ORIGINAL PAPER

STUDY OF THE TIME-FRACTIONAL WAVE EQUATION VIA DOUBLE SHEHU TRANSFORM METHOD

KUMAR DEEPAK[1], MURLI MANOHAR GOUR[1], LOKESH KUMAR YADAV[1], SUNIL DUTT PUROHIT[2]

Manuscript received: 28.10.2023; Accepted paper: 15.04.2024;
Published online: 30.06.2024.

Abstract. *In this study, we have explored the analytical solution for the time-fractional wave equation through the utilization of the double Shehu transform. The time-fractional wave equation holds significant importance, being prevalent in scenarios involving electromagnetic wave propagation, vibrating strings, and other related areas. Employing the proposed technique, we successfully obtained the exact solution. In addition, an illustrative example is provided to demonstrate the validity and accuracy of the presented method. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.*

Keywords: *Time-fractional wave differential equations; Shehu transforms; double Shehu transform.*

1 INTRODUCTION

In the realm of mathematical modeling, many physical and natural phenomena are described using differential equations. However, when it comes to systems with memory and hereditary characteristics, conventional integer order differential equations often fall short of providing accurate representations. To address this limitation and achieve a better understanding of models involving memory and hereditary problems, researchers have turned their attention to fractional-order modeling. Fractional differential equations (FDEs) have proven to be more effective and consistent than their integer-order counterparts in capturing the complexities of hereditary problems across various scientific disciplines such as physics, fluid mechanics, quantum mechanics, and more.

The notable mathematicians Suliman Alfaqeih and Emine Misirli have introduced a groundbreaking contribution to the field. They have successfully generalized the one-dimensional Shehu transform into a two-dimensional variant known as the double Shehu transform (DHT). By establishing essential properties and theorems associated with the DHT, they have demonstrated its efficiency, high accuracy, and practicality through its successful application in solving integral equations and partial differential equations [1].

Exploring different aspects of FDEs, researchers have delved into analytical solutions, numerical solutions [2], uniqueness and multiplicity of solutions, singularity analysis, and

[1] Vivekananda Global University, Department of Mathematics, 302017 Jaipur, India. E-mail: deepak@vgu.ac.in; murlimanohar.gaur@vgu.ac.in; lokesh.yadav@vgu.ac.in.
[2] Rajasthan Technical University, Department of HEAS (Mathematics), 324010 Kota, India.
E-mail: sdpurohit@rtu.ac.in.

https://doi.org/10.46939/J.Sci.Arts-24.2-a09

iterative approaches. Analytical solutions of fractional differential equations [3], particularly fractional partial differential equations [4], have been extensively investigated using diverse methods and techniques. Among the numerous equations that arise in applied sciences, the fractional wave equation [5-6] holds significant importance, being prevalent in scenarios involving electromagnetic wave propagation, vibrating strings, and other related areas. However, obtaining exact analytical solutions for fractional wave equations is challenging in most cases, prompting researchers to employ various algorithms and transforms.

Various integral transforms and methodologies have been introduced by different researchers to address fractional wave equations. These approaches encompass the Shehu transform method and double Sumudu transform [7], q-homotopy analysis transform method [8], reduced differential transform method, homotopy decomposition method [9], Shehu transform decomposition method [10], Laplace Adams–Bashforth method [11] and several others [12-14]. Their primary objective is to discover precise or approximate solutions for fractional wave equations. Note-worthy contributions in this field encompass research on fractional diffusion equations through the Laplace transform method [3], the examination of local fractional wave equations using Fourier series, and the investigation of the role of Mittag-Leffler (ML) function in fractal vibrating strings. Additionally, the utilization of the homotopy analysis method (HAM) and a comparative analysis with the modified trial equation method have been explored for obtaining analytical solutions for fractional wave equations.

In line with these endeavors, our study focuses on applying the double shehu transform [1] to obtain analytical solutions for the time-fractional wave equation [5]. Our aim is to elucidate the behavior of the wave equation and provide a comprehensive understanding of its dynamics using given initial and boundary conditions.

$$\frac{\partial^{\varsigma}\psi(\mu,\tau)}{\partial\tau^{\varsigma}} = c^2\frac{\partial^2\psi(\mu,\tau)}{\partial\mu^2}, (\mu,\tau)\epsilon R_+^2, A < \varsigma \leq 2,$$

$$IC: \ \psi(\mu,0) = \sin\mu, \frac{\partial\psi}{\partial\tau}(\mu,0) = 0, \tag{1}$$

$$BC: \psi(0,\tau) = 0, \frac{\partial\psi}{\partial\tau}(0,\tau) = E_{\varsigma}(-\tau^{\varsigma}).$$

The advantage of this technique is it's facilitating the efficient and rapid solution of both exact and numerical solutions for fractional-order differential equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2 PRELIMINARIES

In this section, we recall some basic definitions and properties of the Shehu transform for onward work

Definition 1. [15] The Shehu transforms (ST) of the function $\psi(\mu,\tau)$ with respect to variables μ and τ respectively, are defined by:

$$H_{\mu}(\psi(\mu,\tau):\eta,u) = \int_0^{\infty} e^{\left(\frac{-\eta\mu}{u}\right)}\psi(\mu,\tau)d\mu\,,$$

$$H_{\tau}(\psi(\mu,\tau):\omega,\vartheta) = \int_0^{\infty} e^{\left(\frac{-\omega\tau}{\vartheta}\right)}\psi(\mu,\tau)d\tau.$$

Definition 2. [1] The Double Shehu transforms (DHT) of the function $\psi(\mu,\tau\,)$ is defined by the double integral as:

$$H_{\mu,\tau}^{2}\big(\psi(\mu,\tau)\big) = G[(\eta,\omega),(u,\vartheta)] = \int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{\eta\mu}{u}+\frac{\omega\tau}{\vartheta}\right)}\psi(\mu,\tau)d\mu d\tau. \tag{2}$$

Definition 3. The inverse double Shehu transforms $\psi(\mu,\tau) = H_{\mu,\tau}^{-2}\left[H_{\mu,\tau}^{2}\big(\psi(\mu,\tau)\big)\right]$ is defined by the complex double integral formula

$$\begin{aligned}\psi(\mu,\tau) &= H_{\mu\tau}^{-2}\left[H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big)\right]\\ &= \frac{1}{2\pi i}\int_{\varsigma-i\infty}^{\varsigma+i\infty}\frac{1}{u}e^{\left(\frac{\eta\mu}{u}\right)}d\eta\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{\vartheta}e^{\left(\frac{\omega\tau}{\vartheta}\right)}H_{\mu\tau}\big(\psi(\mu,\tau)\big)d\omega.\end{aligned} \tag{3}$$

Definition 4. [16-17] The double Shehu transform involves applying the first and second order partial derivatives with respect to μ

$$\begin{aligned}H_{\mu\tau}^{2}\left(\frac{\partial\psi(\mu,\tau)}{\partial\mu}\right) &= \left(\frac{\eta}{u}\right)H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big) - H_{\tau}\big(\psi(0,\tau)\big),\\ H_{\mu\tau}^{2}\left(\frac{\partial^2}{\partial\mu^2}\psi(\mu,\tau)\right) &= \left(\frac{\eta}{u}\right)^2 H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big) - \left(\frac{\eta}{u}\right)H_{\tau}\big(\psi(0,\tau)\big) - H_{\tau}\left(\frac{\partial}{\partial\mu}\psi(0,\tau)\right).\end{aligned} \tag{4}$$

Similarly, with respect to τ

$$\begin{aligned}H_{\mu\tau}^{2}\left(\frac{\partial\psi(\mu,\tau)}{\partial\tau}\right) &= \left(\frac{\omega}{\vartheta}\right)H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big) - H_{\tau}\big(\psi(0,\tau)\big),\\ H_{\mu\tau}^{2}\left(\frac{\partial^2}{\partial\mu^2}\psi(\mu,\tau)\right) &= \left(\frac{\omega}{u}\right)^2 H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big) - \left(\frac{\eta}{\vartheta}\right)H_{\tau}\big(\psi(0,\tau)\big) - H_{\mu}\left(\frac{\partial}{\partial\mu}\psi(\mu,0)\right).\end{aligned} \tag{5}$$

In addition, the Shehu transform is a mathematical technique used for evaluating the mixed double order partial derivative of a function of two variables.

$$\begin{aligned}&H_{\mu\tau}^{2}\left(\frac{\partial^2}{\partial\tau\partial\mu}\psi(\mu,\tau)\right)\\ &= \left(\frac{\eta\omega}{u\vartheta}\right)H_{\mu\tau}^{2}\big(\psi(\mu,\tau)\big) - \left(\frac{\omega}{\vartheta}\right)H_{\tau}\big(\psi(0,\tau)\big) - \left(\frac{\eta}{u}\right)H_{\mu}\big(\psi(\mu,0)\big) + \psi(0,0).\end{aligned} \tag{6}$$

Definition 5. [16-17] The double Shehu transforms of the $n, m \in N$ times partial derivatives $\left(\frac{\partial^n \psi(\mu,\tau)}{\partial \mu^n}\right), \left(\frac{\partial^m \psi(\mu,\tau)}{\partial \tau^m}\right)$ of the function $\psi(\mu,\tau)$ are given by:

$$H^2_{\mu\tau}\left(\frac{\partial^n \psi(\mu,\tau)}{\partial \mu^n}\right) = \left(\frac{\eta}{u}\right)^n H^2_{\mu\tau}\big(\psi(\mu,\tau)\big) - \left(\frac{p}{u}\right)^{n-1} H_\tau\big(\psi(0,\tau)\big) - \sum_{j=1}^{n-1}\left(\frac{\eta}{u}\right)^{n-1-j} H_\tau\left(\frac{\partial^j \psi(0,\tau)}{\partial \mu^j}\right), \tag{7}$$

$$H^2_{\mu\tau}\left(\frac{\partial^m \psi(\mu,\tau)}{\partial \mu^m}\right) = \left(\frac{\omega}{\vartheta}\right)^m H^2_{\mu\tau}\big(\psi(\mu,\tau)\big) - \left(\frac{\omega}{\vartheta}\right)^{m-1} H_\mu\big(\psi(\mu,0)\big) - \sum_{i=1}^{m-1}\left(\frac{\omega}{\vartheta}\right)^{m-1-i} H_\mu\left(\frac{\partial^i \psi(\mu,0)}{\partial \tau^i}\right). \tag{8}$$

Definition 6.[18-19] The Mittag-Leffler function $E_\varsigma, E_{\varsigma,\beta,}$ are defined as

$$E_\varsigma(\tau) = \sum_{n-0}^{\infty} \frac{\tau^n}{\Gamma(n\varsigma+1)}, \varsigma \in R, Re(\varsigma) > 0, \tag{9}$$

$$E_{\varsigma,\beta}(\tau) = \sum_{n-0}^{\infty} \frac{\tau^n}{\Gamma(n\varsigma+\beta)}, \varsigma, \beta \in R, Re(\varsigma), Re(\beta) > 0. \tag{10}$$

Existence and Uniqueness of Double Shehu Transform (DHT)

Theorem 1. [20] Let $g(x,t)$ be a continuous function on every finite interval (0, X), and (0, T), and of exponential oreder, thet is for some $a, b \in R$

$$sup \frac{|g(x,t)|}{e^{ax+bt}} < \infty, \quad x, t > 0.$$

Then the (DHT) $g(x,t)$ exists.

Theorem 2. [20] Let $h(x,t)$ and $l(x,t)$ be continuous functions and having the double Shehu transforms $H^2_{xt}(h(x,t)$ and $H^2_{xt}(l(x,t)$ respectively. If $H^2_{xt}(h(x,t) = H^2_{xt}(l(x,t)$ then $h(x,t) = l(x,t)$.

3. BASIC IDEA OF PROPOSED METHODOLOGY

In this section, we describe the solution of the time fractional wave equation of order $1 < \varsigma \leq 2$, with the help of proposed technique [3]. Our approach involves the application of the double Shehu transform the differential equation presented in equation (1), employing the procedure described in equation (8).

$$
\begin{aligned}
H_{\mu\tau}^2\left\{\frac{\partial^\varsigma}{\partial\tau^\varsigma}\psi(\mu,\tau)\right\}
&= H_{\mu\tau}^2\left\{\frac{\partial^2}{\partial\mu^2}\psi(\mu,\tau)\right\}\left(\frac{\omega}{\vartheta}\right)^\varsigma H_{\mu\tau}^2(\psi(\mu,\tau)) - \left(\frac{\omega}{\vartheta}\right)^{\varsigma-1} H_\mu(\psi(\mu,0)) \\
&- \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} H_\mu\left(\frac{\partial}{\partial\mu}\psi(\mu,0)\right) c^2\left(\frac{\eta}{u}\right)^2 H_{\mu\tau}^2(\psi(\mu,\tau)) \\
&- \left(\frac{\eta}{u}\right) H_\tau\big(\psi(0,\tau)\big) - H_\tau\left(\frac{\partial}{\partial\tau}\psi(0,\tau)\right)
\end{aligned} \tag{11}
$$

,

which implies that

$$
\begin{aligned}
\left[\left(\frac{\omega}{\vartheta}\right)^\varsigma - c^2\left(\frac{\eta}{u}\right)^2\right] H_{\mu\tau}^2(\psi(\mu,\tau))
&= \left(\frac{\omega}{\vartheta}\right)^{\varsigma-1} H_\mu\big(\psi(\mu,0)\big) + \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} H_\mu\left(\frac{\partial}{\partial\mu}\psi(\mu,0)\right) \\
&- \left(\frac{\eta}{u}\right) H_\tau\big(\psi(0,\tau)\big) - H_\tau\left(\frac{\partial}{\partial\tau}\psi(0,\tau)\right)
\end{aligned} \tag{12}
$$

.

$$
\begin{aligned}
\psi(\mu,\tau) = \Bigg[H_{\mu\tau}^{-2}\frac{1}{\left\{\left(\frac{\omega}{\vartheta}\right)^\varsigma - c^2\left(\frac{p}{\eta}\right)^2\right\}}\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1} H_\tau(\psi(\mu,0)) \\
+ \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} H_\mu\left(\frac{\partial}{\partial\mu}\psi(\mu,0)\right) - \left(\frac{\eta}{u}\right) H_\tau\big(\psi(0,\tau)\big) \\
- H_\tau\left(\frac{\partial}{\partial\tau}\psi(0,\tau)\right)\Bigg]
\end{aligned} \tag{13}
$$

.

Using the initial and boundary conditions, we obtain

$$
\begin{aligned}
\psi(\mu,\tau) = H_{\mu\tau}^{-2}\Bigg[\frac{1}{\left\{\left(\frac{\omega}{\vartheta}\right)^\varsigma - c^2\left(\frac{\eta}{u}\right)^2\right\}}\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1} H_\mu(f_0(\mu)) + \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} H_\mu(f_1(\mu)) - \left(\frac{\eta}{u}\right) H_\tau\big(\psi_0(\tau)\big) \\
- H_\tau\big(g_1(\tau)\big)\Bigg]
\end{aligned}
$$

4. EXAMPLE

In this section, we determine the analytical solution of time-fractional wave equation to prove the efficiency and accuracy of proposed technique. Considering $c = 1$ in equation

(1), then the time-fractional wave equation with the initial and boundary conditions takes the form as

$$\frac{\partial^{\varsigma}\psi(\mu,\tau)}{\partial\tau^{\varsigma}} = \frac{\partial^{2}\psi(\mu,\tau)}{\partial\mu^{2}}, (\mu,\tau) \in R_{+}^{2}, 1 < \varsigma \leq 2,$$

$$IC: \psi(\mu,0) = \sin\mu, \frac{\partial\psi}{\partial\tau}(\mu,0) = 0,$$

$$BC: \psi(0,\tau) = 0, \frac{\partial\psi}{\partial\tau}(0,\tau) = E_{\varsigma}(-\tau^{\varsigma}).$$

Further, we have

$$H_{\mu}\{\psi(\mu,0)\} = \frac{u^{2}}{\eta^{2}+u^{2}}, H_{\mu}\left(\frac{\partial\psi(\mu,0)}{\partial\tau}\right) = 0, H_{\tau}\big(\psi(0,\tau)\big) = 0,$$

$$H_{\mu}\left(\frac{\partial\psi}{\partial\tau}(0,\tau)\right) = \frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1}}{1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}}$$

$$\psi(\mu,\tau) = H_{\mu\tau}^{-2}\left[\frac{1}{\left\{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - c^{2}\left(\frac{\eta}{u}\right)^{2}\right\}}\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1} H_{\mu}(f_{0}(\mu)) + \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} H_{\mu}(f_{1}(\mu)) - \left(\frac{p}{u}\right) H_{\tau}\big(g_{0}(\tau)\big) - H_{\tau}\big(g_{1}(\tau)\big)\right]$$

$$= H_{\mu\tau}^{-2}\left[\frac{1}{\left\{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - \left(\frac{\eta}{u}\right)^{2}\right\}}\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1}\frac{1}{1+\left(\frac{\eta}{u}\right)^{2}} + \left(\frac{\omega}{\vartheta}\right)^{\varsigma-2} * 0 - \left(\frac{\eta}{u}\right) * 0 - \frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1}}{1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}}\right]$$

$$= H_{\mu\tau}^{-2}\left[\frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma}}{\left\{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - \left(\frac{\eta}{u}\right)^{2}\right\}}\left[\frac{1}{1+\left(\frac{\eta}{u}\right)^{2}} - \frac{1}{1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}}\right]\right]$$

$$= H_{\mu\tau}^{-2}\left[\frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1}}{\left\{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - \left(\frac{\eta}{u}\right)^{2}\right\}}\left[\frac{1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - 1 - \left(\frac{\eta}{u}\right)^{2}}{\left(1+\left(\frac{\eta}{u}\right)^{2}\right)\left(1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}\right)}\right]\right]$$

$$= H_{\mu\tau}^{-2}\left[\frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma-1}}{\left\{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - \left(\frac{\eta}{u}\right)^{2}\right\}}\left[\frac{\left(\frac{\omega}{\vartheta}\right)^{\varsigma} - \left(\frac{\eta}{u}\right)^{2}}{\left(1+\left(\frac{\eta}{u}\right)^{2}\right)\left(1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}\right)}\right]\right]$$

$$= H_{\mu\tau}^{-2}\left[\frac{\left(\frac{q}{\vartheta}\right)^{\varsigma-1}}{\left(1+\left(\frac{\eta}{u}\right)^{2}\right)\left(1+\left(\frac{\omega}{\vartheta}\right)^{\varsigma}\right)}\right]$$
$$= \sin\mu E_{\varsigma}(-\tau^{\varsigma}) \ .$$

5. CONCLUDING REMARK

Utilizing the double Shehu transform to derive analytical solutions for boundary value problems associated with the one-dimensional time-fractional wave equation can make a significant and original contribution to the domain of fractional calculus. The successful acquisition of analytical solutions not only enhances the existing knowledge but also encourages the exploration of various fractional order equations, such as the fractional order Fokker-Planck equation, fractional order Brownian motion equation, fractional order Langevin equation, and similar fractional order equations. This broadens the potential application of the proposed technique to address a broader spectrum of problems.

Acknowledgment: *The authors express their sincere thanks to the editor and reviewers for their fruitful comments and suggestions that improved the quality of the manuscript.*

REFERENCES

[1] Alfaqeih, S., Misirli, E., *International Journal of Analysis and Applications*, **18**(3), 381, 2020.
[2] McLean, W., Mustapha, K., *Numerische Mathematik*, **105**, 481, 2007.
[3] Khan, A., Khan, T. S., Syam, M. I., Khan, H., *European Physical Journal Plus*, **134**(4), 163, 2019.
[4] Lamb Jr, G. L., *Introductory applications of partial differential equations: with emphasis on wave propagation and diffusion,* John Wiley and Sons, 2011.
[5] Sandev, T., Tomovski, Z., *Journal of Physics A: Mathematical and Theoretical*, **43**(5), 055204, 2010.
[6] Zhang, X., Zhao, J., Liu, J., Tang, B., *Applied Mathematical Modelling*, **38**(23), 5545, 2014.
[7] Maitama, S., Zhao, W., *arXiv preprint arXiv*, **1904**, 11370, 2019.
[8] Yadav, L. K., Agarwal, G., Suthar, D. L., Purohit, S.D., *Arab Journal of Basic and Applied Sciences*, **29**(1), 86, 2022.
[9] He, J. H., *Computer Methods in Applied Mechanics and Engineering*, **178**, 257, 1999.
[10] Yadav, L. K., Agarwal, G., Gour, M. M., Kumari, M., *International Journal of Dynamics and Control*, **12**(1), 271, 2023.
[11] Owolabi, K.M., *Chaos, Solitons & Fractals*, **115**, 160, 2018.
[12] Owolabi, K. M., Agarwal, R. P., Pindza, E., Bernstein S., Osman M. S., *Neural Computing and Applications*, **35**, 1130, 2023.
[13] Alqhtani, M., Owolabi, K. M., Saad, K. M., Pindza, E., *Communications in Nonlinear Science and Numerical Simulation*, **119**, 107118, 2023.
[14] Owolabi, K. M., Atangana, A., *Computational and Applied Mathematics*, **37**, 2166,

2018.
[15] Sameehah, R. A., Ahmad, A. H., Mbroka, S. H., *African Journal of Mathematics and Computer Science Research*, **14**(1), 4, 2021.
[16] Alfaqeih, S., Bakicierler, G., Misirli, E., *Punjab University Journal of Mathematics*, **54**(1), 1, 2022.
[17] Bokhari, A., Baleanu, D., Belgacem, R., *Journal of Mathematics and Computer Science*, **20**, 101, 2020.
[18] Belgacem, R., Baleanu, D., Bokhari, A., *International Journal of Analysis and Applications*, **17**(6), 917, 2019.
[19] Kilbas, A. A., Saigo, M., Saxena, R. K., *Integral Transforms and Special Functions*, **15**(1), 31, 2004.
[20] Alfaqeih, S., Misirli, E., International *Journal of Analysis and Applications*, **18**(3), 381, 2020.