ORIGINAL PAPER

SOME NEW PRESERVATION THEOREMS IN AN IDEAL NANOTOPOLOGICAL SPACE

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Abstract. *In this paper, the concepts of ideal nanotopological spaces called $n\star$-Hausdorff, $n\mathcal{J}_{g^{\#}}$-Hausdorff and $n\mathcal{J}_{g}$-Hausdorff spaces are introduced and the preservation theorems concerning $n\mathcal{J}_{g^{\#}}$-Hausdroff and $n\mathcal{J}_{g}$-Hausdorff spaces are investigated.*

Keywords: *$n\star$-Hausdorff; $n\mathcal{J}_{g^{\#}}$-Hausdorff; $n\mathcal{J}_{g}$-Hausdorff spaces.*

1. INTRODUCTION

Lellis Thivagar et al., [1] defined and studied by the concept of nanotopological spaces.Parimala et al., [2] introduced the notion of nI-open sets and several properties of such sets in ideal nanotopological spaces. Several kinds of nI-openness has been initiated.

An ideal I [3] on a topological space (X,τ) is a non-empty collection of subsets of X which satisfies the following conditions.

1.$A \in I$ and $B \subset A$ imply $B \in I$ and

2.$A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a topological space (X,τ) with an ideal I on X. If $\wp(X)$ is the family of all subsets of X, a set operator $(.)^{\star}:\wp(X)\to\wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A\subset X$, $A^{\star}(I,\tau)=\{x\in X: U\cap A\notin I$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau: x\in U\}$ [4]. The closure operator defined by $cl^{\star}(A)=A\cup A^{\star}(I,\tau)$ [5] is a Kuratowski closure operator which generates a topology $\tau^{\star}(I,\tau)$ called the $\star$-topology finer than τ. The topological space together with an ideal on X is called an ideal topological space or an ideal space denoted by (X,τ,I). We will simply write $A^{\star}$ for $A^{\star}(I,\tau)$ and $\tau^{\star}$ for $\tau^{\star}(I,\tau)$. The purpose of this paper are to introduce new concepts of Hausdorff spaces called $n\star$-Hausdorff, $nI_{g^{\#}}$-Hausdorff and nI_g-Hausdorff spaces and to investigate preservation theorems concerning $nI_{g^{\#}}$-Hausdorff and nI_g-Hausdorff spaces in an ideal nanotopological space.

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2. PRELIMINARIES

Definition 2.1. [6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.
1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U} \{R(x): R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x): R(x) \cap X \neq \phi\}$.
3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [1] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:
1.U and $\phi \in \tau_R(X)$,
2.The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$,
3.The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nanotopology with respect to X and $(U, \tau_R(X))$ is called the nanotopological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed. A nanotopological space $(U, \mathcal{N})$ with an ideal I on U is called [7] an ideal nanotopological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n | x \in G_n, G_n \in \mathcal{N}\}$, denotes [7] the family of nano open sets containing x.

In the future an ideal nanotopological space $(U, \mathcal{N}, I)$ will be simply called a space.

Definition 2.3. A subset H of a space $(U, \mathcal{N})$ is called
1.nano α-open set [1] if $H \subseteq nint(ncl(nint(H)))$.
2.nano g-closed set [8] if $ncl(H) \subseteq G$, whenever $H \subseteq G$ and G is nano open.
3.nano $g\alpha$-closed set [9] if $n\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano α-open.
4.nano αg-closed set [9] if n-$\alpha cl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano open.
5.nano $g^{\#}$-closed set [11] if $ncl(H) \subseteq G$ whenever $H \subseteq G$ and G is nano αg-open.

Definition 2.4. A subset H of a space $(U, \mathcal{N}, I)$ is called
1.nano $\mathcal{I}_g$-closed [10] if $H_n^\star \subseteq G$ whenever $H \subseteq G$ and G is nano open.
2.nano $\mathcal{I}_{g^\#}$-closed [11] if $H_n^\star \subseteq G$ whenever $H \subseteq G$ and G is nano αg-open.
3.nano $\star$-closed [10] if $H_n^\star \subseteq H$.

3. NANO $I_G^{\#}$ -CONTINUITY AND NANO I_G-CONTINUITY

Definition 3.1. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is said to be
1. nano $\mathcal{I}_g$-continuous (briefly $n\mathcal{I}_g$-continuous) if $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U for every nano

closed set M of V;
2. nano $\mathcal{I}_{g^\#}$-continuous (briefly $n\mathcal{I}_{g^\#}$-continuous) if $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U for every nano closed set M of V.

Definition 3.2. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is said to be
1. nano $\mathcal{I}_g$-$\star$-continuous (briefly $n\mathcal{I}_g$-$\star$-continuous) if $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U for every nano $\star$-closed set M of V;
2. nano $\mathcal{I}_{g^\#}$-$\star$-continuous (briefly $n\mathcal{I}_{g^\#}$-$\star$-continuous) if $f^{-1}(M)$ is $\mathcal{I}_{g^\#}$-closed in U for every nano $\star$-closed set M of V.

Theorem 3.3. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is $n\mathcal{I}_g$-continuous if and only if $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U for every nano closed set M of V.

Proof: Suppose that f is $n\mathcal{I}_g$-continuous function. Let M be any nano closed set of V. By assumption of f, $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U. Conversely, suppose that the condition holds. Let M be any nano closed set of V. By the given condition, $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U. Hence f is $n\mathcal{I}_g$-continuous function.

Theorem 3.4. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is $n\mathcal{I}_{g^\#}$-continuous if and only if $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U for every nano closed set M of V.

Proof: Suppose that f is $n\mathcal{I}_{g^\#}$-continuous function. Let M be any nano closed set of V. By assumption of f, $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U. Conversely, suppose that the condition holds. Let M be any nano closed set of V. By the given condition, $f^{-1}(M)$ is $\mathcal{I}_{g^\#}$-closed in U. Hence f is $n\mathcal{I}_{g^\#}$-continuous function.

Definition 3.5. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is said to be
1. nano $\mathcal{I}_g$-irresolute (briefly $n\mathcal{I}_g$-irresolute) if $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U for every $n\mathcal{I}_g$-closed set M of V;
2. nano $\mathcal{I}_{g^\#}$-irresolute (briefly $n\mathcal{I}_{g^\#}$-irresolute) if $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U for every $n\mathcal{I}_{g^\#}$-closed set M of V.

Theorem 3.6. A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is $n\mathcal{I}_g$-irresolute if and only if $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U for every $n\mathcal{I}_g$-closed set M of V.

Proof: Suppose that f is $n\mathcal{I}_g$-irresolute function. Let M be any $n\mathcal{I}_g$-closed set of V. By assumption of f, $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U.Conversely, suppose that the condition holds. Let M be any $n\mathcal{I}_g$-closed set of V. By the given condition, $f^{-1}(M)$ is $n\mathcal{I}_g$-closed in U. Hence f is $n\mathcal{I}_g$-irresolute function.

Theorem 3.7 A function $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is $n\mathcal{I}_{g^\#}$-irresolute if and only if $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U for every $n\mathcal{I}_{g^\#}$-closed set M of V.

Proof: Suppose that f is $n\mathcal{I}_{g^\#}$-irresolute function. Let M be any $n\mathcal{I}_{g^\#}$-closed set of V. By assumption of f, $f^{-1}(M)$ is $n\mathcal{I}_{g^\#}$-closed in U.Conversely, suppose that the condition holds.

Let M be any $n\mathcal{I}_{g^{\#}}$-closed set of V. By the given condition, $f^{-1}(M)$ is $n\mathcal{I}_{g^{\#}}$-closed in U. Hence f is $n\mathcal{I}_{g^{\#}}$-irresolute function.

4.NEW TYPES OF HAUSDORFF SPACES IN IDEAL NANOTOPOLOGICAL SPACE

In this section, we define new concepts of Hausdorff spaces called $n\star$-Hausdorff, $n\mathcal{I}_{g^{\#}}$-Hausdorff and $n\mathcal{I}_{g}$-Hausdorff spaces and investigate preservation theorems concerning $n\mathcal{I}_{g^{\#}}$-Hausdorff and $n\mathcal{I}_{g}$-Hausdorff spaces.

Definition 4.1. A nanotopological space $(U, \tau_R(X))$ is called nano Haudorff if for every two different points u and v of U, there exist disjoint nano open sets E and F of U such that $u \in E$ and $v \in F$.

Definition 4.2. An ideal nanotopological space $(U, \tau_R(X), \mathcal{I})$ is called $n\star$-Haudorff if for every two different points u and v of U, there exist disjoint $n\star$-open sets E and F of U such that $u \in E$ and $v \in F$.

Definition 4.3. An ideal nanotopological space $(U, \tau_R(X), \mathcal{I})$ is called $n\mathcal{I}_{g^{\#}}$-Haudorff if for every two different points u and v of U, there exist disjoint $n\mathcal{I}_{g^{\#}}$-open sets E and F of U such that $u \in E$ and $v \in F$.It is obvious that every $n\star$-Hausdorff space is $n\mathcal{I}_{g^{\#}}$-Hausdorff space.The following Example shows that the converse is not true.

Example 4.4. Let $U = \{a_1, a_2, a_3, a_4\}$ with $U/R = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}\}$ and $X = \{a_1, a_4\}$. Then the nanotopology $\mathcal{N} = \{\phi, \{a_1\}, \{a_4\}, \{a_1, a_4\}, U\}$ and $\mathcal{I} = \{\phi, \{a_4\}\}$. Then $(U, \tau_R(X), \mathcal{I})$ is not $n\star$-Hausdorff space but it is $n\mathcal{I}_{g^{\#}}$-Hausdorff space.

Theorem 4.5. Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nanotopological space and $(V, \tau'_R(Y))$ be nano Hausdorff. If $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is injective and $n\mathcal{I}_{g^{\#}}$-continuous, then $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_{g^{\#}}$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is nano Hausdorff, there exist disjoint nano open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{I}_{g^{\#}}$-continuous and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{I}_{g^{\#}}$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is an $n\mathcal{I}_{g^{\#}}$-Hausdorff.

Theorem 4.6. Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nanotopological space and $(V, \tau'_R(Y))$ be $n\star$-Hausdorff. If $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is injective and $n\mathcal{I}_{g^{\#}}$-$\star$-continuous, then $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_{g^{\#}}$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is $n\star$-Hausdorff, there exist disjoint $n\star$-open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{I}_{g^{\#}}$-$\star$-continuous and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{J}_{g^{\#}}$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is an $n\mathcal{J}_{g^{\#}}$-Hausdorff.

Definition 4.7. An ideal nanotopological space $(U, \tau_R(X), \mathcal{J})$ is called $n\mathcal{J}_g$-Haudorff if for every two different points u and v of U, there exist disjoint $n\mathcal{J}_g$-open sets E and F of U such that $u \in E$ and $v \in F$. It is obvious that every $n\mathcal{J}_{g^{\#}}$-Hausdorff space is $n\mathcal{J}_g$-Hausdorff space. The following Example shows that the converse is not true.

Example 4.8. Let $U = \{a_1, a_2, a_3\}$ with $U/R = \{\{a_2\}, \{a_1, a_3\}\}$ and $X = \{a_1, a_2\}$. Then the nanotopology $\mathcal{N} = \{\phi, \{a_2\}, \{a_1, a_3\}, U\}$ and $\mathcal{J} = \{\phi, \{a_3\}\}$. Then $(U, \tau_R(X), \mathcal{J})$ is not $n\mathcal{J}_{g^{\#}}$-Hausdorff but it is $n\mathcal{J}_g$-Hausdorff space.

Theorem 4.9. Let $(U, \tau_R(X), \mathcal{J})$ be an ideal nanotopological space and $(V, \tau'_R(Y))$ be nano Hausdorff. If $f: (U, \tau_R(X), \mathcal{J}) \to (V, \tau'_R(Y))$ is injective and $n\mathcal{J}_g$-continuous, then $(U, \tau_R(X), \mathcal{J})$ is $n\mathcal{J}_g$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is nano Hausdorff, there exist disjoint nano open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{J}_g$-continuous and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{J}_g$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is an $n\mathcal{J}_g$-Hausdorff.

Theorem 4.10. Let $(U, \tau_R(X), \mathcal{J})$ be an ideal nanotopological space and $(V, \tau'_R(Y)))$ be $n\,\star$-Hausdorff. If $f: (U, \tau_R(X), \mathcal{J}) \to (Y, \tau'_R(Y))$ is injective and $n\mathcal{J}_g$-$\star$-continuous, then $(U, \tau_R(X), \mathcal{J})$ is $n\mathcal{J}_g$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is $n\,\star$-Hausdorff, there exist disjoint $n\,\star$-open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{J}_g$-$\star$-continuous and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{J}_g$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is an $n\mathcal{J}_g$-Hausdorff.

Remark 4.11. We have the following implications for the properties of spaces. nano Hausdorff $\to n\,\star$-Hausdorff $\to n\mathcal{J}_{g^{\#}}$-Hausdorff $\to n\mathcal{J}_g$-Hausdorff None of the above implications is reversible.

5. PRESERVATION THEOREMS

Theorem 5.1. Let $(U, \tau_R(X), \mathcal{J})$ be an ideal nanotopological space and $(V, \tau'_R(Y))$ be $n\mathcal{J}_{g^{\#}}$-Hausdorff. If $f: (U, \tau_R(X), \mathcal{J}) \to (V, \tau'_R(Y))$ is injective and $n\mathcal{J}_{g^{\#}}$-irresolute, then $(U, \tau_R(X), \mathcal{J})$ is $n\mathcal{J}_{g^{\#}}$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is $n\mathcal{J}_{g^{\#}}$-Hausdorff, there exist disjoint $n\mathcal{J}_{g^{\#}}$-open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{J}_{g^{\#}}$-irresolute and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{I}_{g^\#}$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is an $n\mathcal{I}_{g^\#}$-Hausdorff.

Theorem 5.2. Let $(U, \tau_R(X), \mathcal{I})$ be an ideal nanotopological space and $(V, \tau'_R(Y))$ be $n\mathcal{I}_g$-Hausdorff. If $f: (U, \tau_R(X), \mathcal{I}) \to (V, \tau'_R(Y))$ is injective and $n\mathcal{I}_g$-irresolute, then $(U, \tau_R(X), \mathcal{I})$ is $n\mathcal{I}_g$-Hausdorff.

Proof: Let u and v be any two different points of U. Then $f(u)$ and $f(v)$ are different points of V because f is injective. Since V is $n\mathcal{I}_g$-Hausdorff, there exist disjoint $n\mathcal{I}_g$-open sets E and F in V containing $f(u)$ and $f(v)$ respectively. Since f is $n\mathcal{I}_g$-irresolute and $E \cap F = \phi$, we have $f^{-1}(E)$ and $f^{-1}(F)$ are disjoint $n\mathcal{I}_g$-open sets in U such that $u \in f^{-1}(E)$ and $v \in f^{-1}(F)$. Hence U is a $n\mathcal{I}_g$-Hausdorff.

6. CONCLUSION

In this paper, new notions of ideal nanotopological spaces namelyn $\star$-Hausdorff, n$\mathcal{I}_{\mathrm{g}^\#}$-Hausdorff and n$\mathcal{I}_{\mathrm{g}}$-Hausdorff spaces are introduced and studied the preservation theorems concerning n$\mathcal{I}_{\mathrm{g}^\#}$-Hausdroff and n$\mathcal{I}_{\mathrm{g}}$-Hausdorff spaces are obtained.

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