ORIGINAL PAPER

SOLVING FUZZY INITIAL VALUE PROBLEM WITH DIRAC DELTA FUNCTION USING FUZZY LAPLACE TRANSFORMS

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Abstract. *In this paper we investigate the solutions of second-order fuzzy initial value problem using the fuzzy Laplace transforms with Dirac delta function under the generalized differentiabilitiy. The related theorems and properties are given in detail and the method is illustrated by solving some examples.*

Keywords: *fuzzy initial value problem; fuzzy Laplace transform; Dirac delta function*

1. INTRODUCTION

Fuzzy differential equations (FDEs) are utilized for the modelling problems in science and engineering. Most of the problems have uncertain structural parameters. Instead, many researchers have modeled these uncertain structural parameters as fuzzy numbers in this area [1-3]. Fuzzy initial value problems (FIVPs) are one of the simplest FDEs that may appear in many applications.

There are several approaches to solving the FIVPs. The properties of differentiable fuzzy set-valued functions by means of the concept of H-differentiability due to Puri and Ralescu [4] were discussed by Kaleva [3]. Seikkala [5], defined the fuzzy derivative which is the generalization of the Hukuhara derivative, and showed that the fuzzy initial value problem has a unique solution. Strongly generalized differentiability was introduced by Bede and Gal [6] and studied by Bede et al. [7]. The strongly generalized derivative is defined for a larger class of fuzzy-valued functions than the H-derivative. So in this paper, we use this differentiability method.

The Dirac-delta function, which was first introduced by theoretical physicist Paul Dirac [8] in 1958, is a generalized singularity function whose value is zero everywhere except one point, with an integral of one over the entire domain. The delta function is represented with the Greek lowercase symbol delta which has the following properties:

$$\delta(x-x_0)=\begin{cases}\infty, & x=x_0\\ 0, & x\neq x_0.\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(x-x_0)dx=1,$$

$$\int_{-\infty}^{+\infty}f(x)\delta(x-x_0)dx=f(x_0),$$

where f is Riemann integrable function.

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Many physical and mechanical phenomena can be well described by means of the Dirac-delta function. For instance, the bending and vibration behavior of structures under concentrated loads, impulsive loading, moving loads, and impact loading; and thermoelastic vibration behavior of structures under heat source points can be mathematically modeled by means of the Dirac-delta function. In general, such problems can be handled using analytical and/or numerical approaches. In order to solve such type of problems, we use the fuzzy Laplace transform.

Fuzzy Laplace transform is useful to solve FIVPs with the Dirac delta function. Allahviranloo and Barkhordari Ahmadi first introduced fuzzy Laplace transform [9]. Later, Salahshour and Allahviranloo point out that under what conditions the fuzzy-valued functions can possess the fuzzy Laplace transform and they consider the important properties and related theorems for solving FIVPs [10]. Citil researched solutions of FIVPs by using fuzzy laplace transform [11, 12].

In this paper, we investigate the fuzzy solutions of the FIVPs by the fuzzy Laplace transform with the Dirac delta function under the concept of generalized differentiability.

2. NOTATION AND PRELIMINARIES

We now recall the basic definitions and the theorems utilized in this study. We denote the set of all real numbers represented by $\mathbb{R}$ and all the fuzzy numbers set on $\mathbb{R}$ by $\mathbb{R}_F$.

Definition 1. [13] A fuzzy number on real numbers $\mathbb{R}$ is a function like $\hat{u}: \mathbb{R} \to [0,1]$ with the following properties:

1. $\hat{u}$ is normal, i.e., $\exists t_0 \in R$ for which $\hat{u}(t_0) = 1$;
2. $\hat{u}$ is convex fuzzy set, i.e., $\mu(\lambda t + (1-\lambda)s) \geq \min\{\hat{u}(t), \hat{u}(s)\}$ for all $t, s \in R$, $\lambda \in [0,1]$,
3. $\hat{u}$ is upper semi-continuous on $\mathbb{R}$,
4. $cl\{x \in \mathbb{R}: u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

The α −level set of a fuzzy number $\hat{u} \in \mathbb{R}_F$, $0 \leq \alpha \leq 1$; denoted by $[\hat{u}]^\alpha$, is defined in [9] as follows

$$[\hat{u}]_a = \begin{cases} \{x \in R: u(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ cl(\text{supp}\hat{u}) \text{if} & \alpha = 0. \end{cases}$$

Definition 2. [14] An arbitrary fuzzy number $\hat{u}$ in the parametric form is represented by an ordered pair of functions $[u_\alpha^-, u_\alpha^+]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements

i. u_α^- is a non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,

ii. u_α^+ is non- increasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,

iii. $u_\alpha^- \leq u_\alpha^+$, $0 < \alpha \leq 1$.

Definition 3. [14] For $\hat{u}, \hat{v} \in \mathbb{R}_F$, and $\lambda \in \mathbb{R}$, the sum $\hat{u} \oplus \hat{v}$ and the $\lambda \odot \hat{u}$ are defined as

$$[\hat{u} \oplus \hat{v}]^\alpha = [\hat{u}]^\alpha + [\hat{v}]^\alpha = \{x + y: x \in [\hat{u}]^\alpha, y \in [\hat{v}]^\alpha\},$$

$$[\lambda \odot \hat{u}]^\alpha = [\lambda \odot \hat{u}]^\alpha = \{\lambda x: x \in [\hat{u}]^\alpha\}$$

for all $\alpha \in [0,1]$.

Definition 4. [14] A fuzzy number $\hat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ level is of the form for $a_1 < a_2 < a_3$ which $a_1, a_2, a_3 \in \mathbb{R}$, $[\hat{A}]^\alpha = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$, for all $\alpha \in [0,1]$.

Definition 5. [4] Let $\hat{u}, \hat{v} \in \mathbb{R}_F$. If there exists $\hat{w} \in \mathbb{R}_F$ such that $\hat{u} = \hat{v} + \hat{w}$, then w is called the Hukuhara difference of $\hat{u}$ and $\hat{v}$ and it is denoted by $\hat{u} \ominus_h \hat{v}$. If $\hat{u} \ominus_h \hat{v}$ exists, its $\alpha - levels$ are

$$[\hat{u} \ominus_h \hat{v}]^\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+]$$

for $\alpha \in [0,1]$.

In the present work, the sign ”$\ominus_h$” always stands for Hukuhara difference (H-difference). Note that the function $\hat{f}: [a,b] \to \mathbb{R}_F$ is called the fuzzy-valued function.

Definition 6. [15] Let $\hat{f}: (a,b) \to \mathbb{R}_F$ and $x_0 \in [a,b]$. If there exists $\hat{f}'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \hat{f}(x_0 + h) \ominus_h \hat{f}(x_0)$, $\hat{f}(x_0) \ominus_h \hat{f}(x_0 - h)$ and the limits hold

$$\lim_{h\to 0} \frac{\hat{f}(x_0 + h) \ominus_h \hat{f}(x_0)}{h} = \lim_{h\to 0} \frac{\hat{f}(x_0) \ominus_h \hat{f}(x_0 - h)}{h} = \hat{f}'(x_0)$$

$\hat{f}$ is Hukuhara differentiable at x_0.

Definition 7. [15] Let $\hat{f}: (a,b) \to \mathbb{R}_F$ and $x_0 \in [a,b]$. If there exists $\hat{f}'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \hat{f}(x_0 + h) \ominus_h \hat{f}(x_0)$, $\hat{f}(x_0) \ominus_h \hat{f}(x_0 - h)$ and the limits hold when f is $(1)-$ differentiable at x_0

$$\lim_{h\to 0} \frac{\hat{f}(x_0 + h) \ominus_h \hat{f}(x_0)}{h} = \lim_{h\to 0} \frac{\hat{f}(x_0) \ominus_h \hat{f}(x_0 - h)}{h} = \hat{f}'(x_0),$$

If there exists $f'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists \hat{f}(x_0) \ominus_h \hat{f}(x_0 + h)$, $\hat{f}(x_0 - h) \ominus_h \hat{f}(x_0)$ and the limits hold when f is $(2)-$ differentiable at x_0

$$\lim_{h\to 0} \frac{\hat{f}(x_0) \ominus_h \hat{f}(x_0 + h)}{-h} = \lim_{h\to 0} \frac{\hat{f}(x_0 - h) \ominus_h \hat{f}(x_0)}{-h} = \hat{f}'(x_0).$$

Theorem 1. [16] Let $\hat{f}: [a,b] \to \mathbb{R}_F$ be fuzzy-valued function and for each $\alpha \in [0,1]$

$$[\hat{f}(x)]^\alpha = [f_\alpha^-(x), f_\alpha^+(x)].$$

We say that

- if $\hat{f}$ is $(i)-$differentiable ,

(i) $[\hat{f}'(x)]^\alpha = [\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}],$

- If $\hat{f}$ is $(ii)-$differentiable

(ii) $[\hat{f}'(x)]^{\alpha} = [\{(f_{\alpha}^{+})'(x), (f_{\alpha}^{-})'(x)\}]$.

Theorem 2. [16] Let $\hat{f}': [a,b] \to \mathbb{R}_F$ be fuzzy-valued function and for each $\alpha \in [0,1]$

$$[\hat{f}(x)]^{\alpha} = [f_{\alpha}^{-}(x), f_{\alpha}^{+}(x)],$$

$\hat{f}$ and $\hat{f}'$ are (i) or (ii) differentiable. We say that

- if $\hat{f}$ and $\hat{f}'$ are (i) −differentiable

(i) $[\hat{f}''(x)]^{\alpha} = [(f_{\alpha}^{-})''(x), (f_{\alpha}^{+})''(x)]$,

- If $\hat{f}$ is (i) −differentiable and f' is (ii) −differentiable

(ii) $[\hat{f}''(x)]^{\alpha} = [(f_{\alpha}^{+})''(x), (f_{\alpha}^{-})''(x)]$,

- If $\hat{f}$ is (ii) −differentiable and $\hat{f}'$ is (i) −differentiable

(iii) $[\hat{f}''(x)]^{\alpha} = [(f_{\alpha}^{+})''(x), (f_{\alpha}^{-})''(x)]$,

- if $\hat{f}$ and $\hat{f}'$ are (ii) −differentiable

(iv) $[\hat{f}''(x)]^{\alpha} = [(f_{\alpha}^{-})''(x), (f_{\alpha}^{+})''(x)]$.

Theorem 3. [17] Let $\hat{f}(x)$ be a fuzzy-valued function on $[a,\infty)$ represented by $\left((f_{\alpha}^{-}(t)), (f_{\alpha}^{+}(t))\right)$. For any fixed $\alpha \in [0,1]$,assume $f_{\alpha}^{-}(t)$ and $f_{\alpha}^{+}(t)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive functions $M_{\alpha}^{-}$and M_{α}^{+} such that $\int_a^b |f_{\alpha}^{-}(t)|dt \leq M_{\alpha}^{-}$ and $\int_a^b |f_{\alpha}^{+}(t)|dt \leq M_{\alpha}^{+}$ for every $b \geq a$. Then $\hat{f}(x)$ is improper fuzzy Riemann-integrable on $[a,\infty)$ and the improper fuzzy Riemann-integral is a fuzzy number. Furthermore, we have

$$\int_a^{\infty} \hat{f}(t)dt = \left(\int_a^{\infty} f_{\alpha}^{-}(t)dt, \int_a^{\infty} f_{\alpha}^{+}(t)dt\right).$$

Definition 8. [18] The Heaviside or unit step function, denoted here by $u_c(t)$, is zero for $t < c$ and is one for $t \geq c$; that is,

$$U_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

The Heaviside function can be used to represent a translation of a function $f(t)$ a distance c in the positive t direction. We have

$$U_c(t)f(t-c) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c. \end{cases}$$

Suppose that $\hat{f}$ is a fuzzy-valued function and s is a real parameter. The fuzzy Laplace transform of $\hat{f}$ is defined as following:

Definition 9. [10] The fuzzy Laplace transform of fuzzy-valued function $\hat{f}$ is defined as follows:

$$\hat{F}(s) = L\left(\hat{f}(t)\right) = \int_0^\infty e^{-st}\hat{f}(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\hat{f}(t)dt$$

i,e.,

$$\hat{F}(s) = \left[\lim_{\tau\to\infty}\int_0^\tau e^{-st}f_\alpha^-(t)dt, \lim_{\tau\to\infty}\int_0^\tau e^{-st}f_\alpha^+(t)dt\right]$$

whenever the limits exist.

Consider the fuzzy-valued function $\hat{f}$; then the lower and upper fuzzy Laplace transform of this function are denoted based on the lower and upper of the fuzzy-valued function $\hat{f}$ as follows:

$$\hat{F}(s;\alpha) = L\left(\hat{f}(t;\alpha)\right) = [l(f_\alpha^-(t)), l(f_\alpha^+(t))]$$

where

$$l(f_\alpha^-(t)) = \int_0^\infty e^{-st}f_\alpha^-(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}f_\alpha^-(t)dt,$$

$$l(f_\alpha^+(t)) = \int_0^\infty e^{-st}f_\alpha^+(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}f_\alpha^+(t)dt.$$

Theorem 4. [10] If $\hat{F}(s) = L\left(\hat{f}(t)\right)$ for $s > 0$, then $\hat{F}(s-a) = L\left(e^{at}\hat{f}(t)\right)$, a reel and $s > a$.

Theorem 5. [10] If $\hat{F}(s) = L\left(\hat{f}(t)\right)$ then $L\left(u_a(t)\hat{f}(t-a)\right) = e^{-as}\hat{F}(s)$, $a \geq 0$.

Theorem 6. [10] Suppose that $\hat{f}$ is continuous fuzzy-valued function on $[a,\infty)$ and exponential order α and that $\hat{f}'$ is piecewise continuous fuzzy-valued function on $[a,\infty)$, then

$$L\left(\hat{f}'(t)\right) = sL\left(\hat{f}(t)\right) \ominus_h \hat{f}(0),$$

if $\hat{f}$ is (1) $-$differentiable,

$$L\left(\hat{f}'(t)\right) = \left(-\hat{f}(0)\right) \ominus_h \left(-sL\left(\hat{f}(t)\right)\right),$$

if $\hat{f}$ is (2) $-$differentiable.

In order to solve the second-order fuzzy differential equation, we need the fuzzy Laplace transform of second-order derivatives under generalized H-differentiability. So, we give the following:

Theorem 7. [10] Suppose that $\hat{f}$ and $\hat{f}'$ are continuous fuzzy-valued functions on $[a,\infty)$ and of exponential order and that $\hat{f}''$ is piecewise continuous fuzzy-valued function on $[a,\infty)$, then

$$L\left(\hat{f}''(t)\right) = s^2L\left(\hat{f}(t)\right) \ominus_h s\hat{f}(0) \ominus_h' \hat{f}(0), \quad (1)$$

if $\hat{f}$ and $\hat{f}'$ are (1) −differentiable,

$$L\left(\hat{f}''(t)\right) = -\hat{f}'(0) \ominus_h (-s^2)L\left(\hat{f}(t)\right) - s\hat{f}(0), \qquad (2)$$

if $\hat{f}$ is (1) −differentiable and f' is (2) −differentiable,

$$L\left(\hat{f}''(t)\right) = -s\hat{f}(0) \ominus_h (-s^2)L\left(\hat{f}(t)\right) \ominus \hat{f}'(0), \qquad (3)$$

if $\hat{f}$ is (2) −differentiable and $\hat{f}'$ is (1) −differentiable,

$$L\left(\hat{f}''(t)\right) = s^2 L\left(\hat{f}(t)\right) \ominus_h s\hat{f}(0) - \hat{f}'(0), \qquad (4)$$

if $\hat{f}$ and $\hat{f}'$ are (2) −differentiable.

Theorem 8. [9] Let $\hat{f}(t)$ and $\hat{g}(t)$ be continuous fuzzy-valued functions and c_1 and c_2 constants, then

$$L(c_1\hat{f}(t) + c_2\hat{g}(t)) = (c_1 L(\hat{f}(t))) + (c_2 L(\hat{g}(t))).$$

3. SOLUTION OF FIVPS

In this section, we will research the solution of FIVPs with Dirac delta function under generalized H-differentiability which has been proposed in Salahshour and Allahviranloo ([10]).

We consider the fuzzy initial value problem of the form

$$\hat{u}''(t) + p\hat{u}' + q\hat{u} = g(t), \quad \hat{u}(0) = \hat{u}_0, \qquad \hat{u}'(0) = \hat{z}_0,$$

where $g(t)$ is a Dirac delta function, $\hat{u}(t) = (u_\alpha^-(t), u_\alpha^+(t))$ is a fuzzy-valued function of t, p and q are the classic continuous functions, $\hat{u}_0 = (\hat{u}_{01}, \hat{u}_{02}, \hat{u}_{03})$ and $\hat{z}_0 = (\hat{z}_{01}, \hat{z}_{02}, \hat{z}_{03})$ are triangular fuzzy numbers.

By using the fuzzy Laplace transform method the following can be given:

Case 1

If $\hat{u}$ and $\hat{u}'$ are (i) −differentiability, then :

$$\begin{aligned} &s^2 L(\hat{u}(t;\alpha)) \ominus_h s\hat{u}(0;\alpha) \ominus_h \hat{u}'(0;\alpha) \\ &\oplus p(sL\hat{u}(t;\alpha) \ominus_h \hat{u}(0;\alpha)) \oplus q(L\hat{u}(t;\alpha)) \\ &= L(g(t)), \end{aligned} \qquad (5)$$

Case 2

If $\hat{u}$ is (i) −differentiability and $\hat{u}'$ are (ii) −differentiability, then

$$\begin{aligned} &-\hat{u}'(0;\alpha) \ominus_h (-s^2)L(\hat{u}(t;\alpha)) - s\hat{u}(0;\alpha) \\ &\oplus p(sL\hat{u}(t;\alpha) \ominus_h \hat{u}(0;\alpha)) \oplus q(L\hat{u}(t;\alpha)) \\ &= L(g(t)), \end{aligned} \qquad (6)$$

Case 3

If $\hat{u}$ is (ii) −differentiability and $\hat{u}'$ are (i) −differentiability, then

$$-s\hat{u}(0;\alpha) \ominus_h (-s^2)L(\hat{u}(t;\alpha)) \ominus_h \hat{u}'(0;\alpha)$$
$$\oplus\, p\big((-\hat{u}(0;\alpha)) \ominus_h (-sL\hat{u}(t;\alpha))\big) \oplus q(L\hat{u}(t;\alpha)) \quad (7)$$
$$= L(g(t)),$$

Case 4

If $\hat{u}$ and $\hat{u}'$ are (ii) $-$differentiability, then we

$$s^2L(\hat{u}(t;\alpha)) \ominus_h s\hat{u}(0;\alpha) - \hat{u}'(0;\alpha)$$
$$\oplus\, p\big((-\hat{u}(0;\alpha)) \ominus_h (-sL\hat{u}(t;\alpha))\big) \oplus q(L\hat{u}(t;\alpha)) \quad (8)$$
$$= L(g(t)).$$

3. EXAMPLES

In the following examples, the proposed method will be illustrated.

Example 1. [19] Consider the following FIVP:

$$\hat{u}''(t) + 2\hat{u}'(t) + 4\hat{u}(t) = 50\cos(t)\delta(t - pi),$$
$$\hat{u}(0) = \hat{5}, \qquad \hat{u}'(0) = \hat{1}$$

where $\hat{5} = (4,5,6)$ and $\hat{1} = (0,1,2)$, δ is a Dirac delta function. Now we consider this example by using the fuzzy Laplace method in four cases as following:

Case 1

Let us consider $\hat{u}(t)$ and $\hat{u}'(t)$ are (i) $-$differentiable; then by applying (5), we have

$$s^2L(\hat{u}(t;\alpha)) \ominus_h s\hat{u}(0;\alpha) \ominus_h \hat{u}'(0;\alpha)$$
$$\oplus\, 2(sL\hat{u}(t;\alpha) \ominus_h \hat{u}(0;\alpha)) \oplus 4(L\hat{u}(t;\alpha))$$
$$= L(50 * \cos(t) * \delta(t - pi);\alpha),$$

then we get the α $-$level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = 4 * e94(-t) * (cos(394(1/2) * t) - (394(1/2) * sin(394(1/2) * t))/3)$$
$$+(8 * 394(1/2) * e94(-t) * sin(394(1/2) * t))/3$$
$$-(50 * 394(1/2) * e94(pi - t) * sin(394(1/2) * (t - pi)) * U(t - pi))/3,$$

$$u_0^+(t) = 6 * e94(-t) * (cos(394(1/2) * t) - (394(1/2) * sin(394(1/2) * t))/3) + (14$$
$$* 394(1/2) * e94(-t) * sin(394(1/2) * t))/3$$
$$-(50 * 394(1/2) * e94(pi - t) * sin(394(1/2) * (t - pi)) * U(t - pi))/3.$$

According to Definition 2, $\hat{u}(t)$ is a valid fuzzy-valued function for $t \in (0,1.4)$ in Fig. 1.

Case 2

Let us consider $\hat{u}(t)$ is (i) $-$differentiable and $\hat{u}'(t)$ are (ii) $-$differentiable; then by applying (6), we have

$$u_0^-(t) = 200*U(t-pi)*((e94(pi-t)*(cos(394(1/2)*(t-pi)) - (394(1/2) * sin(394(1/2)*(t-pi)))/3))/16 - (e94(t-pi)*(cosh(594(1/2)*(t-pi)) - (3*594(1/2)*sinh(594(1/2)*(t-pi)))/5))/16) - 100*U(t-pi)*((e94(pi-t)*(cos(394(1/2)*(t-pi)) + (394(1/2) * sin(394(1/2)*(t-pi)))/3))/8 - (e94(t-pi)*(cosh(594(1/2)*(t-pi)) - (594(1/2)*sinh(594(1/2)*(t-pi)))/5))/8) - 50*U(t-pi) * ((394(1/2)*e94(pi-t)*sin(394(1/2)*(t-pi)))/6 + (594(1/2) * e94(t-pi)*sinh(594(1/2)*(t-pi)))/10) - (11*e94(t) * (cosh(594(1/2)*t) - (594(1/2)*sinh(594(1/2)*t))/5))/2 + 16 * e94(t)*(cosh(594(1/2)*t) + (594(1/2)*sinh(594(1/2)*t))/5) + (5*e94(t)*(cosh(594(1/2)*t) - (3*594(1/2)*sinh(594(1/2) * t))/5))/2 + (27*e94(-t)*(cos(394(1/2)*t) - (394(1/2) * sin(394(1/2)*t))/3))/2 + (11*e94(-t)*(cos(394(1/2)*t) + (394(1/2)*sin(394(1/2)*t))/3))/2,$$

$$u_0^+(t) = 200*U(t-pi)*((e94(pi-t)*(cos(394(1/2)*(t-pi)) - (394(1/2) * sin(394(1/2)*(t-pi)))/3))/16 - (e94(t-pi)*(cosh(594(1/2)*(t-pi)) - (3*594(1/2)*sinh(594(1/2)*(t-pi)))/5))/16) - 100*U(t-pi)*((e94(pi-t)*(cos(394(1/2)*(t-pi)) + (394(1/2) * sin(394(1/2)*(t-pi)))/3))/8 - (e94(t-pi)*(cosh(594(1/2)*(t-pi)) - (594(1/2)*sinh(594(1/2)*(t-pi)))/5))/8) - 50*U(t-pi) * ((394(1/2)*e94(pi-t)*sin(394(1/2)*(t-pi)))/6 + (594(1/2) * e94(t-pi)*sinh(594(1/2)*(t-pi)))/10) - 5*e94(t) * (cosh(594(1/2)*t) - (594(1/2)*sinh(594(1/2)*t))/5) + 24 * e94(t)*(cosh(594(1/2)*t) + (594(1/2)*sinh(594(1/2)*t))/5) + 3 * e94(t)*(cosh(594(1/2)*t) - (3*594(1/2)*sinh(594(1/2)*t))/5) + 21*e94(-t)*(cos(394(1/2)*t) - (394(1/2)*sin(394(1/2) * t))/3) + 5*e94(-t)*(cos(394(1/2)*t) + (394(1/2)*sin(394(1/2) * t))/3) + (594(1/2)*e94(t)*sinh(594(1/2)*t))/5 + (394(1/2) * e94(-t)*sin(394(1/2)*t))/3.$$

According to Definition 2, $\hat{u}(t)$ has not a valid fuzzy-valued function in Fig. 2.

Case 3

Let us consider $\hat{u}(t)$ is (ii) $-$differentiable and $\hat{u}'(t)$ are (i) $-$differentiable; then by applying (7), we have

$$-s\hat{u}(0;\alpha) \ominus_h (-s^2)L(\hat{u}(t;\alpha)) \ominus_h \hat{u}'(0;\alpha) \oplus 2\big((-\hat{u}(0;\alpha)) \ominus_h (-sL\hat{u}(t;\alpha))\big) \oplus 4(L\hat{u}(t;\alpha)) = L(50*\cos(t)*\delta(t-pi);\alpha),$$

then we get the α $-$level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = (77*e94(-(5*t)/2))/4 - (57*e94(-(3*t)/2))/4 + 3*\delta(t) - 150 * U(t-pi)*(e94((3*pi)/2 - (3*t)/2)/4 - e94((5*pi)/2 - (5*t)/2)/4 + 100 * U(t-pi)*((3*e94((3*pi)/2 - (3*t)/2))/8 - (5*e94((5*pi)/2 - (5*t)/2))/8),$$

$$u_0^+(t) = \frac{113*e94\left(-\frac{5*t}{2}\right)}{2} - 43*e94(-2*t) + 2*\delta(t) + 100*U(t-pi)*\left(e94(2*pi-2*t) - \frac{5*e94\left(\frac{5*pi}{2}-\frac{5*t}{2}\right)}{4}\right) - 200*U(t-pi)*\left(\frac{e94(2*pi-2*t)}{2} - \frac{e94\left(\frac{5*pi}{2}-\frac{5*t}{2}\right)}{2}\right)$$

According to Definition 2, $\hat{u}(t)$ has not a valid fuzzy-valued function in Fig. 1.

Case 4
$\hat{u}(t)$ has not a valid fuzzy-valued function.

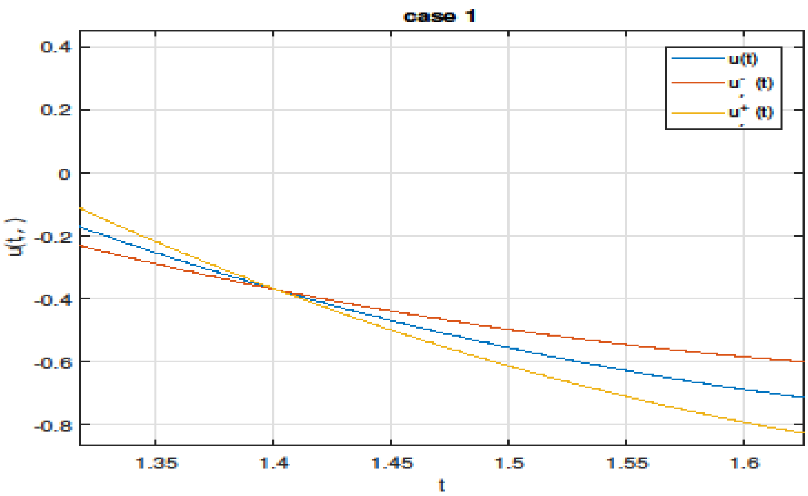

Figure 1. Level sets of the case 1 solution of the Example 1

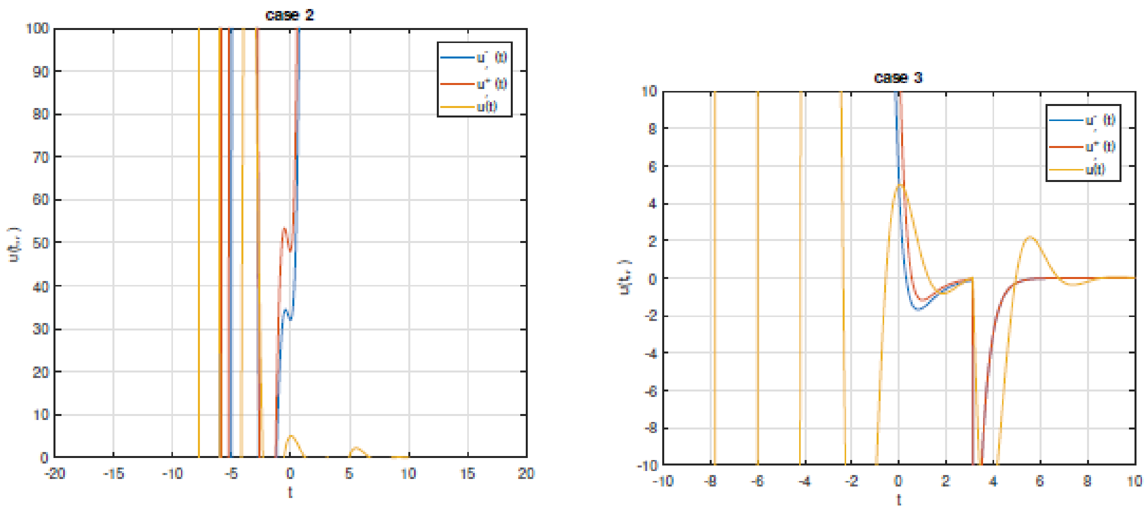

Figure 2. Level sets of the case 2 and case 3 solution of the Example 1

Example 2. Consider the following FIVP:

$$\hat{u}''(t) + 9\hat{u}(t) = 3U(t-3), \qquad \hat{u}(0) = \hat{1}, \qquad \hat{u}'(0) = \hat{0}$$

where $\hat{1} = (0.5,1,1.5)$ and $\hat{0} = (-1,0,1)$, U is a Heaviside funcion. Now we consider this example by using fuzzy laplace method in four cases as following:

Case 1

Let us consider $\hat{u}(t)$ and $\hat{u}'(t)$ are (i) −differentiable; then by applying (5), we have

$$s^2L(\hat{u}(t;\alpha)) \ominus_h s\hat{u}(0;\alpha) \ominus_h \hat{u}'(0;\alpha) \oplus 9*(L\hat{u}(t;\alpha)) = L(t*U(t-3)),$$

then we get the α −level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = cos(3*t)/2 - sin(3*t)/3 - sin(3*t)*U(t-pi)$$
$$u_0^+(t) = (3*cos(3*t))/2 + sin(3*t)/3 - sin(3*t)*U(t-pi)$$

According to Definition 2, $\hat{u}(t)$ is a valid fuzzy-valued function for $t \in (0,0.72) \cup (1.8,2.8)$ in Fig. 2

Case 2

Let us consider $\hat{u}(t)$ is (i) −differentiable and $\hat{u}'(t)$ are (ii) −differentiable; then by applying (6), we have

$$-\hat{u}'(0;\alpha) \ominus_h (-s^2)L(\hat{u}(t;\alpha)) - s\hat{u}(0;\alpha) \oplus 9*(L\hat{u}(t;\alpha)) = L(t*U(t-3)),$$

then we get the α −level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = cos(3*t)/2 + e94(-3*t)/6 - e94(3*t)/6 - (U(t-pi)*((3*sin(3*t))/2 + (3*e94(3*pi-3*t))/4 - (3*e94(3*t-3*pi))/4))/3 - 3*U(t-pi)*(sin(3*t)/6 - e94(3*pi-3*t)/12 + e94(3*t-3*pi)/12)$$

$$u_0^+(t) = cos(3*t) + e94(-3*t)/12 + (5*e94(3*t))/12 - (U(t-pi)*((3*sin(3*t))/2 + (3*e94(3*pi-3*t))/4 - (3*e94(3*t-3*pi))/4))/3 - 3*U(t-pi)*(sin(3*t)/6 - e94(3*pi-3*t)/12 + e94(3*t- *pi)/12)$$

According to Definition 2, $\hat{u}(t)$ is a valid fuzzy-valued function for $t \in (-0.25,1)$ in Fig. 3.

Case 3

Let us consider $\hat{u}(t)$ is (ii) −differentiable and $\hat{u}'(t)$ are (i) −differentiable; then by applying (7), we have

$$-s\hat{u}(0;\alpha) \ominus_h (-s^2)L(\hat{u}(t;\alpha)) \ominus_h \hat{u}'(0;\alpha) \oplus 9*(L\hat{u}(t;\alpha)) = L(t*U(t-3)),$$

then we get the α −level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = cos(3*t) + (2*e94(-3*t))/3 - e94(3*t)/6 + sin(3*t)/2 - (U(t-pi)*((3*sin(3*t))/2 + (3*e94(3*pi-3*t))/4 - (3*e94(3*t-3*pi))/4))/3 - 3*U(t-pi)*(sin(3*t)/6 - exp(3*pi-3*t)/12 + exp(3*t-3*pi)/12)$$

$$u_0^+(t) = (10*t)/9 + cos(3*t)/6 - sin(3*t)/27 - sin(3*t)*U(t-pi) + 24*U(t-pi)*(t/9 - pi/9 + sin(3*t)/27) + 1/3$$

According to Definition 4, $\hat{u}(t)$ is a valid fuzzy-valued function for $t \in (0.45,0.69)$ in Fig. 3.

Case 4

Let us consider $\hat{u}(t)$ and $\hat{u}'(t)$ are (ii) −differentiable; then by applying (8), we have

$$s^2L(\hat{u}(t;\alpha)) \ominus_h s\hat{u}(0;\alpha) - \hat{u}'(0;\alpha) \oplus 9 * (L\hat{u}(t;\alpha)) = L(t * U(t-3)),$$

then we get the α −level representation of solution as following for $\alpha = 0$

$$u_0^-(t) = cos(3*t)/2 + sin(3*t)/3 - sin(3*t) * U(t-pi)$$
$$u_0^+(t) = (3*cos(3*t))/2 - sin(3*t)/3 - sin(3*t) * U(t-pi)$$

According to Definition 2, $\hat{u}(t)$ is a valid fuzzy-valued function for $t \in (0,0.3) \cup (1.4,2.4)$ in Fig.2.

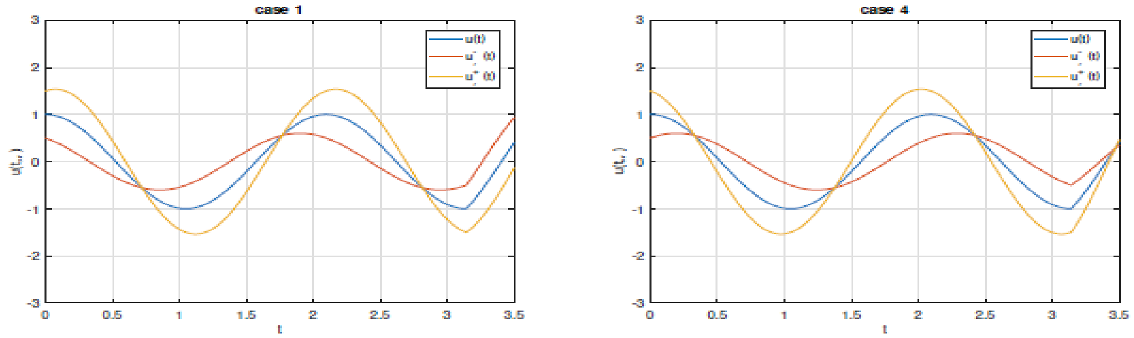

Figure 3. Level sets of the case 1 and case 4 solution of the Example 2

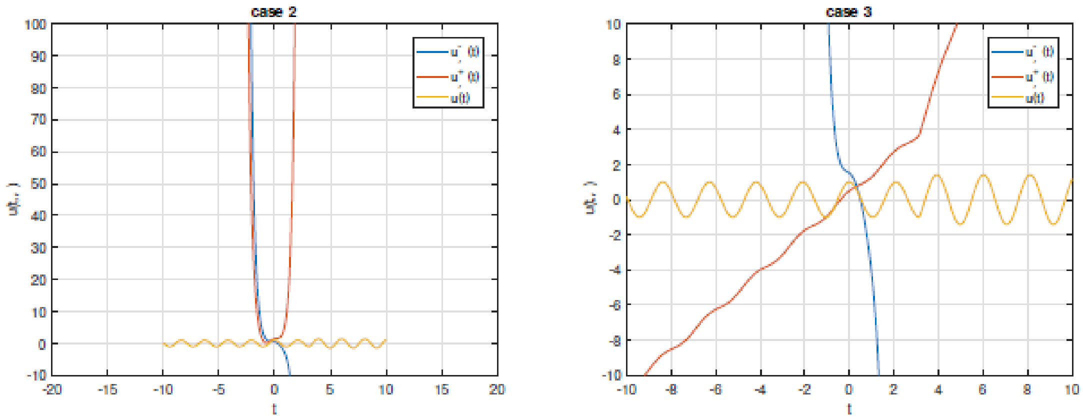

Figure 4. Level sets of the case 2 and case 3 solution of the Example 2

From, Examples 1 and 2, we see that the solution of a FDE is dependent on the election of the derivative in the (i) −differentiable or in the (ii) −differentiable. We saw that for four cases all the obtained solutions were not acceptable. Some intervals did not provide the fuzzy solution conditions. So we can choose an adequately fuzzy solution.

3. CONCLUSION

In this paper, we applied the fuzzy Laplace transforms provided solutions to fuzzy second-order FIVPs which are interpreted by using the generalized differentiability concept with the Dirac delta function. We have shown graphs in which intervals the solutions of the problem provide the fuzzy conditions. The efficiency of the method was illustrated by two numerical examples.

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