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SPINOR REPRESENTATIONS OF PAFORS IN E^3

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Abstract. *In this paper, we introduce the spinor representations of PAFORS for the trajectories endowed with PAFORS on regular surfaces of Euclidean* 3*-space* $\mathbb{E}^3$. *We find the spinor equations of PAFORS vectors. Moreover, we obtain the relations between spinor representations of PAFORS and Darboux frame. Then, we give some geometric interpretations and results concerned with this relationship.*

Keywords: *Kinematics of a particle; positional adapted frame on regular surfaces; spinors.*

1. INTRODUCTION

For many years, to study on the surface theory in different dimensions and spaces has been an issue of interest for researchers. The contribution of Jean Gaston Darboux to this theory is very big [1]. The concept of moving frames was adapted to the curves on surfaces by Darboux. He introduced a moving frame for surface curves which is well known today as Darboux frame by everyone. Darboux frame can be defined everywhere for the curves lying on a regular surface since regular surfaces do not have any umbilic points [2, 3]. Many researchers have used the Darboux frame as a useful tool to study many different and interesting topics in the theory of surfaces up to the present. One can find some of these interesting works in [4-8].

Recently, by making use of Darboux frame, Özen and Tosun have presented a new moving frame on regular surfaces in Euclidean 3-space for the trajectories having non-vanishing angular momentum [9] by inspring the study [10]. They have called this frame PAFORS shortly. The main purpose of this frame is to enable the researchers to study kinematics and inverse kinematics more convenient observation environment in the future. Also, in the study [11], some characterizations on asymptotic, geodesic, and slant helical trajectories are given by Özen and Tosun according to PAFORS. Then, PAFORS is extended to the 3-dimensional Minkowski space by Gürbüz in the study [12]. This study discusses the evolution of an electric field concerning PAFORS in Minkowski 3-space.

Another topic of this study is spinors. Spinors are physical concepts having an important role in various areas such as quantum mechanics, the theory of relativity, and physics. The term “spinor” is used first by Paul Ehrenfest in quantum physics in the 1920s [13]. On the other hand, the first mathematician who tackled the spinors in a geometrical sense was the famous French mathematician Élie Cartan [14]. When Cartan was examining the representation of groups, he found the mathematical forms of spinors in 1913. By Cartan, it was specified that spinors satisfy a linear representation of the groups of rotations of a space

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of any dimension. Because of that spinors are directly concerned with geometry in addition to their relationship with physics [15]. In light of the study [14], we know that the set of isotropic vectors of the vector space $\mathbb{C}^3$ constructs a two-dimensional surface in the space $\mathbb{C}^2$. On the contrary, these vectors in $\mathbb{C}^2$ represent the same isotropic vectors. Cartan expressed that these vectors are complex as two-dimensional in space $\mathbb{C}^2$ [14, 16]. On the other hand, the triads of unit vectors which are orthogonal by twos were expressed in terms of a single vector that has two complex components, which is called a spinor [14, 17, 18]. On the other hand, Pauli matrices which explain the electron spin in quantum theory were introduced by W. Pauli in [19]. By Pauli, it was stated that the wave function of an electron can be represented by a vector with two complex components in 1927, and also this vector is called a spinor [15].

Another thing that can be of importance is the spinor equations of the moving frames. The paper [18] performed by del Castillo and Barrales was a milestone for many researchers. In this paper, the spinor equations of the Serret-Frenet frame were introduced by these authors. After this paper, many studies similar to it were performed by many authors on this topic. For example, spinor representations of Darboux frame, Sabban frame, and Bishop frame were investigated in the studies [20-22], respectively. Then, the spinor equations of involute evolute curves [23], Bertrand curves [16], and successor curves [24] were investigated. Moreover, spinor representations of framed Bertrand and framed Mannheim curves were determined in [25] and [26], respectively. Then, spinor representations of framed curves in the 3-dimensional Lie groups were determined [27]. Also, spinor representations of positional adapted frame were introduced in [28]. Additionally, the hyperbolic spinors were examined and combined with the different frames [29-31].

This article is organized as follows. In Section 2, we present the required information concerning the spinors and PAFORS to be understood in the following sections. In Section 3, the spinor representations of PAFORS in the Euclidean 3-space are given. Also, the relations between spinor representations of PAFORS and the Darboux frame are obtained. In Section 4, a numerical example which comprises an illustrative figure is provided. Then, we give conclusions in Section 5.

2. PRELIMINARIES

In this section, we review some necessary information concerning PAFORS and spinors.

2.1. PAFORS

Let Euclidean 3-space E^3 be considered with the standard inner product which is defined as $\langle \mathcal{M}, \mathcal{N} \rangle = m_1 n_1 + m_2 n_2 + m_3 n_3$ where $\mathcal{M} = (m_1, m_2, m_3)$, $\mathcal{N} = (n_1, n_2, n_3)$ are arbitrary vectors of E^3. The norm of the vector $\mathcal{M}$ is given by $\|\mathcal{M}\| = \sqrt{\langle \mathcal{M}, \mathcal{M} \rangle}$. Based on this definition, a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ is called a unit speed curve if the condition $\|d\alpha/ds\| = 1$ is satisfied for every $s \in I$. If the derivative of a differentiable curve never equals zero, this curve is said to be a regular curve. These kinds of curves have always a unit speed parameterization [32]. During the study, we will denote the differentiation concerning the parameter of the curve by a prime.

Suppose that a point particle moves on the unit speed trajectory $\alpha : I \subset R \to M \subset E^3$ lying on a regular surface M. As it is well known, a regular surface curve has well defined Darboux frame denoted by $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$. So, the Darboux frame $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$ is exist along $\alpha = \alpha(s)$ where $\mathbf{U}(s)$ is the unit normal vector of M restricted to α and $\mathbf{T}(s)$ is the unit tangent vector of α. Moreover, since $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$ is an orthonormal vector system, the remaining basis vector $\mathbf{Y}(s)$ is calculated as $\mathbf{Y}(s) = \mathbf{U}(s) \times \mathbf{T}(s)$.

On the other hand, the derivative formulas of the Darboux frame are given in the following equation:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{Y}'(s) \\ \mathbf{U}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix}$$

here, $\tau_g(s)$, $k_g(s)$, and $k_n(s)$ indicate the geodesic torsion, geodesic curvature, and normal curvature of the curve α, respectively [2, 4].

Let the angular momentum vector of the aforesaid particle about the origin not be equal to zero during the motion of this particle. In such a case, PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is well-defined for the motion of this particle. The first basis vector $\mathbf{T}(s)$ is a unit tangent vector and it is common with the Darboux frame, while the other ones are calculated as follows [9]:

$$\begin{cases} \mathbf{H}(s) = \dfrac{\langle -\alpha(s), \mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \mathbf{Y}(s) + \dfrac{\langle \alpha(s), \mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \mathbf{U}(s), \\ \mathbf{G}(s) = \dfrac{\langle \alpha(s), \mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \mathbf{Y}(s) + \dfrac{\langle \alpha(s), \mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \mathbf{U}(s). \end{cases}$$

There is a relation between PAFORS and Darboux frame as in the following:

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega(s) & -\sin\Omega(s) \\ 0 & \sin\Omega(s) & \cos\Omega(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix}, \tag{1}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{Y}(s)$ and $\mathbf{G}(s)$ which is positively oriented from $\mathbf{Y}(s)$ to $\mathbf{G}(s)$. For PAFORS, the derivative formulas are given by

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{G}'(s) \\ \mathbf{H}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix}, \tag{2}$$

where

$$\begin{cases} k_1(s) = k_g(s)\cos\Omega(s) - k_n(s)\sin\Omega(s), \\ k_2(s) = k_g(s)\sin\Omega(s) + k_n(s)\cos\Omega(s), \\ k_3(s) = \tau_g(s) - \Omega'(s). \end{cases} \tag{3}$$

The rotation angle $\Omega(s)$ is specified as

$$\Omega(s)=\begin{cases}\arctan\left(-\dfrac{\langle \alpha(s),\mathbf{Y}(s)\rangle}{\langle \alpha(s),\mathbf{U}(s)\rangle}\right), & if\ \langle \alpha(s),\mathbf{U}(s)\rangle>0,\\[2ex] \arctan\left(-\dfrac{\langle \alpha(s),\mathbf{Y}(s)\rangle}{\langle \alpha(s),\mathbf{U}(s)\rangle}\right)+\pi, & if\ \langle \alpha(s),\mathbf{U}(s)\rangle<0,\\[2ex] -\dfrac{\pi}{2}\ if\ \langle \alpha(s),\mathbf{U}(s)\rangle=0,\ \langle \alpha(s),\mathbf{Y}(s)\rangle>0, & \\[2ex] \dfrac{\pi}{2}\ if\ \langle \alpha(s),\mathbf{U}(s)\rangle=0,\ \langle \alpha(s),\mathbf{Y}(s)\rangle<0. & \end{cases}$$

Additionally, any element of the set $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s),k_1(s),k_2(s),k_3(s)\}$ is said to be the PAFORS apparatus of the trajectory $\alpha=\alpha(s)$ [9].

2.2. SPINORS

A spinor is expressed as two-dimensional complex vectors and represented as:

$$\mu=\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$$

using the vectors $u,v,w\in\mathbb{R}^3$ such that

$$\begin{cases}u+iv=\mu^t\sigma\mu,\\ w=-\hat{\mu}^t\sigma\mu.\end{cases}\tag{4}$$

Here "t" indicates the transposition, $\hat{\mu}$ indicates the conjugation [14] (mate [17]) of μ, $\bar{\mu}$ indicates the complex conjugation of μ. In addition to these, we must emphasize that $u+iv$ is an isotropic vector and w is a real vector. The following can be presented [18]:

$$\hat{\mu}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\bar{\mu}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\bar{\mu}_1\\ \bar{\mu}_2\end{pmatrix}=\begin{pmatrix}-\bar{\mu}_2\\ \bar{\mu}_1\end{pmatrix}.$$

Also, the following 2×2 matrices which are Cartesian components for the vector $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ can be given as follows [18]:

$$\sigma_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.\tag{5}$$

Let $\rho = (\rho_1, \rho_2, \rho_3) \in \mathbb{C}^3$ be an isotropic vector (that is $\langle \rho, \rho \rangle = 0$) where $\mathbb{C}^3$ denotes the three-dimensional complex vector space. The set of these kinds of vectors of $\mathbb{C}^3$ generates a two-dimensional surface in $\mathbb{C}^2$. If the surface is parameterized through μ_1 and μ_2, in that case

$$\rho_1 = \mu_1^2 - \mu_2^2, \quad \rho_2 = i(\mu_1^2 + \mu_2^2), \quad \rho_3 = -2\mu_1\mu_2,$$

and

$$\mu_1 = \pm\sqrt{\frac{\rho_1 - i\rho_2}{2}}, \quad \mu_2 = \pm\sqrt{\frac{-\rho_1 - i\rho_2}{2}} \tag{6}$$

can be written [14]. Due to the equations (4) and (5), the followings can be given

$$\rho_1 = \mu^t \sigma_1 \mu = \mu_1^2 - \mu_2^2, \quad \rho_2 = \mu^t \sigma_2 \mu = i(\mu_1^2 + \mu_2^2), \quad \rho_3 = \mu^t \sigma_3 \mu = -2\mu_1\mu_2,$$

and

$$\begin{cases} u + iv = \left(\mu_1^2 - \mu_2^2, i(\mu_1^2 + \mu_2^2), -2\mu_1\mu_2\right), \\ w = \left(\mu_1\bar{\mu}_2 + \bar{\mu}_1\mu_2, i(\mu_1\bar{\mu}_2 - \bar{\mu}_1\mu_2), |\mu_1|^2 - |\mu_2|^2\right). \end{cases} \tag{7}$$

Since $u + iv \in \mathbb{C}^3$ is an isotropic vector, u, v, w are mutually orthogonal, and in this case $|u| = |v| = |w| = \bar{\mu}^t \mu$ and $\langle u \wedge v, w \rangle = \det(u, v, w) > 0$. On the other hand, if u, v, w are mutually orthogonal vectors of the same magnitude in $\mathbb{R}^3 \left(\det(u, v, w) > 0\right)$, in that case, there is a spinor which is defined as indicated in equation (4) [14, 17, 18].

Let μ and ϕ be two spinors. In this case, the equations

$$\mu^t \sigma \phi = \phi^t \sigma \mu, \tag{8}$$

$$\overline{\mu^t \sigma \phi} = -\hat{\mu}^t \sigma \hat{\phi}, \tag{9}$$

$$\left(\varrho_1 \mu + \varrho_2 \phi\right) = \bar{\varrho}_1 \hat{\mu} + \bar{\varrho}_2 \hat{\phi}, \tag{10}$$

$$\hat{\hat{\mu}} = -\mu, \tag{11}$$

hold where $\varrho_1, \varrho_2 \in \mathbb{C}$ [18]. Another thing that can be of importance is that equation (8) is satisfied, since the matrices $\sigma_1, \sigma_2, \sigma_3$ in the equation (5) are symmetric. Also, since the spinors μ and $-\mu$ correspond to the same ordered orthogonal basis $\{u, v, w\}$ with $|u| = |v| = |w|$ and $\det(u, v, w) > 0$, the correspondence between the spinors and orthogonal bases given in equation (4) is two-to-one. We must emphasize that the ordered triads $\{u, v, w\}, \{v, w, u\}, \{w, u, v\}$ correspond to different spinors. Moreover, if $\mu \neq 0$, in this case, the set $\{\mu, \bar{\mu}\}$ is linearly independent [14, 17, 18].

Finally, we must emphasize the study [20] which has an important place for the present paper. In [20], Kişi and Tosun investigated the spinor representations of the Darboux frame and obtained a spinor ϕ which satisfies the following:

$$\mathbf{Y} + i\,\mathbf{U} = \phi^t \sigma \phi, \tag{12}$$

$$\mathbf{T}=-\hat{\phi}^t\sigma\phi, \tag{13}$$

where $\overline{\phi}^t\phi=1$ and the spinor ϕ corresponds to the triad $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$. Then, the derivative formulas of Darboux frame are equivalent to the following single spinor equation [20]:

$$\frac{d\phi}{ds}=\frac{1}{2}\left[-i\tau_g\phi+\left(k_g+ik_n\right)\hat{\phi}\right]. \tag{14}$$

For more details on the concept of spinors, the readers are referred to the works [13, 14, 17, 18, 33-35].

3. THE SPINOR REPRESENTATIONS OF PAFORS

In this section, we introduce the spinor formulas of each PAFORS vectors $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ of a trajectory α which is a unit-speed curve, separately. Moreover, we give the relations between the spinor representations of PAFORS and the Darboux frame. Also, we obtain some geometric properties and results concerning them. Afterward, a numerical example comprising an illustrative figure is constructed related to the spinor representations of PAFORS.

Definition 3.1. Let P be the moving point particle of constant mass that moves on the regular surface M and along the unit speed trajectory $\alpha=\alpha(s)$. Let the PAFORS triad $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ of this trajectory be given and the spinor μ corresponds to the triad. Then, the following spinor representations of PAFORS vectors can be given:

$$\mathbf{G}+i\,\mathbf{H}=\mu^t\sigma\,\mu, \tag{15}$$

$$\mathbf{T}=-\hat{\mu}^t\sigma\,\mu, \tag{16}$$

where $\overline{\mu}^t\mu=1$.

Theorem 3.1. Let α be the unit speed trajectory equipped with PAFORS which lies on the regular surface M and the spinor μ corresponds to the triad $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$. The PAFORS equations are equivalent to the following single spinor equation:

$$\frac{d\mu}{ds}=\frac{1}{2}\left[-ik_3\mu+\left(k_1+ik_2\right)\hat{\mu}\right]. \tag{17}$$

Proof: If we differentiate the equation (15) with respect to the parameter s, then we have:

$$\frac{d\mathbf{G}}{ds}+i\frac{d\mathbf{H}}{ds}=\left(\frac{d\mu}{ds}\right)^t\sigma\mu+\mu^t\sigma\frac{d\mu}{ds}. \tag{18}$$

Since $\{\mu,\hat{\mu}\}$ is a basis for spinors, we obtain:

$$\frac{d\mu}{ds}=\zeta\mu+\gamma\hat{\mu}, \tag{19}$$

where ζ and γ are complex-valued functions. With the help of the equations (2), (8), (17) and (18), we have:

$$-k_1\mathbf{T}+k_3\mathbf{Y}+i\left(-k_2\mathbf{T}-k_3\mathbf{M}\right)=2\zeta\mu^t\sigma\mu+2\gamma\hat{\mu}^t\sigma\mu.$$

With the aid of the equations (15) and (16), we can write:

$$-ik_3\left(\mathbf{M}+i\mathbf{Y}\right)-\left(k_1+ik_2\right)\mathbf{T}\quad=2\zeta\left(\mathbf{M}+i\mathbf{Y}\right)-2\gamma\mathbf{T}.$$

Therefore, we have $\zeta=-ik_3/2$ and $\gamma=(k_1+ik_2)/2$. If the functions ζ and γ are substituted in the equation (19), we find the equation (17).

Theorem 3.2. Let α be the unit speed trajectory equipped with PAFORS which lies on the regular surface M and the spinor μ corresponds to the triad $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$. Then, the spinor equations of the PAFORS vectors are given as follows:

$$\mathbf{T}=-\hat{\mu}^t\sigma\mu, \tag{20}$$

$$\mathbf{G}=\frac{1}{2}\left(\mu^t\sigma\mu-\hat{\mu}^t\sigma\hat{\mu}\right), \tag{21}$$

$$\mathbf{H}=-\frac{i}{2}\left(\mu^t\sigma\mu+\hat{\mu}^t\sigma\hat{\mu}\right). \tag{22}$$

Proof: Let the spinor μ corresponds to the triad $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ of the trajectory α on the regular surface M. With the help of the equation (15), we can derive the equalities $\mathbf{G}=Re(\mu^t\sigma\,\mu)$ and $\mathbf{H}=Im(\mu^t\sigma\,\mu)$. We have already $\mathbf{T}=-\hat{\mu}^t\sigma\,\mu$ from the equation (16). Thus, we can write:

$$\mathbf{G}=\frac{1}{2}\left(\mu^t\sigma\mu+\overline{\mu^t\sigma\mu}\right),$$
$$\mathbf{H}=-\frac{i}{2}\left(\mu^t\sigma\mu-\overline{\mu^t\sigma\mu}\right),$$

where we utilize the properties of complex numbers: $Re(\epsilon)=\frac{1}{2}\left(\epsilon+\overline{\epsilon}\right)$ and $iIm(\epsilon)=\frac{1}{2}\left(\epsilon-\overline{\epsilon}\right)$ for all $\epsilon\in\mathbb{C}$. Then, by using the last two equations and (9), we have:

$$\mathbf{G}=\frac{1}{2}\left(\mu^t\sigma\mu-\hat{\mu}^t\sigma\hat{\mu}\right),$$
$$\mathbf{H}=-\frac{i}{2}\left(\mu^t\sigma\mu+\hat{\mu}^t\sigma\hat{\mu}\right).$$

This finishes the proof.

Corollary 3.1 Let α be the unit speed trajectory equipped with PAFORS which lies on the regular surface M and the spinor μ corresponds to the triad $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$. Then, the spinor components of PAFORS vectors are written as follows:

$$\mathbf{T}=\left(\mu_1\overline{\mu}_2+\overline{\mu}_1\mu_2,i\left(\mu_1\overline{\mu}_2-\overline{\mu}_1\mu_2\right),|\,\mu_1\,|^2-|\,\mu_2\,|^2\right),$$

$$\mathbf{G}=\frac{1}{2}\left(\mu_1^2-\mu_2^2+\bar{\mu}_1^2-\bar{\mu}_2^2, i\left(\mu_1^2+\mu_2^2-\bar{\mu}_1^2-\bar{\mu}_2^2\right), -2\left(\mu_1\mu_2+\bar{\mu}_1\bar{\mu}_2\right)\right),$$
$$\mathbf{H}=-\frac{i}{2}\left(\mu_1^2-\mu_2^2-\bar{\mu}_1^2+\bar{\mu}_2^2, i\left(\mu_1^2+\mu_2^2+\bar{\mu}_1^2+\bar{\mu}_2^2\right), 2\left(\bar{\mu}_1\bar{\mu}_2-\mu_1\mu_2\right)\right).$$

Proof: According to the equation (20)-(22), we have the followings:

- For spinor equation $\mu^t\sigma\,\mu$, we obtain:

$$\begin{cases} \mu^t\sigma_1\mu=\begin{pmatrix}\mu_1 & \mu_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=\mu_1^2-\mu_2^2,\\ \mu^t\sigma_2\mu=\begin{pmatrix}\mu_1 & \mu_2\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=i(\mu_1^2+\mu_2^2),\\ \mu^t\sigma_3\mu=\begin{pmatrix}\mu_1 & \mu_2\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=-2\mu_1\mu_2.\end{cases} \tag{22}$$

- Also, for the spinor equation $\hat{\mu}^t\sigma\,\hat{\mu}$, we have:

$$\begin{cases} \hat{\mu}^t\sigma_1\hat{\mu}=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\bar{\mu}_2\\ \bar{\mu}_1\end{pmatrix}=\bar{\mu}_2^2-\bar{\mu}_1^2,\\ \hat{\mu}^t\sigma_2\hat{\mu}=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}-\bar{\mu}_2\\ \bar{\mu}_1\end{pmatrix}=i\left(\bar{\mu}_2^2+\bar{\mu}_1^2\right),\\ \hat{\mu}^t\sigma_3\hat{\mu}=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\bar{\mu}_2\\ \bar{\mu}_1\end{pmatrix}=2\bar{\mu}_1\,\bar{\mu}_2.\end{cases} \tag{24}$$

- Also, for the spinor equation $\hat{\mu}^t\sigma\,\mu$, we get:

$$\begin{cases} \hat{\mu}^t\sigma_1\mu=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mu_2\\ \mu_1\end{pmatrix}=-\bar{\mu}_1\mu_1-\bar{\mu}_1\mu_2,\\ \hat{\mu}^t\sigma_2\mu=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=i\left(-\bar{\mu}_2\mu_1+\bar{\mu}_1\mu_2\right),\\ \hat{\mu}^t\sigma_3\mu=\begin{pmatrix}-\bar{\mu}_2 & \bar{\mu}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}=-|\bar{\mu}_1|^2+|\bar{\mu}_2|^2.\end{cases} \tag{25}$$

Substituting the equations (23), (24) and (25) in the equations (20), (21), (22), the proof is finished. In the following theorem, we present the spinor relations between the PAFORS and Darboux frame.

Theorem 3.3. Let the spinors μ and ϕ corresponds to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. Then, the following relations are satisfied:

$$\begin{gathered}\mu^t\sigma\,\mu=e^{i\Omega}(\phi^t\sigma\,\phi),\\ \mathbf{T}=\mathbf{T},\end{gathered} \tag{26}$$

where the angle Ω is the Euclidean angle between the vector $\mathbf{U}$ and the vector $\mathbf{H}$.

Proof: With the help of the equation (1), we get:

$$\begin{aligned} \mathbf{G}+i\mathbf{H} &= \cos\Omega\mathbf{Y}-\sin\Omega\mathbf{U}+i\left(\sin\Omega\mathbf{Y}+\cos\Omega\mathbf{U}\right) \\ &= (\mathbf{Y}+i\mathbf{U})(\cos\Omega+i\sin\Omega) \\ &= (\mathbf{Y}+i\mathbf{U})e^{i\Omega}. \end{aligned} \tag{27}$$

According to the equations (1), (12) and (13), we have the desired results.

Theorem 3.4. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. Then, the following relation can be given:

$$\mu=\pm e^{\frac{i\Omega}{2}}\phi. \tag{28}$$

Proof: According to the equation (26) and (27), we can say that the angle between the vectors $\mathbf{Y}+i\mathbf{U}$ and $\mathbf{G}+i\mathbf{H}$ is Ω. By using the equations (26) and (7), we have

$$\begin{aligned} \mu^t\sigma\mu &= \left(\mu_1^2-\mu_2^2, i\left(\mu_1^2+\mu_2^2\right), -2\mu_1\mu_2\right), \\ \phi^t\sigma\phi &= \left(\phi_1^2-\phi_2^2, i\left(\phi_1^2+\phi_2^2\right), -2\phi_1\phi_2\right). \end{aligned}$$

Then, we get $\mu_1^2=e^{i\Omega}\phi_1^2$ and $\mu_2^2=e^{i\Omega}\phi_2^2$. Therefore, the equalities $\mu_1=e^{i\frac{\Omega}{2}}\phi_1$ and $\mu_2=e^{i\frac{\Omega}{2}}\phi_2$ can be given. Because the spinors μ and $-\mu$ correspond to the same ordered orthonormal basis, the spinor μ and $-\mu$ correspond to the vector $\mathbf{G}+i\mathbf{H}$, and the spinor ϕ and $-\phi$ correspond to the vector $\mathbf{Y}+i\mathbf{U}$. Consequently, we have $\mu=\pm e^{\frac{i\Omega}{2}}\phi$.

Corollary 3.2. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. Then the angle between the spinors μ and ϕ is $\Omega/2$.

Theorem 3.5. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. In that case, the following relation is satisfied:

$$\hat{\mu}=\pm e^{-i\frac{\Omega}{2}}\hat{\phi}. \tag{29}$$

Proof: Taking the conjugate of both sides of the equality (28), we get: $\hat{\mu}=\pm e^{i\frac{\Omega}{2}}\phi.$ By (10), we find the equation $\hat{\mu}=\pm e^{-i\frac{\Omega}{2}}\hat{\phi}$.

Corollary 3.3. Let the spinors μ and ϕ represent the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. While μ makes a rotation angle as $\Omega/2$ with ϕ, $\hat{\mu}$ makes the same rotation angle in the opposite direction with $\hat{\phi}$.

Theorem 3.6. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. The derivative of the spinor μ can be written with the help of the curvatures of the Darboux frame k_g, k_n and τ_g as follows:

$$\frac{d\mu}{ds}=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left(k_g+ik_n\right)e^{i\Omega}\hat{\mu}\right]. \tag{30}$$

Proof: By using the equations (3) and (17), we get:

$$\begin{aligned}
\frac{d\mu}{ds}&=\frac{1}{2}\left[-ik_3\mu+\left(k_1+ik_2\right)\hat{\mu}\right]\\
&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left[\left(k_g\cos\Omega-k_n\sin\Omega\right)+\left(k_g\sin\Omega+k_n\cos\Omega\right)\right]\hat{\mu}\right]\\
&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left[k_g\left(\cos\Omega+i\sin\Omega\right)+ik_n\left(\cos\Omega+i\sin\Omega\right)\right]\hat{\mu}\right]\\
&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left(k_ge^{i\Omega}+ik_ne^{i\Omega}\right)\hat{\mu}\right]\\
&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left(k_g+ik_n\right)e^{i\Omega}\hat{\mu}\right].
\end{aligned}$$

Corollary 3.4. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. The angle between $d\mu/ds$ and $\hat{\mu}$ is Ω if $\tau_g=\Omega'$.

Theorem 3.7. Let the spinors μ and ϕ correspond to the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$ related to α, respectively. Then, the following equation yields:

$$\frac{d\mu}{ds}=\pm\frac{1}{2}e^{i\frac{\Omega}{2}}\left[-i\left(\tau_g-\Omega'\right)\phi+\left(k_g+ik_n\right)\hat{\phi}\right].$$

Proof: Using the equations (28), (29) and (30) gives us the desired result:

$$\begin{aligned}
\frac{d\mu}{ds}&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\mu+\left(k_g+ik_n\right)e^{i\Omega}\hat{\mu}\right]\\
&=\frac{1}{2}\left[-i\left(\tau_g-\Omega'\right)\left(\pm e^{i\frac{\Omega}{2}}\phi\right)+\left(k_g+ik_n\right)e^{i\Omega}\left(\pm e^{-i\frac{\Omega}{2}}\hat{\phi}\right)\right]\\
&=\pm\frac{1}{2}e^{i\frac{\Omega}{2}}\left[-i\left(\tau_g-\Omega'\right)\phi+\left(k_g+ik_n\right)\hat{\phi}\right].
\end{aligned}$$

Corollary 3.5 Let the spinors μ and ϕ represent the triads $\{\mathbf{G},\mathbf{H},\mathbf{T}\}$ and $\{\mathbf{Y},\mathbf{U},\mathbf{T}\}$, respectively. The angle between $d\mu/ds$ and $\hat{\phi}$ is $\Omega/2$ on condition that $\Omega'=\tau$.

4. APPLICATION

In this section, we give a numerical example in order to support constructed theorems and results with respect to the spinor representations of PAFORS in Euclidean 3-space.

Example 4.1 Let us consider a right-handed circular helix that lies on the cylinder

$$M = \{(x, y, z) : x^2 + y^2 = R^2,\ z \geq 0\}.$$

Suppose that an electron (with an electrical charge $-e$ and mass m) under a constant magnetic field $(0,0,B)$ along the z-axis moves on the aforementioned helix. In Cartesian coordinates, the position vector of the electron is written as the following:

$$\mathbf{x} = \left(R\cos(\omega t), R\sin(\omega t), v_z t\right).$$

Note that v_z and $w = \dfrac{eB}{m}$ are positive constants.

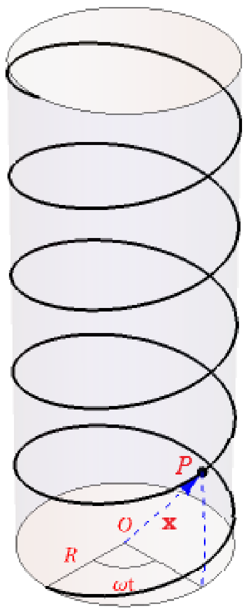

Figure 1. The trajectory of the electron P under the constant magnetic field

It should be noted that Figure 1 is drawn with the help of the website Wolfram Mathematica (Wolfram Cloud). The unit speed parametrization of the curve traced out by the aforementioned electron is found as follows:

$$\gamma(s) = \left(R\cos\frac{\omega s}{\beta}, R\sin\frac{\omega s}{\beta}, \frac{v_z s}{\beta}\right) \tag{31}$$

by Özen et al. [36] where $s = s(t) = \beta t$ and $\beta = \sqrt{R^2\omega^2 + v_z^2}$. By straightforward calculations, we have the following Darboux apparatus:

$$\begin{cases} \mathbf{T}(s) = \left(-\sin\alpha \sin\frac{\omega s}{\beta}, \sin\alpha\cos\frac{\omega s}{\beta}, \cos\alpha \right), \\ \mathbf{U}(s) = \left(\cos\frac{\omega s}{\beta}, \sin\frac{\omega s}{\beta}, 0 \right), \\ \mathbf{Y}(s) = \left(\cos\alpha\sin\frac{\omega s}{\beta}, -\cos\alpha\cos\frac{\omega s}{\beta}, \sin\alpha \right), \end{cases} \quad \text{and} \quad \begin{cases} k_g(s) = 0, \\ k_n(s) = -\frac{\omega}{\beta}\sin\alpha, \\ \tau_g(s) = \frac{\omega}{\beta}\cos\alpha. \end{cases}$$

Here the helix axis is z -axis and α is the helix angle calculated by $\tan\alpha = \frac{R\omega}{v_z}$. Via the equation (6), we get:

$$\phi_1 = \pm\sqrt{\frac{i}{2}(\cos\alpha+1)e^{-i\frac{\omega s}{\beta}}}, \quad \phi_2 = \pm\sqrt{\frac{i}{2}(\cos\alpha-1)e^{i\frac{\omega s}{\beta}}}.$$

With the help of equation (14), we obtain:

$$\frac{d\phi}{ds} = -\frac{i\omega}{2\beta}\left(\cos\alpha\phi + \sin\alpha\hat{\phi}\right). \tag{32}$$

Since $\langle \gamma(s), \mathbf{Y} \rangle = \frac{v_z s}{\beta}\sin\alpha$ and $\langle \gamma(s), \mathbf{U} \rangle = R > 0$, we have $\Omega(s) = \arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right)$. Then, we have:

$$\mathbf{T}(s) = \left(-\sin\alpha\sin\frac{\omega s}{\beta}, \sin\alpha\cos\frac{\omega s}{\beta}, \cos\alpha \right),$$

$$\mathbf{G}(s) = \begin{pmatrix} \cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\alpha\sin\frac{\omega s}{\beta} - \sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\frac{\omega s}{\beta}, \\ -\cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\alpha\cos\frac{\omega s}{\beta} - \sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\sin\frac{\omega s}{\beta}, \\ \cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\sin\alpha \end{pmatrix},$$

$$\mathbf{H}(s) = \begin{pmatrix} \sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\alpha\sin\frac{\omega s}{\beta} + \cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\frac{\omega s}{\beta}, \\ -\sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\cos\alpha\cos\frac{\omega s}{\beta} + \cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\sin\frac{\omega s}{\beta}, \\ \sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha \right) \right)\sin\alpha \end{pmatrix},$$

and

$$k_1(s)=\frac{\omega}{\beta}\sin\alpha\sin\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha\right)\right),$$
$$k_2(s)=-\frac{\omega}{\beta}\sin\alpha\cos\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha\right)\right),$$
$$k_3(s)=\frac{\omega}{\beta}\cos\alpha+\frac{v_z\beta R\sin\alpha}{\beta^2R^2+v_z^2s^2\sin^2\alpha}.$$

Using the equation (6) gives us the following:

$$\mu_1=\pm\sqrt{\frac{i}{2}(\cos\alpha+1)e^{i\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha\right)-\frac{\omega s}{\beta}\right)}},$$
$$\mu_2=\pm\sqrt{\frac{i}{2}(\cos\alpha-1)e^{i\left(\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha\right)+\frac{\omega s}{\beta}\right)}}.$$

Finally, using the equation (17) yields:

$$\frac{d\mu}{ds}=-\frac{i}{2}\left[\left(\frac{\omega}{\beta}\cos\alpha+\frac{v_z\beta R\sin\alpha}{\beta^2R^2+v_z^2s^2\sin^2\alpha}\right)\mu+\frac{\omega}{\beta}\sin\alpha e^{i\arctan\left(-\frac{v_z s}{\beta R}\sin\alpha\right)}\hat{\mu}\right].$$

One can easily check the last equation with the help of the equation (30) if desired.

4. CONCLUSIONS

The purpose of this study is to investigate the spinor representations of PAFORS for the trajectories equipped with PAFORS on regular surfaces in $\mathbb{E}^3$. For this purpose, we found the spinor formulas of PAFORS vectors and also, presented some geometric interpretations. Also, we gave the relations between spinor representations of PAFORS and the Darboux frame. Additionally, we presented an illustrative example concerned with the spinor representation of PAFORS.

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