ORIGINAL PAPER

ON SUBULTRA-GROUPS

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Abstract. *In this work subultra-groups of an ultra-group of a subgroup over a finite group are studied. Also, a subultra-group generated by a subset of an ultra-group is worked and then all the elements are determined in terms of the subset. Moreover, in which conditions the intersection of two normal subultra-groups is normal subultra-group is given. Quotient ultra-groups are also studied.*

Keywords: *Ultra-groups; congruence; transversal.*

1. INTRODUCTION AND PRELIMINARIES

Among this work G is considered as a finite group and e is the identity of G. Before introducing ultra-groups we firstly should mention about the transversal property of set pairs. For a given group G and subsets H and U if $h = h'$ and $u = u'$, for every $h \in H$ and $u \in U$ while $uh = h'u'$ then we call the pair of subsets (H, U) transversal. We can apply the transversal property to subgroups. If H and U are subgroups then (H, U) is transversal iff $H \cap U = \{e\}$. In this work, we study the case where H is a subgroup and U is a nonempty subset. In that case the pair (H, U) being transversal is when $|U \cap H_g| \leq 1$ for all g of G . In [2] an ultra-group U is of the subgroup H over the group G is defined by the operations $\alpha : U \times U \to U$ and $\beta_h : U \to U$ where (H, U) is transversal.

There are few publications on this new definition. The concept was first introduced in [1]. In the paper ultra-group homomorphism and fundamental properties are presented over the definition. The definition led to the investigation the category of ultra-groups in [2]. Then free ultra-groups, generators, and relations are studied in [3]. Finally in [4] some examples are given for ultra-groups with the help of GAP programming language.

In Section 2 we give the basic definitions and theorems which is used in this work, which were introduced in [2].

In Section 3 some subultra-group properties are presented, and the subultra-group of a generating set is introduced. It is well known that the normal subgroup intersection of a group is a normal subgroup. But for normal subultra-groups, we noticed that the intersection of normal subultra-groups need not be a normal subultra-group.

Finally in Section 4 quotient ultra-groups are studied. Throughout this paper, all fundamental definitions and theorems can be found in [5].

From now on we will give several definitions and theorems which are presented in [1].

Definition 1.1. A transversal is a set that contains only one element from each part of the partition. If $H \leq G$, then a transversal for the partition is denoted by $H \setminus G$ (G/H).

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Let H be a subgroup of G and U a subset of G. It is shown in [2] that $G = HU = UH$ when $|U \cap H_g| = 1$ for all $g \in G$. It means, for every $g = uh \in UH$ we have $uh = h'u'$ for unique $h' \in H$ and $u' \in U$.

Definition 1.2. For given subgroup H of the group G a subset U of G is said to be (right unitary) complementary set of the subgroup H, if for any elements $u \in U$ and $h \in H$ there are the unique elements $h' \in H$ and $u' \in U$ such that $uh = h'u'$ and $e \in U$. We will denote h' by ${}^{u}h$ and u' by u^h.
Since $U \subseteq G$ for any elements u_1, $u_2 \in U$ there are unique elements $[u_1, u_2] \in U$ and $(u_1, u_2) \in H$ such that $u_1 u_2 = (u_1, u_2)[u_1, u_2]$. If $x \in U$ there exists $u^{-1} \in G$. As $G = HU$, there is $x^{(-1)} \in H$ and $x^{[-1]} \in U$ such that $x^{-1} = x^{(-1)}x^{[-1]}$.

Definition 1.3. A complementary set ${}_HU$ is is called (right) ultra-group of H over group G with the operations $\alpha : {}_HU \times {}_HU \to {}_HU$ and $\beta_h : {}_HU \to {}_HU$ with $\alpha((u_1, u_2)) := [u_1, u_2]$ and $\beta_h(u) := u^h$ for all $h \in H$.

From now on, we will continue with the right ultra group and will denote it by U. If $[u, v] = [v, u]$ for all elements $u, v \in U$ then the ultra-group U is called abelian. One can easily observe that every group is an ultra-group but the converse does not hold. A remarkable property of right ultra-groups is that every element does not have a right inverse, but they have a left inverse. The α operation of an ultra-group U has the right cancellation. Namely if $[v, u] = [w, u]$ for $u, v, w \in U$, then we conclude $v = w$. We observe that we do not have left cancellations for the right ultra-groups.

Proposition 1.4. The following statements hold for any ultra-group U of subgroup H over the group G.

(i) $u^{hh'} = (u^h)^{h'}$,
(ii) $[u, v]^h = [u^{({}^{v}h)}, v^h]$,
(iii) $[[u, v], w] = [u^{(v,w)}, [v, w]]$,
(iv) $e^h = e, u^e = u$,
(v) $[e, u] = u = [u, e]$,
(vi) $[u^{-1}, u] = e = \left[u^{(u^{(-1)})}, u^{[-1]}\right]$, for $u, v, w \in U$ and $h, h' \in H$.

Definition 1.5. If U is an ultra-group of H over G then $S \subset U$ with e, is called the subultra-group of H over G, if S is an ultra-group with operations α and β_h of U.
If X, Y are two subsets of the ultra-group U, then $[X, Y]$ is the set of all $[x, y]$, where $x \in X$ and $y \in Y$. If Y is a singleton $\{y\}$ then we use $[X, y]$ instead of $[X, y]$. Moreover, if X is a subultra-group of ultra-group U and $y \in U$, then the subset $[X, y]$ is called a right coset of X in U.

Lemma 1.6. For given subultra-group S of an ultra-group U over the subgroup H of the group G we have,

(i) $[x^{(y^{(-1)})}, y^{[-1]}] \in S \Leftrightarrow x = [s, y]$ for some $s \in S$.
(ii) The relation σ defined as: $x\sigma y \Leftrightarrow x = [s, y]$ for some $s \in S$ on U is an equivalence relation.

Theorem 1.7. If U is an ultra-group over the subgroup H of the group G and T is a subgroup of G which satisfies $T \subseteq H$, then $T = HS$ for some subultra-group S of U. Moreover if S_i , $(i = 1,2,3,\dots,n)$ are subultra-groups of the ultra-group Usatisfying $S_i \subseteq S_{i+1}$ for then $K_i = HS_i$ for subgroups with $K_i \subseteq K_{i+1}$ of G.

Definition 1.8. For given ultra-group U over the subgroup H of the group G and a congruence σ over U, the set $U/\sigma = \{[u]_\sigma \mid u \in U\}$ with the operations α_σ and β_{σ_h} given as,

(i) $\alpha_\sigma([u]_\sigma, [u']_\sigma) = [\alpha(u, u')]_\sigma$
(ii) $\beta_{\sigma_h}([u]_\sigma) = [\beta_h(u)]_\sigma$ is an ultra-group of K over the group G, where $H \leq K \leq G$.
is called a quotient ultra-group.

Definition 1.9. The subultra-group N of an ultra-group U over the subgroup H of the group G is said to be normal if $[u,[N,v]] = [N,[u,v]]$, for all $a, b \in U$.

Lemma 1.10. The following are holds for a normal subultra-group N of an ultra-group U over the subgroup H of the group G,

(i) for all $u \in U, [u,N] = [N,u]$,
(ii) for all $u, v \in U$, $[[N,u],[N,v]] = [N,[u,v]]$,
(iii) If $[N,v] = N$, then $v \in N$,
(iv) If S is a subultra-group of U, then $[N,S]$ is a subultra-group of U. Also if S is a normal subultra-group of U then $[N,S]$ is a normal subultra-group of U.

Theorem 1.11. If S is a subultra-group of an ultra-group U, then the equivalence relation σ is a congruence if and only if S is a normal subultra-group U.

2. MAIN RESULTS

2.1. ON SUBULTRA-GROUPS

Lemma 2.1.1. Let U be an ultra-group of the subgroup H over the group G and S_1, S_2 be subultra-groups of U. Then $S_1 \cap S_2$ is also a subultra-group of U.

Proof: Let S_1, S_2 be subultra-groups of U. Obviously $e \in S_1 \cap S_2$. If $s_1, s_2 \in S_1 \cap S_2$ then

$$\alpha(s_1, s_2) \in S_1 \text{ and } \beta_h(s_1) \in S_1$$

for every $h \in H$ since S_1 is a subultra-group of U. As well as

$$\alpha(s_1, s_2), \beta_h(s_2) \in S_2$$

because S_2 is a subultra-group of U. So, we have

$$\alpha(s_1, s_2), \beta_h(s_2) \in S_1 \cap S_2$$

and $S_1 \cap S_2$ is a subultra-group of U.

Corollary 2.1.2. The subultra-group mentioned in Theorem (1.7) is an ultra-group of H over the group K.

Proof: Clearly $e \in S$. Since $= HS$, S is a complementary set with operations α and β_h. Hence S is an ultra-group of H over the group K.

Corollary 2.1.3. Let U be an ultra-group of H over the group G, and S_i for $i = 1,2,3,\ldots,n$ be subultra-groups of the ultra-group U such that $S_i \subseteq S_{i+1}$ for $i = 1,2,3,\ldots,n-1$. Then we have subgroups $K_i = HS_i$ for $i = 1,2,3,\ldots,n$ of group G such that $K_i \subseteq K_{i+1}$ for $i = 1,2,3,\ldots,n-1$ and $\leq K_i \leq G$. If H is a maximal subgroup of G then one of the followings hold:

(i) U has no subultra-group except itself.
(ii) All the subultra-groups of U are ultra-groups of H over the group G
(iii) Subultra-group S is a complementary set of H in G.

Proof: Seen by Theorem (1.7).

If we define the smallest of the complementary sets that contain e and are closed under the binary and unary operations α and β_h as minimal ultra-group then the (iii) would be equivalent (i) or (ii).

Definition 2.1.4. Let U be an ultra-group of the subgroup H over the group G. The set $Z(U) = \{u \in U \mid [u,u'] = [u',u], \forall u' \in U\}$ is called the center of U.

Corollary 2.1.5. Let U be an ultra-group of the subgroup H over the group G. Then

$$U \cap Z(G) \subseteq Z(U).$$

Proof: Let $u \in U \cap Z(G)$. For every $u' \in U$ we can write:

$$uu' = (u,u')[u,u'] \text{ and } u'u = (u',u)[u',u]$$

Since U is an ultra-group by uniqueness property for ultra-groups implies

$$uu' = (u,u')[u,u'] = (u',u)[u',u] = u'u$$

$\Rightarrow [u,u'] = [u',u]$

Therefore $u \in Z(U)$ and $U \cap Z(G) \subseteq Z(U)$.

Definition 2.1.6. Let U be an ultra-group of the subgroup H over the group G, $e \in X \subseteq U$ and S_i be subultra-groups of U. Then the intersection of subultra-groups S_i is denoted by;

$$\langle X \rangle = \bigcap_{X \subseteq S_i} S_i$$

Corollary 2.1.7. $\langle X \rangle$ defined in Definition (2.1.6) is a subultra-group of U.

Proof: It is clear by Lemma (2.1.1).

Theorem 2.1.8. Let U be an ultra-group of the subgroup H over the group G and $\langle X\rangle$ be the subultra-group defined in Definition (2.1.6). Then

$$\langle X\rangle = \left\{\left[\cdots\left[[[x_1,\ x_2],x_3]\cdots\right],x_n\right]\middle|\ x_i \in X\right\}$$

Here the element $\left[\cdots\left[[[x_1,\ x_2],x_3]\cdots\right],x_n\right]$ is denoted by α^n in [3] where α is the binary operation of U.

Proof: Let

$$T = \left\{\left[\cdots\left[[[x_1,\ x_2],x_3]\cdots\right],x_n\right]\middle|\ x_i \in X\right\}$$

Firstly, since for every $x_i \in X$ we have $x_i \in S_i$ and S_i are subultra-groups then for arbitrary

$$k = \left[\cdots\left[[[x_1,\ x_2],x_3]\cdots\right],x_n\right] \in S_i \Rightarrow k \in \bigcap_{X\subseteq S_i} S_i = \langle X\rangle$$

In this case $T \subseteq \langle X\rangle$. In contrary to every $x \in \langle X\rangle \Rightarrow x \in S_i$ and by Proposition (1.4) (v) $x = [e,x] \in T$ and therefore $\langle X\rangle \subseteq T$. Then we conclude

$$\langle X\rangle = T = \left\{\left[\cdots\left[[[x_1,\ x_2],x_3]\cdots\right],x_n\right]\middle|\ x_i \in X\right\}.$$

Definition 2.1.9. The subultra-group defined in Definition (2.1.6) is called the subultra-group generated by X.

Theorem 2.1.10. Let U be an ultra-group of the subgroup H over the group G and $N_1,\ N_2$ are normal subultra-group of U. If $K = N_1 \cap N_2$ then for every $a,b \in U$,

$$\left[a,[K,b]\right] \subseteq \left[K,[a,b]\right].$$

Proof: Since $N_1,\ N_2$ are normal subultra-group of, then for every $a,b \in U$,

$$[a,[N_1,b]] = [N_1,[a,b]] \text{ and } [a,[\,N_2,b]] = [\,N_2,[a,b]]$$

by Definition (1.9). Let $[a,[k,b]] \in [a,[K,b]]$ for some $k \in K = N_1 \cap N_2$. Then

$$[a,[k,b]] \in [a,[N_1,b]] = [N_1,[a,b]] \text{ and } [a,[k,b]] \in [a,[N_2,b]] = [N_2,[a,b]]$$

since $k \in N_1$ and $k \in N_2$. Therefore,

$$\left[a,[k,b]\right] \in \left[N_1,[a,b]\right] \cap \left[N_2,[a,b]\right].$$

Hence

$$[a,[k,b]] = [n_1,[a,b]] \text{ and } [a,[k,b]] = [n_2,[a,b]]$$

for some $n_1 \in N_1$ and $n_2 \in N_2$. In this case
$$[a,[k,b]] = [n_1,[a,b]] = [n_2,[a,b]]$$

and right cancellation implies $n_1 = n_2 \in N_1 \cap N_2 = K$. So we have,

$$[a,[k,b]] \in [N_1 \cap N_2,[a,b]] = [K,[a,b]] \Rightarrow [a,[K,b]] \subseteq [K,[a,b]].$$

Lemma 2.1.11. Let U be an ultra-group of the subgroup H over the group G. If $U \subset C_G(H)$ then U has the left cancellation. Here $C_G(H) = \{g \in G | hg = gh,\ for\ \forall h \in H\}$ is the centralizer of H in G.

Proof: Let $[a,b] = [a,c]$ for $a,b,c \in U$. We can write

$$ab = (a,b)[a,b] \text{ and } ac = (a,c)[a,c].$$

Therefore

$$(a,b)^{-1}ab = (a,c)^{-1} \text{ ac since } [a,b] = [a,c].$$

Because $a \in U \subseteq C_G(H)$ and $(a,b)^{-1}, (a,c)^{-1} \in H$ we conclude

$$(a,b)^{-1}b = (a,c)^{-1}c$$

and then by transversal property $b = c$.

Theorem 2.1.12. Let U be an ultra-group of the subgroup H over the group G and $U \subset C_G(H)$. If N_1, N_2 are normal subultra-groups of U, then $N_1 \cap N_2$ is a normal subultra-group of U.

Proof: Let $= N_1 \cap N_2$, $a,b \in U$ and $[k,[a,b]] \in [K,[a,b]]$ for some $k \in K$.

$$[k,[a,b]] \in [N_1,[a,b]] = [a,[N_1,b]]$$

and

$$[k,[a,b]] \in [N_2,[a,b]] = [a,[N_2,b]]$$

since $k \in K = N_1 \cap N_2$, so

$$[k,[a,b]] \in [a,[N_1,b]] \cap [a,[N_2,b]].$$

We can write

$$[k,[a,b]] = [a,[n_1,b]] \text{ and } [k,[a,b]] = [a,[n_2,b]]$$

for some $n_1 \in N_1$ and $n_2 \in N_2$. Hence

$$[k,[a,b]] = [a,[n_1,b]] = [a,[n_2,b]]$$

and since $U \subset C_G(H)$, $[n_1,b] = [n_2,b]$ by Lemma (2.1.11). By left cancellation

$$[n_1, b] = [n_2, b] \Rightarrow n_1 = n_2 \in N_1 \cap N_2 = K$$

and then

$$[k, [a, b]] \in [a, [N_1 \cap N_2, b]] = [a, [K, b]] \Rightarrow [K, [a, b]] \subseteq [a, [K, b]].$$

In contrary $[a, [K, b]] \subseteq [K, [a, b]]$ for every $a, b \in U$ by Theorem (2.1.10). Finally, we have $[a, [K, b]] = [K, [a, b]]$ for every $a, b \in U$ and $K = N_1 \cap N_2$ is a normal subultra-group of U.

2.2. ON QUOTIENT ULTRA-GROUPS

If σ which is defined in Lemma (1.6) preserves the operations of U then σ would be a congruence. The operations of U are also preserved for a subultra-group S of U, in this case S can be considered as a congruence. In particular, a normal subultra-group is a congruence.

Theorem 2.2.1. Let U be an ultra-group, S, is a subultra-group of U and σ be a congruence on U. The set $S/\sigma = \{[s]_\sigma \mid s \in S\}$ is an ultra-group by the operations defined in Definition (1.3). If S/σ is an abelian ultra-group, then $[s_1, s_2]\sigma[s_2, s_1]$ for all $s_1, s_2 \in S$.

Proof: Let S/σ is an abelian ultra-group then for every $[s_1]_\sigma, [s_2]_\sigma \in S/\sigma$, where s_1, s_2 are arbitrary elements of S, we have:

$[[s_1]_\sigma, [s_2]_\sigma] = [[s_2]_\sigma, [s_1]_\sigma]$
$\Leftrightarrow \alpha([s_1]_\sigma, [s_2]_\sigma) = \alpha([s_2]_\sigma, [s_1]_\sigma)$
$\Leftrightarrow [\alpha(s_1, s_2)]_\sigma = [\alpha(s_2, s_1)]_\sigma$
$\Leftrightarrow \big[[s_1, s_2]\big]_\sigma = \big[[s_2, s_1]\big]_\sigma$
$\Leftrightarrow [s_1, s_2]\sigma[s_2, s_1]$.

Corollary 2.2.2. Let U be an ultra-group of the subgroup H over the group G, N be a normal subultra-group of U. If U/N is abelian then

$$[u_1, u_2] \in [u_2, [N, u_1]] \text{ and } [u_2, u_1] \in [u_1, [N, u_2]]$$

for every $u_1, u_2 \in U$.

Proof: Since U/N is an abelian then for every $u_1, u_2 \in U$ we have

$$[u_1, u_2]N[u_2, u_1] \Rightarrow [u_1, u_2] = [n, [u_2, u_1]]$$

for some $n \in N$, then

$$[u_1, u_2] \in [N, [u_2, u_1]].$$

By Lemma (1.10),

$$[u_1, u_2] \in [[N, u_2], [N, u_1]] \text{ as well as } [u_2, u_1] \in [[N, u_1], [N, u_2]]$$

and we have

$$[u_1,\ u_2] \in [\ u_2, [N, u_1]] \text{ and } [\ u_2, u_1] \in [\ u_1, [N, u_2]]$$

by the Definition (1.2).

Corollary 2.2.3. Let U be an ultra-group of the subgroup H over the group G. If $U/N = [e]_N$ for every normal subultra-group N of U then U is abelian ultra-group.

Proof: If $U/N = [e]_N$ we have $[u_1,\ u_2][e]_N[\ u_2, u_1]$ for every $u_1,\ u_2 \in U$. Then

$$[u_1,\ u_2][e]_N[\ u_2, u_1] \Rightarrow [u_1,\ u_2] = [e, [\ u_2, u_1]]$$

and by the Proposition (1.4)

$$[u_1,\ u_2] = [e, [\ u_2, u_1]] = [\ u_2, u_1].$$

Therefore U is abelian ultra-group.

3. CONCLUSIONS

In this article, it was obtained some algebraic structural elements which are derived from the definitions of ultra-groups and (normal) subultra-groups. Moreover, we studied the quotient ultra-groups and determined some results. Since it is a new field for researchers only a few papers have been made on this subject, such basic algebraic results will help to construct more on this new concept.

By the center definition and quotient ultra-group properties presented in this paper, researchers could focus on defining and adapting basic result of nilpotent or soluble subultra-groups.

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