partial v}N_1(s)+\frac{\partial z(s,v_0)}{\partial v}N_2(s). \tag{11}$$

Let $\Phi(s,v)=\overline{\alpha}(s)+vu(s)$ be the normal surface of the curve $\overline{\alpha}(s)$ where $u(s)=\cos\varphi N_1(s)+\sin\varphi N_2(s)$ and φ is the angle between u and N_1 .According to Theorem 1 $\overline{\alpha}(s)$ is a line of curvature on the surface $\Phi(s,v)$ if and only if $\Phi(s,v)$ is developable and $n(s,v_0)$ is parallel to $u(s)$. The surface $\Phi(s,v)$ is developable if and only if

$$\det(\overline{\alpha}',u,u')=0 \Leftrightarrow \varphi'(1-\lambda k_1)=0$$

From this equation $\varphi'=0 \;\; or \;\; 1-\lambda k_1=0.$

Thus, $\varphi=constant \;\; or \;\; \lambda=\dfrac{1}{k_1}$ is obtained. Here k_1 is the bishop curvature and is different from zero.

$n(s,v_0)//u(s)$ if and only if

$$\begin{cases} \left.\dfrac{\partial \mathrm{g}}{\partial \mathrm{v}}\right|_{\mathrm{v}_0}=-\beta(\mathrm{s})\sin\varphi \\ \left.\dfrac{\partial \mathrm{h}}{\partial \mathrm{v}}\right|_{\mathrm{v}_0}=\beta(\mathrm{s})\cos\varphi \end{cases}, \;\; \beta(\mathrm{s})\neq 0$$

So, we can present the following theorem:

Theorem 6. Let $\alpha(s)$, $L_1\le s\le L_2$, be a unit speed curve with nonvanishing curvature and $\overline{\alpha}(s)$, $L_1\le s\le L_2$, be a Mannheim B-curve. $\overline{\alpha}$ to be both parametric curve and curvature line on the surface (7) if and only if

$$\begin{cases} x(s,v_0) = y(s,v_0) \equiv 0, \ z(s,v_0) = -\lambda, \ \lambda = constant \\ \dfrac{\partial y(s,v_0)}{\partial v} = -\beta(s)\cos\varphi(s), \quad \beta(s) \neq 0 \\ \dfrac{\partial z(s,v_0)}{\partial v} = \beta(s)\sin\varphi(s), \\ \varphi = constant \ \ or \ \lambda = \dfrac{1}{k_1}. \end{cases} \tag{12}$$

where $K_1 \leq v, v_0 \leq K_2 \, (v_0 \ fixed)$.

3.2. RULED SURFACES WITH A COMMON MANNHEIM-B CURVATURE LINE

Ruled surfaces are one of the simplest objects in geometric modeling as they are generated basically by moving a line in space. A surface φ is called a ruled surface in Euclidean space, if it is a surface swept out by a straight line l moving along a curve α. The generating line l and the curve α are called the rulings and the base curve of the surface, respectively.

We show how to derive the formulations of a ruled surfaces family such that the common Mannheim-B curvature line is also the base curve of ruled surfaces.

Let $\varphi = \varphi(s,v)$ be a ruled surface with the Mannheim-B curvature line base curve. From the definition of ruled surface, there is a vector $R = R(s)$ such that;

$$\varphi(s,v) - \varphi(s,v_0) = (v - v_0)R(s)$$

From (9), we get

$$(v - v_0)R(s) = x(s,v)\mathrm{T}(\mathrm{s}) + (\lambda + z(s,v))N_1(\mathrm{s}) + y(s,v)N_2(\mathrm{s})$$

By using the conditions (12), we obtain

$$R(s) = x(s)T(s) + \beta(s)\cos\theta(s)N_1(s) + \beta(s)\sin\theta(s)N_2(s) \quad , \beta(s) \neq 0$$

So, the ruled surfaces family with common Mannheim B curvature line given by;

$$\varphi(s,v) = \alpha + (v - v_0)\left[x(s)T(s) + \beta(s)\cos\theta(s)N_1(s) + \beta(s)\sin\theta(s)N_2(s)\right], \tag{13}$$

So, we can present the following theorem:

Theorem 7. Given an arc length the curve $\overline{\alpha}(s)$, there exists a ruled surface family possessing $\overline{\alpha}(s)$ as a common Mannheim B curvature line.

Remark 8. Observe that, changing $x(s)$ and $\beta(s)$ in Eqn. (13) yields different ruled surfaces interpolating $\overline{\alpha}(s)$ as a common Mannheim B curvature line.

Corollary 9. Ruled surface (13) is a developable if and only if $x(s)=0$ *or* $\theta=\left(\dfrac{\pi}{2}+\varphi\right)+2k\pi,\ k\in\mathbb{Z}$.

3.3. EXAMPLES OF GENERATING PARAMETRIC SURFACES AND RULED SURFACES WITH COMMON MANNHEİM-B CURVATURE LINE

Example 10. Let $\alpha(s)=(\cos s,\sin s,0)$ be a unit speed circle curve. Then it is easy to show that

$$\begin{cases} T(s)=(-\sin s,\cos s,0), \\ N_1(s)=(-\cos s,-\sin s,0), \\ N_2(s)=(0,0,1), \\ \kappa(s)=1\ ,\ \tau(s)=0. \end{cases}$$

From Eqn.(4), $\tau(s)=-\dfrac{d\theta}{ds}\Rightarrow\theta=c,\ c=constant$. Here $c=0$ can be taken.

a) If we take $x(s,v)=0,\ \ y(s,v)=-\sin v\ ,\ z(s,v)=-\lambda+v\sin v$ and $v_0=0,\ \lambda=1,$ $\beta(s)=1,\varphi=\dfrac{\pi}{2}$ then the Eqn. (12) is satisfied. Thus, we obtain a member of the surface with a common Mannheim-B curvature line as

$$\varphi_1(s,v)=\left(\cos s\left(1-v\sin v\right),\sin s\left(1-v\sin v\right),\sin v\right)$$

where $-\pi\le s\le\pi,\ 0\le v\le\pi$ (Fig. 2).

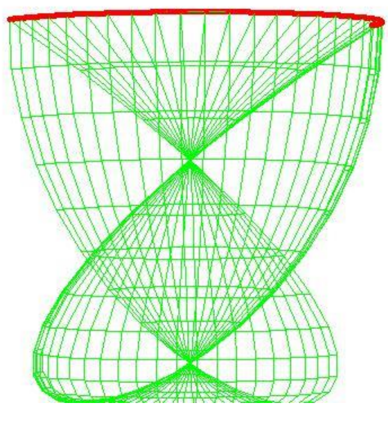

Figure 2. $\varphi_1(s,v)$ **as a member of the surface and its common Mannheim-B curvature line.**

b) If we take $x(s,v)=0,\quad y(s,v)=v\cos v,\; z(s,v)=-\lambda+v\cos v$ and $v_0=0,\; \lambda=1,$ $\beta(s)=1,\; \varphi=\dfrac{7\pi}{4}$ then the Eqn. (12) is satisfied. Thus, we obtain a member of the surface with a common Mannheim-B curvature line as

$$\varphi_2(s,v)=\left(\cos s\left(1-v\cos v\right),\sin s\left(1-v\cos v\right),v\cos v\right)$$

where $0\le s\le 2\pi,\; 0\le v\le 2\pi$ (Fig. 3).

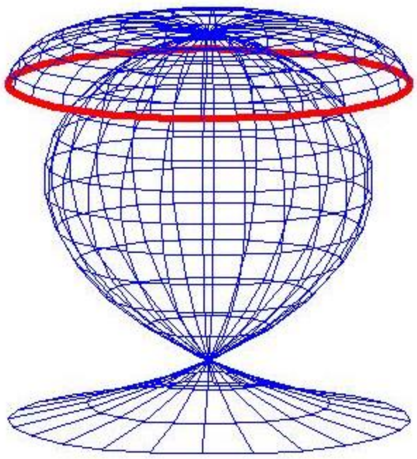

Figure 3. $\varphi_2(s,v)$ **as a member of the surface and its common Mannheim-B curvature line.**

For the same curve let us find a ruled surface. In Eqn. (13), if we take $x(s)=s, \beta(s)=-\cos s,\; \varphi=\dfrac{\pi}{4}$ and $v_0=0$, then we obtain the following ruled surface with a common Mannheim-B curvature line as

$$\varphi_3(s,v)=\left(\cos s+v(\sin^2 s+\frac{\sqrt{2}}{2}\cos^2 s),\sin s+v(\frac{\sqrt{2}}{2}-1)\sin s\cos s,\; v\frac{\sqrt{2}}{2}\cos s\right)$$

where $0\le s\le 2\pi,\; 0\le v\le 2\pi$ (Fig. 4).

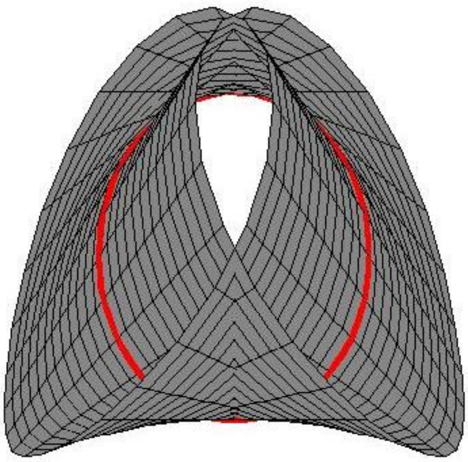

Figure 4. $\varphi_3(s,v)$ **as a member of the developable ruled surface family and its Mannheim-B curvature line.**

In Eqn. (13), if we take $x(s)=0$, then we obtain the following developable ruled surface with a common Mannheim-B curvature line as

$$\varphi_4(s,v)=\left(\cos s\left(1+v\frac{\sqrt{2}}{2}\cos s\right),\sin s\left(1+v\frac{\sqrt{2}}{2}\cos s\right),\ v\frac{\sqrt{2}}{2}\cos s\right)$$

where $-\pi \le s \le \pi$, $-1 \le v \le 1$ (Fig. 5).

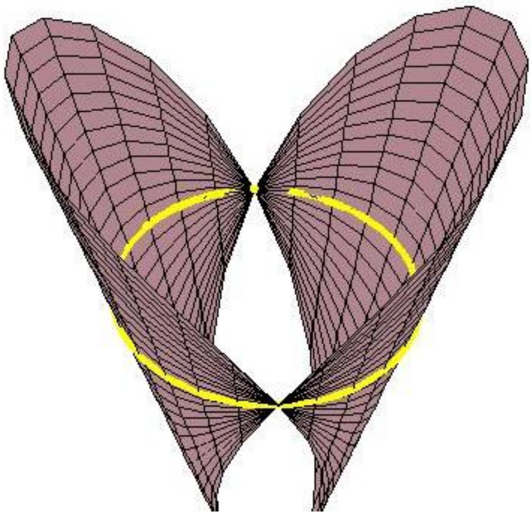

Figure 5. $\varphi_4(s,v)$ **as a member of the developable ruled surface family and its Mannheim-B curvature line.**

4. CONCLUSIONS

Necessary and sufficient conditions were obtained for parametric and ruled surfaces passing through the Mannheim B curve pair of the given unit speed curve given by the Bishop framework in 3-dimensional Euclidean space and accepting this curve as both a parameter curve and a curvature line. The condition that it can be developed specifically from within the ruled surfaces was also stated.

In addition, it has been shown with various examples that members of the surface family passing through the same Mannheim B curvature line can be obtained each time the deviation functions in the writing of the surface equations are chosen differently to meet the conditions. The results obtained from this study can be investigated in various spaces such as Galileo, Minkowski, or high-dimensional spaces.

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