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DETERMINANTS AND INVERSES OF GENERALIZED LOWER HESSENBERG MATRICES

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Abstract. *Hessenberg matrices arise in various mathematical and engineering applications due to their special properties and efficient algorithms for solving linear systems. This study aims to compute the determinants and inverses of generalized lower Hessenberg matrices composed of conditional polynomial sequences which are defined based on specific conditions and exhibit intriguing behavior. Moreover, we present more generalized results compared to some earlier works using the factorization properties of some special matrices and some analytical methods.*

Keywords: *Hessenberg matrix; matrix factorization; generalized conditional polynomials; determinant, inverse.*

1. INTRODUCTION

Matrices are fundamental mathematical objects which are used in a variety of fields such as mathematics, physics, computer science, engineering, and data analysis. Matrices are a cornerstone of modern mathematics because they give a simple and powerful representation of linear transformations, systems of linear equations, and mathematical structures. Determinants and inverses of the matrices are two fundamental ideas in linear algebra which are crucial in understanding the underlying structure of linear systems. These powerful tools reveal hidden secrets of matrices, allowing us to study, solve, and modify complicated systems with elegance and accuracy.

Hessenberg matrices are especially observed in eigenvalue computations and the solution of systems of linear equations. They have some advantageous properties over general matrices when dealing with specific algorithms. For example, the QR algorithm for computing eigenvalues of a matrix often converges faster for Hessenberg matrices than for general matrices. Overall, Hessenberg matrices are a critical concept in numerical linear algebra and play an important role in many numerical methods, particularly dealing with eigenvalue problems.

In recent years, several researchers have studied determinants, inverses and some properties of some special type matrices whose elements are special number sequences [1-7, 14-16]. For example, Shen et al. obtained determinants and inverses of circulant matrices with Fibonacci and Lucas numbers [1]. The authors expressed the determinants of two types of circulant matrices by utilizing only Fibonacci and Lucas numbers. Stanimirovic *et al.* studied the inverses of the generalized Fibonacci matrix whose elements are the general second-order non-degenerated sequence [2]. Stanimirovic *et al.* computed the pseudoinverse of singular type generalized Fibonacci matrix whose nonzero elements are generalized Fibonacci

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numbers [3]. The authors derived some combinatorial identities involving generalized Fibonacci numbers and binomial coefficients. Kılıç and Arıkan presented various results on the calculation of large classes of Hessenberg matrices, whose entries are the terms of any sequence, using the function generation method [4]. Radicic considered the Moore - Penrose inverse of singular type k-circulant matrix whose elements are the classical binomial coefficients [5]. Köme gave factorizations of the *(r, k)*-bonacci and inverse *(r, k)*-bonacci matrices [6]. He obtained upper and lower bounds of the eigenvalues of the symmetric *(r, k)*-bonacci matrices by using doubly stochastic matrices and the theory of majorization. Shen and Liu computed the determinants and inverses of lower Hessenberg matrix whose elements are the Horadam numbers [7].

In recent years, there have been several applications and generalizations of the Fibonacci and Lucas numbers [8-13]. For example, Falcon and Plaza studied the $k-$Fibonacci sequence, $\{F_{k,n}\}_{n=0}^{\infty}$, by analysing the recursive application of two geometrical transformations used in the (4TLE) partition [9]. Edson and Yayenie proposed a notable generalization of the Fibonacci numbers, which is called as bi-periodic Fibonacci sequence, and then the authors obtained the extended Binet formula, generating function and several identities [10]. Bilgici examined the bi-periodic Lucas numbers and modified generalized Lucas numbers and he obtained the generating functions, the Binet formulas and some special identities [11]. Tan and Leung defined the generalized bi-periodic Horadam sequence and the authors gave some binomial identities and congruence relations for the generalized bi-periodic Horadam sequence by using the method of Carlitz and Ferns [12]. Morever, Varma and Bala defined the generalized bivariate bi-periodic Fibonacci polynomials [13], which has some interesting properties as it is a more general form of many sequences in the literature such as [10,11]. The authors derived Catalan's identity, d'Ocagne's identity, Cassini's identity and Gelin Cesaro identity, generating functions and Binet formula of the generalized bivariate bi-periodic Fibonacci polynomials that we call as generalized conditional polynomial sequence in this paper.

Definition 1.1. [13] For $n \geq 2$ and any nonzero real numbers a, b and c, generalized conditional polynomial sequence is defined by

$$W_n = \begin{cases} axW_{n-1} + cyW_{n-2} & \text{if } n \text{ is even} \\ bxW_{n-1} + cyW_{n-2} & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.1}$$

with the arbitrary initial conditions W_0 and W_1.

Here we note that, throughout this paper, we use $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function.

Definition 1.2. [2] The generalized Fibonacci matrix $\mathbb{U}_n^{(a,b,s)} = u_{i,j}^{(a,b,s)}$ is defined by

$$u_{i,j}^{(a,b,s)} = \begin{cases} U_{i-j+1}^{(a,b)}, & i-j+s \geq 0 \\ 0, & i-j+s < 0 \end{cases}, \tag{1.2}$$

where $U_n^{(a,b)}$ is the second order recurrent sequence.

Definition 1.3. The generalized conditional lower Hessenberg matrix is defined by

$$\mathbb{W}_n[x,y]=[w_{i,j}]=\begin{cases}\left(\dfrac{b}{a}\right)^{\frac{\xi(i-j)}{2}}W_{i-j+1}, & i-j+1\geq 0\,,\\ 0, & i-j+1<0\end{cases} \tag{1.3}$$

where W_n is the generalized conditional polynomial sequence.

Until this time, several researchers have studied the inverses, determinants and applications of the Toeplitz and Hessenberg matrices with special recurrence sequences such as Fibonacci, Lucas and Horadam sequences [1-7,14-16]. By analogy to the earlier works, we compute the determinants and inverses of the generalized Hessenberg matrices whose elements are the conditional polynomial sequence. Moreover, we provide more generalized results for the specific values of a, b, c, x, y, W_0 and W_1.

2. MAIN RESULTS

In this section, we consider the determinants and inverses of the conditional Hessenberg matrices whose elements are the generalized conditional polynomial sequence. The next theorem explains the determinants of the matrix $\mathbb{W}_n[x,y]=[w_{i,j}]$.

Theorem 2.1. For the generalized conditional lower Hessenberg matrix, $\mathbb{W}_n[x,y]$, we have

$$\det\left[\mathbb{W}_n[x,y]\right]=\frac{(W_1-bxW_0)^{n-2}(aW_1^2-abxW_0W_1-bcyW_0^2)}{a}. \tag{1.4}$$

Proof: First, we investigate the case $W_1=0$. Let $\mathbb{D}_n[\mathrm{x},\mathrm{y}]=\left[d_{i,j}\right]$ and $\mathbb{E}_n=\left[e_{i,j}\right]$ be two $n\times n$ matrices which are as follows:

$$\mathbb{D}_n[\mathrm{x},\mathrm{y}]=\left[d_{i,j}\right]=\begin{cases}1, & i=j,\\ -\sqrt{ab}x, & i=j+1, j>1,\\ -cy, & i=j+2,\\ 0, & \text{otherwise.}\end{cases},\quad \mathbb{E}_n=\left[e_{i,j}\right]=\begin{cases}1, & i+j=3, j\in\{1,2\}\\ & \text{or } i=j\geq 3,\\ 0, & \text{otherwise.}\end{cases}.$$

Thus, we can verify $\mathbb{K}_n[x,y]=\mathbb{D}_n[x,y]\mathbb{W}_n[x,y]\mathbb{E}_n$, where

$$\mathbb{K}_n[x,y]=\begin{pmatrix}\frac{\sqrt{b}W_0}{\sqrt{a}} & 0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\sqrt{b}cyW_0}{\sqrt{a}} & \frac{\sqrt{b}W_0}{\sqrt{a}} & 0 & \cdots & 0\\ 0 & 0 & -bxW_0 & \frac{\sqrt{b}W_0}{\sqrt{a}} & \cdots & 0\\ 0 & 0 & 0 & -bxW_0 & \ddots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{b}W_0}{\sqrt{a}}\\ 0 & 0 & 0 & 0 & \cdots & -bxW_0\end{pmatrix}.$$

From the definition of the matrices $\mathbb{D}_n[\mathrm{x},\mathrm{y}]$, $\mathbb{E}_n$ and $\mathbb{K}_n[x,y]$, we know that

$$\det \mathbb{D}_n[\mathrm{x},\mathrm{y}] = 1,$$

$$\det \mathbb{E}_n = -1$$

and

$$\det \mathbb{K}_n[x,y] = \frac{(-1)^{n-2} W_0^n b^{n-1} x^{n-2} cy}{a}.$$

Therefore, we get $\det[\mathbb{W}_n[x,y]] = \frac{(-1)^{n-1} W_0^n b^{n-1} x^{n-2} cy}{a}$ which satisfies the formula (1.4).

Next, we investigate the case $W_1 \neq 0$. So, we define an $n \times n$ matrix be of the form

$$\mathbb{Q}_n[x,y] = [q_{ij}] = \begin{cases} 1, & i = j, \\ -\dfrac{\sqrt{b}(cyW_0 + axW_1)}{\sqrt{a}W_1}, & i = 2, j = 1 \\ -\sqrt{ab}x, & i = j+1, j > 1 \\ -cy, & i = j+2 \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have $\mathbb{R}_n[x,y] = \mathbb{Q}_n[x,y]\mathbb{W}_n[x,y]$, where

$$\mathbb{R}_n[x,y] = \begin{pmatrix} W_1 & \frac{\sqrt{b}W_0}{\sqrt{a}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-bcyW_0^2 - abxW_0W_1 + aW_1^2}{aW_1} & \frac{\sqrt{b}W_0}{\sqrt{a}} & 0 & \cdots & 0 \\ 0 & 0 & -bxW_0 + W_1 & \frac{\sqrt{b}W_0}{\sqrt{a}} & \cdots & 0 \\ 0 & 0 & 0 & -bxW_0 + W_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{b}W_0}{\sqrt{a}} \\ 0 & 0 & 0 & 0 & \cdots & -bxW_0 + W_1 \end{pmatrix}.$$

As $\det \mathbb{Q}_n[x,y] = 1$ and $\det \mathbb{R}_n[x,y] = \frac{(-bxW_0+W_1)^{n-2}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{a}$, we obtain

$$\det[\mathbb{W}_n[x,y]] = \frac{(-bxW_0 + W_1)^{n-2}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{a},$$

which proves the theorem.

Lemma 2. 1. For $-bxW_0 + W_1 \neq 0$, $bcyW_0^2 + abxW_0W_1 - aW_1^2 \neq 0$ and $W_1 \neq 0$, the inverse of the matrix $\mathbb{R}_n[x,y]$, $\mathbb{R}_n^{-1}[x,y] = [r'_{ij}]$, is equal to

$$r'_{ij} = \begin{cases} \dfrac{1}{W_1}, & i = j = 1 \\ -\left(\dfrac{b}{a}\right)^{\frac{j-i}{2}-1} \dfrac{b\,(-W_0)^{j-i} {W_1}^{i-1}}{(-bxW_0+W_1)^{j-2}(bcyW_0^2+abxW_0W_1-aW_1^2)}, & 1 \le i \le 2, j > 1 \\ \left(\dfrac{b}{a}\right)^{\frac{j-i}{2}} \dfrac{(-W_0)^{j-i}}{(-bxW_0+W_1)^{j-i+1}}, & j \ge i > 2 \\ 0, & \text{otherwise} \end{cases}.$$

Proof: Let $h_{i,j} = \sum_{l=1}^{n} r_{i,l} r'_{l,j}$. It is clear from Lemma 2.1 that $h_{1,1} = h_{2,2} = 1$, and for $i > j$ $h_{i,j} = 0$. For the case $i = j > 2$, we have

$$h_{i,i} = r_{i,i} r'_{i,i} = (-bxW_0 + W_1) \cdot \left(\frac{1}{-bxW_0 + W_1}\right) = 1.$$

In the case $j > 1$, we obtain

$$\begin{aligned} h_{1,j} &= r_{1,1} r'_{1,j} + r_{1,2} r'_{2,j} \\ &= -W_1 \cdot \left(\frac{b}{a}\right)^{\frac{j-3}{2}} \frac{b\,(-W_0)^{j-1}}{(-bxW_0+W_1)^{j-2}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\ &\quad - \frac{\sqrt{b}W_0}{\sqrt{a}} \cdot \left(\frac{b}{a}\right)^{\frac{j-4}{2}} \frac{b\,(-W_0)^{j-2}\mathrm{W}_1}{(-bxW_0+W_1)^{j-2}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\ &= 0. \end{aligned}$$

For $j > 2$, we get

$$\begin{aligned} h_{2,j} &= \sum_{l=1}^{n} r_{2,l} r'_{l,j} = r_{2,2} r'_{2,j} + r_{2,3} r'_{3,j} \\ &= -\frac{(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{aW_1} \cdot \left(\frac{b}{a}\right)^{\frac{j-4}{2}} \frac{b\,(-W_0)^{j-2} W_1}{(-bxW_0+W_1)^{j-2}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\ &\quad + \frac{\sqrt{b}W_0}{\sqrt{a}} \cdot \left(\frac{b}{a}\right)^{\frac{j-3}{2}} \frac{(-W_0)^{j-3}}{(-bxW_0+W_1)^{j-2}}. \\ &= 0. \end{aligned}$$

For the last case $n \ge j > i > 2$, we have

$$\begin{aligned} h_{i,j} &= r_{i,i} r'_{i,j} + r_{i,i+1} r'_{i+1,j} \\ &= (-bxW_0 + W_1) \cdot \left(\frac{b}{a}\right)^{\frac{j-i}{2}} \frac{(-W_0)^{j-i}}{(-bxW_0+W_1)^{j-i+1}} \\ &\quad + \frac{\sqrt{b}W_0}{\sqrt{a}} \cdot \left(\frac{b}{a}\right)^{\frac{j-i-1}{2}} \frac{(-W_0)^{j-i-1}}{(-bxW_0+W_1)^{j-i}} \\ &= 0. \end{aligned}$$

Therefore, we can obtain $\mathbb{R}_n[x,y]\mathbb{R}_n^{-1}[x,y] = \mathbb{I}_n$, where $\mathbb{I}_n$ is an $n \times n$ identity matrix. Similarly, we can verify $\mathbb{R}_n^{-1}[x,y]\mathbb{R}_n[x,y] = \mathbb{I}_n$. Hence, the proof is completed.

Theorem 2. 2. For $-bxW_0 + W_1 \neq 0$ and $(bcyW_0^2 + abxW_0W_1 - aW_1^2) \neq 0$**,** the inverse of the matrix $\mathbb{W}_n[x,y]$, $\mathbb{W}_n^{-1}[x,y] = [w'_{i,j}]$, is equal to

$$\mathbb{W}_n^{-1}[x,y] = \left[w'_{i,j}\right]$$
$$= \begin{cases} \dfrac{\sqrt{a}\sqrt{b}x}{bxW_0 - W_1}, & i = j+1, j \in \{1, n-1\}, \\ \dfrac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}} (-W_0)^{j-i}\left(W_1^{i-1}\right)}{(W_1 - bW_0x)^j}, & 1 \le i \le 2, i \le j \le n-2, \\ \dfrac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}-1} b(-W_0)^{j-i}W_1^{n+i-j-1}}{(W_1 - bW_0x)^{n-2}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}, & 1 \le i \le 2, j \in \{n-1, n\}, \\ \dfrac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}} (-W_0)^{j-i}\left(W_1^{n-j}\right)}{(W_1 - bW_0x)^{n-i+1}}, & 3 \le i \le j, j \in \{n-1, n\}, \\ \dfrac{cy}{bxW_0 - W_1}, & i = j+2, 1 \le j \le n-2, \\ \dfrac{\sqrt{b}\left((abx^2 + cy)W_0 - axW_1\right)}{\sqrt{a}(-bxW_0 + W_1)^2}, & i = j+1, 2 \le j \le n-2, \\ \dfrac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}} (-W_0)^{j-i}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{a(W_1 - bW_0x)^{j-i+3}}, & 3 \le i \le j \le n-2, \\ 0, & \text{otherwise}. \end{cases}$$

Proof: Since

$$\det\left[\mathbb{W}_n[x,y]\right] = \frac{(-bxW_0 + W_1)^{n-2}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{a} \neq 0$$

we can say that the matrix $\mathbb{W}_n[x,y]$ is invertible. Clearly, Theorem 2.2 is valid for $W_1 = 0$. Now, we consider the case $W_1 \neq 0$. By virtue of the equality $\mathbb{Q}_n[x,y]\mathbb{W}_n[x,y] = \mathbb{R}_n[x,y]$, we can write that $\mathbb{W}_n^{-1}[x,y] = \mathbb{R}_n^{-1}[x,y]\mathbb{Q}_n[x,y]$. Explicitly, we can conclude that the last column elements of the matrix $\mathbb{W}_n^{-1}[x,y]$ are $w'_{i,n} = r'_{i,n}$ $(i = 1, \dots, n)$ and

$$w'_{1,1} = \frac{1}{-bxW_0 + W_1},\ w'_{2,1} = w'_{n,n-1} = \frac{\sqrt{a}\sqrt{b}x}{bxW_0 - W_1}.$$

For the case $1 \le i \le 2$, we have

$$\begin{aligned}
w'_{i,n-1} &= r'_{i,n-1}q_{n-1,n-1} + r'_{i,n}q_{n,n-1} \\
&= -\left(\frac{b}{a}\right)^{\frac{n-i-3}{2}} \frac{b\,(-W_0)^{n-i-1}{W_1}^{i-1}}{(-bxW_0+W_1)^{n-3}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\
&\quad + \left(\sqrt{ab}x\right)\left(\frac{b}{a}\right)^{\frac{n-i-2}{2}} \frac{b\,(-W_0)^{n-i}{W_1}^{i-1}}{(-bxW_0+W_1)^{n-2}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{n-i-3}{2}} b(-W_0)^{n-i-1}W_1^i}{(W_1-bW_0x)^{n-2}(-bcyW_0^2-abxW_0W_1+aW_1^2)}.
\end{aligned}$$

For the case $3 \le i \le n-1$, we can verify that

$$\begin{aligned}
w'_{i,n-1} &= \left(\frac{b}{a}\right)^{\frac{n-i-1}{2}} \frac{(-W_0)^{n-i-1}}{(-bxW_0+W_1)^{n-i}} + \left(-\sqrt{ab}x\right)\left(\frac{b}{a}\right)^{\frac{n-i}{2}} \frac{(-W_0)^{n-i}}{(-bxW_0+W_1)^{n-i+1}} \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{n-i-1}{2}}(-W_0)^{n-i-1}W_1}{(W_1-bW_0x)^{n-i+1}}.
\end{aligned}$$

In the case $1 \le i \le 2, i \le j \le n-2$, we have

$$\begin{aligned}
w'_{i,j} &= \sum_{l=1}^{n} r'_{i,l}q_{l,j} = r'_{i,j}q_{j,j} + r'_{i,j+1}q_{j+1,j} + r'_{i,j+2}q_{j+2,j} \\
&= -\left(\frac{b}{a}\right)^{\frac{j-i-2}{2}} \frac{b\,(-W_0)^{j-i}{W_1}^{i-1}}{(-bxW_0+W_1)^{j-2}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\
&\quad +(\sqrt{ab}x)\left(\frac{b}{a}\right)^{\frac{j-i-1}{2}} \frac{b\,(-W_0)^{j-i+1}{W_1}^{i-1}}{(-bxW_0+W_1)^{j-1}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\
&\quad +(cy)\left(\frac{b}{a}\right)^{\frac{j-i}{2}} \frac{b\,(-W_0)^{j-i+2}{W_1}^{i-1}}{(-bxW_0+W_1)^{j}(bcyW_0^2+abxW_0W_1-aW_1^2)} \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}}(-W_0)^{j-i}(W_1^{i-1})}{(W_1-bW_0x)^j}.
\end{aligned}$$

For the case $i = j+2, 1 \le j \le n-2$, we obtain

$$w'_{j+2,j} = r'_{j+2,j+2}q_{j+2,j} = \frac{-cy}{(-bxW_0+W_1)}.$$

For the case $i = j+1, 2 \le j \le n-2$, we get

$$\begin{aligned} w'_{j+1,j} &= r'_{j+1,j+1}q_{j+1,j} + r'_{j+1,j+2}q_{j+2,j} \\ &= \frac{-\sqrt{ab}x}{-bxW_0 + W_1} + \left(\frac{b}{a}\right)^{\frac{1}{2}} \frac{W_0 cy}{(-bxW_0 + W_1)^2} \\ &= \frac{\sqrt{b}((abx^2 + cy)W_0 - axW_1)}{\sqrt{a}(-bxW_0 + W_1)^2}. \end{aligned}$$

For the last case $3 \le i \le j \le n-2$, we have

$$w'_{i,j} = r'_{i,j}q_{j,j} + r'_{i,j+1}q_{j+1,j} + r'_{i,j+2}q_{j+2,j}$$

$$= \left(\frac{b}{a}\right)^{\frac{j-i}{2}} \frac{(-W_0)^{j-i}}{(-bxW_0 + W_1)^{j-i+1}} + \left(-\sqrt{ab}x\right) \cdot \left(\frac{b}{a}\right)^{\frac{j-i+1}{2}} \frac{(-W_0)^{j-i+1}}{(-bxW_0 + W_1)^{j-i+2}}$$

$$+(-cy) \cdot \left(\frac{b}{a}\right)^{\frac{j-i+2}{2}} \frac{(-W_0)^{j-i+2}}{(-bxW_0 + W_1)^{j-i+3}}$$

$$= \frac{\left(\frac{b}{a}\right)^{\frac{j-i}{2}} (-W_0)^{j-i}(-bcyW_0^2 - abxW_0W_1 + aW_1^2)}{a(W_1 - bW_0x)^{j-i+3}}.$$

It's clear that $w'_{i,j} = 0$ for $i \ge j + 3, 1 \le j \le n - 3$. Therefore, the proof is completed.

Example 2. 1. For $n = 4$, $-bxW_0 + W_1 \neq 0$ and $(bcyW_0^2 + abxW_0W_1 - aW_1^2) \neq 0$**,** the inverse $\mathbb{W}_4^{-1}[x,y] = [w'_{i,j}]$ of the matrix $\mathbb{W}_4[x,y]$ is equal to

$$\left(\begin{array}{cccc} \frac{1}{-bxW_0 + W_1} & -\frac{\sqrt{b}W_0}{\sqrt{a}(-bxW_0 + W_1)^2} & -\frac{bW_0^2W_1}{(-bxW_0 + W_1)^2(bcyW_0^2 + abxW_0W_1 - aW_1^2)} & \frac{\cdots}{\sqrt{a}(-\cdots} \\ \frac{\sqrt{a}\sqrt{b}x}{bxW_0 - W_1} & \frac{W_1}{(-bxW_0 + W_1)^2} & \frac{\sqrt{a}\sqrt{b}W_0W_1^2}{(-bxW_0 + W_1)^2(bcyW_0^2 + abxW_0W_1 - aW_1^2)} & -\frac{\cdots}{(-\cdots} \\ \frac{cy}{bxW_0 - W_1} & \frac{\sqrt{b}\left((abx^2 + cy)W_0 - axW_1\right)}{\sqrt{a}(-bxW_0 + W_1)^2} & \frac{W_1}{(-bxW_0 + W_1)^2} & \\ 0 & \frac{cy}{bxW_0 - W_1} & \frac{\sqrt{a}\sqrt{b}x}{bxW_0 - W_1} & \end{array}\right.$$

Corollary 2. 1. [7] For $n = 5, a = b = A,\ c = B,\ x = y = 1,\ W_0 = a$ and $W_1 = b$ in Theorem 2.2, we obtain the inverse of the lower Hessenberg matrix, $\mathcal{U}_n^{-1^{(a,b,1)}} = \left[u_{i,j}'^{(a,b,1)}\right]$, involving classical Horadam numbers of type 1 as follows:

$$u_{i,j}'^{(a,b,1)} = \begin{cases} -\dfrac{A}{b-aA}, & i = j+1, j \in \{1, n-1\}, \\ \dfrac{(-a)^{j-i} b^{i-1}}{(b-aA)^{j}}, & 1 \le i \le 2, i \le j \le n-2, \\ \dfrac{(-a)^{j-i} b^{n+i-j-1}}{(b-aA)^{-2}(b^2 - abA - a^2B)}, & 1 \le i \le 2, j \in \{n-1, n\}, \\ \dfrac{(-a)^{j-i} b^{n-j}}{(b-aA)^{n-i+1}}, & 3 \le i \le j, j \in \{n-1, n\}, \\ -\dfrac{B}{b-aA}, & i = j+2, 1 \le j \le n-2, \\ \dfrac{aA^2 - bA + aB}{(b-aA)^2}, & i = j+1, 2 \le j \le n-2, \\ \dfrac{(-a)^{j-i}(b^{\,2} - abA - a^2B)}{(b-aA)^{j-i+3}}, & 3 \le i \le j \le n-2, \\ 0, & \text{otherwise.} \end{cases}$$

3. CONCLUSION

Investigation of the determinants and inverses of lower Hessenberg matrices whose elements are conditional polynomial sequences has enriched our understanding of these special structured matrices. In this study, we give some general properties of lower Hessenberg matrices, which are formed by polynomial sequences whose elements are conditionally defined. If we take specific values for different parameters in our results, we reduce them to some earlier works such as [7]. In this context, this study eliminates some important problems for the determinants and inverses of lower Hessenberg matrices.

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