ORIGINAL PAPER

SOME NOTES ON THE PROBLEM WITH NEGATIVE TRIANGULAR FUZZY COEFFICIENT

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Abstract. *This work is on solutions of the fuzzy problem with negative triangular fuzzy coefficient under strong generalized differentiability. Four different solutions are found via fuzzy Laplace transform method. Numerical example is given to explain the problem. We draw the graphics of the four different solutions for alfa level sets. Some notes are given on the comparison results of the solutions.*

Keywords: *Fuzzy problem; fuzzy differential equation; strong generalized differentiability.*

1. INTRODUCTION

The concept of a fuzzy derivative was first introduced by Chang and Zadeh [1]. Dubois and Prade followed up [2]. They used the extension principle in their approach. Puri and Ralescu and Goetschel and Vaxman proposed the other fuzzy derivative concepts [3-4].

The theory of fuzzy differential equations has been rapidly growing. Allahviranloo et al. proposed a novel method for solving fuzzy linear differential equations [5]. Motivated by their work, in 2015, Melliani at al. developed and extended [6]. In many articles, fuzzy differential equations was examined [7-12].

In 2010, Allahviranloo and Ahmadi introduced the fuzzy Laplace transform [13]. To solve first-order fuzzy differential equations, they used fuzzy Laplace transform method under the strongly generalized differentiability. Salahshour and Allahviranloo gave results on the solution fuzzy initial value problems. Also, they gave the conditions for the existence of the fuzzy Laplace transform of fuzzy processes. [14]. The fuzzy Laplace transform method was studied in many works [15-18].

The aim of this work is to research the solutions of the initial value problem with negative triangular fuzzy coefficient under the strong generalized differentiability using the fuzzy Laplace transform method.

This paper is organized as follows: in the second section, we give some basic definitions and theorems which will be used later; in the third section, we introduce and investigate our problem. In the fourth section, we give a numerical example. At the end of the paper, we present some conclusions.

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2. MATERIALS AND METHODS

Definition 1. [19] A fuzzy number is a function $\hat{u}:\mathbb{R}\rightarrow[0,1]$ satisfying the properties:
1. $\hat{u}$ is upper semi-continuous on $\mathbb{R}$, normal and convex fuzzy set.
2. $\overline{\{x\in\mathbb{R}|\hat{u}(x)>0\}}$ is compact.
3. $\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2. [20] Let $\hat{u}\in\mathbb{R}_F$. The α-level set of $\hat{u}$ is $[\hat{u}]^\alpha=\left[\underline{\hat{u}}_\alpha,\overline{\hat{u}}_\alpha\right]=\{x\in\mathbb{R}|\hat{u}(x)\geq\alpha\}$, $0<\alpha\leq 1$.

Definition 3. [21] A fuzzy number $\hat{u}$ is called positive (negative), denoted by $\hat{u}>0$ ($\hat{u}<0$), if its membership function $\hat{u}(x)$ satisfies $\hat{u}(x)=0$, $\forall x<0$ ($x>0$).

Definition 4. [20]

$$[\hat{u}]^\alpha=\left[\underline{u}+\left(\frac{\overline{u}-\underline{u}}{2}\right)\alpha,\overline{u}-\left(\frac{\overline{u}-\underline{u}}{2}\right)\alpha\right]$$

is the $\alpha-$level set of symmetric triangular fuzzy number $\hat{u}$, where $\left[\underline{u},\overline{u}\right]$ is the support of $\hat{u}$.

Definition 5. [13] The α-level set $\left[\underline{\hat{u}}_\alpha,\overline{\hat{u}}_\alpha\right]$ of $\hat{u}$ fuzzy number satisfy the following conditions:
1. $\underline{\hat{u}}_\alpha$ is right-continuous for $\alpha=0$ and bounded, non-decreasing and left-continuous on $(0,1]$,
2. $\overline{\hat{u}}_\alpha$ is right-continuous for $\alpha=0$ and bounded, non-increasing and left-continuous on $(0,1]$,
3. $\underline{\hat{u}}_\alpha\leq\overline{\hat{u}}_\alpha$, $0\leq\alpha\leq 1$.

Definition 6. [22] Let $\hat{u},\hat{v}\in\mathbb{R}_F$. The generalized Hukuhara difference between $\hat{u}$ and $\hat{v}$ is the set $\hat{w}\in\mathbb{R}_F$ which $\hat{u}\ominus_g\hat{v}=\hat{w}$ if and only if $\hat{u}=\hat{v}+\hat{w}$ or $\hat{v}=\hat{u}+(-1)\hat{w}$.

Definition 7. [23] Let $\hat{g}:[a_1,a_2]\longrightarrow\mathbb{R}_F$ and $x_0\in[a_1,a_2]$.
1) If there exists $\hat{g}'(x_0)\in\mathbb{R}_F$ such that for all h > 0 sufficiently small, $\exists\hat{g}(x_0+h)\ominus\hat{g}(x_0)$, $\exists\hat{g}(x_0)\ominus\hat{g}(x_0-h)$ and the limits

$$\lim_{h\longrightarrow 0^+}\frac{\hat{g}(x_0+h)\ominus\hat{g}(x_0)}{h}=\lim_{h\longrightarrow 0^+}\frac{\hat{g}(x_0)\ominus\hat{g}(x_0-h)}{h}=\hat{g}'(x_0),$$

$\hat{g}$ is said to be (1)-differentiable at x_0.
2) If there exists $\hat{g}'(x_0)\in\mathbb{R}_F$ such that for all h > 0 sufficiently small, $\exists\hat{g}(x_0)\ominus\hat{g}(x_0+h)$, $\exists\hat{g}(x_0-h)\ominus\hat{g}(x_0)$ and the limits

$$\lim_{h\longrightarrow 0^+}\frac{\hat{g}(x_0)\ominus\hat{g}(x_0+h)}{-h}=\lim_{h\longrightarrow 0^+}\frac{\hat{g}(x_0-h)\ominus\hat{g}(x_0)}{-h}=\hat{g}'(x_0),$$

$\hat{g}$ is said to be (2)-differentiable at x_0.

Definition 8. [14] Let $\hat{g}:[a_1,a_2]\longrightarrow\mathbb{R}_F$. The fuzzy Laplace transform of $\hat{g}$ is

$$\hat{G}(s)=\hat{L}(\hat{g}(x))=\int_0^\infty e^{-sx}\,\hat{g}(x)dx=\left[\lim_{\rho\to\infty}\int_0^\rho e^{-sx}\,\underline{\hat{g}}(x)dx\,,\lim_{\rho\to\infty}\int_0^\rho e^{-sx}\,\overline{\hat{g}}(x)dx\right],$$

$$\hat{G}(s,\alpha)=\hat{L}([\hat{g}(x)]^\alpha)=\left[\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right),L\left(\overline{\hat{g}}_\alpha(x)\right)\right],$$

$$\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right)=\int_0^\infty e^{-sx}\,\underline{\hat{g}}_\alpha(x)dx=\lim_{\rho\to\infty}\int_0^\rho e^{-sx}\,\underline{\hat{g}}_\alpha(x)dx,$$

$$\hat{L}\left(\overline{\hat{g}}_\alpha(x)\right)=\int_0^\infty e^{-sx}\,\overline{\hat{g}}_\alpha(x)dx=\lim_{\rho\to\infty}\int_0^\rho e^{-sx}\,\overline{\hat{g}}_\alpha(x)dx.$$

Theorem 1. [24] Let $\hat{g}''(x)$ be an integrable fuzzy function and $\hat{g}(x)$, $\hat{g}'(x)$ are primitive of $\hat{g}'(x)$, $\hat{g}''(x)$ on $[0,\infty)$.

1. If the functions $\hat{g}$, $\hat{g}'$ are (1)-differentiable,

$$\hat{L}(\hat{g}''(x))=s^2\hat{L}(\hat{g}(x))\ominus s\hat{g}(0)\ominus\hat{g}'(0).$$

2. If the functions $\hat{g}$, $\hat{g}'$ are (2)-differentiable,

$$\hat{L}(\hat{g}''(x))=s^2\hat{L}(g(x))\ominus s\hat{g}(0)-\hat{g}'(0).$$

3. If $\hat{g}$ is (1)-differentiable, then $\hat{g}'$ is (2)-differentiable,

$$\hat{L}(\hat{g}''(x))=\ominus(-s^2)\hat{L}(\hat{g}(x))-s\hat{g}(0)-\hat{g}'(0).$$

4. If $\hat{g}$ is (2)-differentiable, then $\hat{g}'$ is (1)-differentiable,

$$\hat{L}(\hat{g}''(x))=\ominus(-s^2)\hat{L}(\hat{g}(x))-s\hat{g}(0)\ominus\hat{g}'(0).$$

3. RESULTS AND DISCUSSION

We examine the solutions of the fuzzzy initial value problem

$$\begin{cases}\hat{y}''\oplus(-[\hat{\mu}]^\alpha)\hat{y}=0\\ \hat{y}(0)=[\hat{\beta}]^\alpha\\ \hat{y}'(0)=[\hat{\delta}]^\alpha\end{cases}\qquad(1)$$

where $\hat{y}$ is positive fuzzy function, $[\hat{\mu}]^\alpha=\left[\underline{\hat{\mu}}_\alpha,\overline{\hat{\mu}}_\alpha\right]$, $[\hat{\beta}]^\alpha=\left[\underline{\hat{\beta}}_\alpha,\overline{\hat{\beta}}_\alpha\right]$, $[\hat{\delta}]^\alpha=\left[\underline{\hat{\delta}}_\alpha,\overline{\hat{\delta}}_\alpha\right]$ are positive symmetric triangular fuzzy numbers. Also, $\hat{L}(\hat{y}(t))=\hat{Y}(s)$ is the fuzzy Laplace transform of $\hat{y}$ and (i,j)-solution means that $\hat{y}$ is (i)-differentiable and $\hat{y}'$ is (j)-differentiable, $i,j=1,2$.

3.1. (1,1)-SOLUTION

Since $\hat{y}'$ and $\hat{y}$ and are (1)-differentiable, using the teorem 1, we have the equations

$$s^2\underline{\hat{Y}}_\alpha(s) - \overline{\hat{\mu}}_\alpha \overline{\hat{Y}}_\alpha(s) - s\underline{\hat{y}}_\alpha(0) - \underline{\hat{y}}'_\alpha(0) = 0, \ (2)$$
$$s^2\overline{\hat{Y}}_\alpha(s) - \underline{\hat{\mu}}_\alpha \underline{\hat{Y}}_\alpha(s) - s\overline{\hat{y}}_\alpha(0) - \overline{\hat{y}}'_\alpha(0) = 0. \ (3)$$

Using the initial conditions and the equations (2) and (3) yield

$$\underline{\hat{Y}}_\alpha(s) = \frac{1}{s^4 - \underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha}\left(s\left(s^2\underline{\hat{\beta}}_\alpha + s\underline{\hat{\delta}}_\alpha + \overline{\hat{\mu}}_\alpha \overline{\hat{\beta}}_\alpha\right) + \overline{\hat{\mu}}_\alpha \overline{\hat{\delta}}_\alpha\right).$$

Similarly, $\overline{Y}_\alpha(s)$ are obtained as

$$\overline{Y}_\alpha(s) = \frac{1}{s^4 - \underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha}\left(s\left(s^2\overline{\hat{\beta}}_\alpha + s\overline{\hat{\delta}}_\alpha + \underline{\hat{\mu}}_\alpha \underline{\hat{\beta}}_\alpha\right) + \underline{\hat{\mu}}_\alpha \underline{\hat{\delta}}_\alpha\right).$$

From this, we obtain the lower and upper solutions of the $(1,1)$-solution as

$$\underline{\hat{y}}_\alpha(t) = \left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha\right)\left(\frac{cos\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$
$$+\left(\frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{sin\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$
$$+\left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha + \frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right)$$
$$+\left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha - \frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} - \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{-\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right),$$

$$\overline{\hat{y}}_\alpha(t) = \left(\overline{\hat{\beta}}_\alpha + \left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2} \underline{\hat{\beta}}_\alpha\right)\left(\frac{cos\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$
$$+\left(\frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4} \underline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{sin\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$
$$+\left(\overline{\hat{\beta}}_\alpha + \left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2} \underline{\hat{\beta}}_\alpha + \frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4} \underline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right)$$

$$+\left(\overline{\hat{\beta}}_\alpha+\left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2}\underline{\hat{\beta}}_\alpha-\frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\overline{\hat{\mu}}_\alpha\right)^{1/4}}-\frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4}\underline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{-\left(\underline{\hat{\mu}}_\alpha\overline{\hat{\mu}}_\alpha\right)^{1/4}t}}{4}\right).$$

3.2. (1,2)-SOLUTION

Using teorem 1, the following equations

$$s^2\underline{\hat{Y}}_\alpha(s)-\underline{\hat{\mu}}_\alpha\underline{\hat{Y}}_\alpha(s)-s\underline{\hat{y}}_\alpha(0)-\underline{\hat{y}}'_\alpha(0)=0,$$
$$s^2\overline{\hat{Y}}_\alpha(s)-\overline{\hat{\mu}}_\alpha\overline{\hat{Y}}_\alpha(s)-s\overline{\hat{y}}_\alpha(0)-\overline{\hat{y}}'_\alpha(0)=0$$

are obtained, where $\hat{y}$ is (1)-differentiable and $\hat{y}'$ is (2)-differentiable. Using the initial conditions, we have $\underline{\hat{Y}}_\alpha(s)$ and $\overline{\hat{Y}}_\alpha(s)$ as

$$\begin{cases}\underline{\hat{Y}}_\alpha(s)=\dfrac{s\underline{\hat{\beta}}_\alpha+\underline{\hat{\delta}}_\alpha}{s^2-\underline{\hat{\mu}}_\alpha}\\[2ex] \overline{\hat{Y}}_\alpha(s)=\dfrac{s\overline{\hat{\beta}}_\alpha+\overline{\hat{\delta}}_\alpha}{s^2-\overline{\hat{\mu}}_\alpha}\end{cases}.$$

So, we obtain $(1,2)$-solution as

$$\begin{cases}\underline{\hat{y}}_\alpha(t)=\dfrac{1}{2}\left(\left(\underline{\hat{\beta}}_\alpha+\dfrac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}}\right)e^{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}t}+\left(\underline{\hat{\beta}}_\alpha-\dfrac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}}\right)e^{-\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}t}\right),\\[2ex] \overline{\hat{y}}_\alpha(t)=\dfrac{1}{2}\left(\left(\overline{\hat{\beta}}_\alpha+\dfrac{\overline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}}\right)e^{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}t}+\left(\overline{\hat{\beta}}_\alpha-\dfrac{\overline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}}\right)e^{-\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}t}\right)\end{cases}$$
$$[\hat{y}(t)]^\alpha=\left[\underline{\hat{y}}_\alpha(t),\overline{\hat{y}}_\alpha(t)\right].$$

3.3. (2,1)-SOLUTION

Similar to solution (1,2), (2,1)-solution of the problem (1) is

$$\begin{cases} \underline{\hat{y}}_\alpha(t) = \frac{1}{2}\left(\left(\underline{\hat{\beta}}_\alpha + \frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}}\right) e^{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2} t} + \left(\underline{\hat{\beta}}_\alpha - \frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{1/2}}\right) e^{-\left(\underline{\hat{\mu}}_\alpha\right)^{1/2} t}\right), \\ \overline{\hat{y}}_\alpha(t) = \frac{1}{2}\left(\left(\overline{\hat{\beta}}_\alpha + \frac{\underline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}}\right) e^{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2} t} + \left(\overline{\hat{\beta}}_\alpha - \frac{\underline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{1/2}}\right) e^{-\left(\overline{\hat{\mu}}_\alpha\right)^{1/2} t}\right) \end{cases}$$

$$[\hat{y}(t)]^\alpha = \left[\underline{\hat{y}}_\alpha(t), \overline{\hat{y}}_\alpha(t)\right].$$

3.4. (2,2)-SOLUTION

Similar to solution (1,1), (2,2)-solution of the problem (1) is

$$\underline{\hat{y}}_\alpha(t) = \left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha\right)\left(\frac{cos\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$

$$+\left(\frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{sin\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$

$$+\left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha + \frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right)$$

$$+\left(\underline{\hat{\beta}}_\alpha + \left(\frac{\overline{\hat{\mu}}_\alpha}{\underline{\hat{\mu}}_\alpha}\right)^{1/2} \overline{\hat{\beta}}_\alpha - \frac{\overline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} - \frac{\left(\overline{\hat{\mu}}_\alpha\right)^{1/4} \underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{-\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right),$$

$$\overline{\hat{y}}_\alpha(t) = \left(\overline{\hat{\beta}}_\alpha + \left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2} \underline{\hat{\beta}}_\alpha\right)\left(\frac{cos\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$

$$+\left(\frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{sin\left(\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t\right)}{2}\right)$$

$$+\left(\overline{\hat{\beta}}_\alpha + \left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2} \underline{\hat{\beta}}_\alpha + \frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} + \frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right)$$

$$+\left(\overline{\hat{\beta}}_\alpha + \left(\frac{\underline{\hat{\mu}}_\alpha}{\overline{\hat{\mu}}_\alpha}\right)^{1/2} \underline{\hat{\beta}}_\alpha - \frac{\underline{\hat{\delta}}_\alpha}{\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4}} - \frac{\left(\underline{\hat{\mu}}_\alpha\right)^{1/4} \overline{\hat{\delta}}_\alpha}{\left(\overline{\hat{\mu}}_\alpha\right)^{3/4}}\right)\left(\frac{e^{-\left(\underline{\hat{\mu}}_\alpha \overline{\hat{\mu}}_\alpha\right)^{1/4} t}}{4}\right),$$

$$[\hat{y}(t)]^\alpha = \left[\underline{\hat{y}}_\alpha(t), \overline{\hat{y}}_\alpha(t)\right].$$

Example 1. Consider the following fuzzy problem

$$\begin{cases} \hat{y}''\oplus\left(-\left[\hat{1}\right]^{\alpha}\right)\hat{y}=0, \\ \hat{y}(0)=[1+\alpha,3-\alpha], \\ \hat{y}'(0)=[2+\alpha,4-\alpha] \end{cases}$$

where $\left[\hat{1}\right]^{\alpha}=[\alpha,2-\alpha]$.

(1,1)-solution is

$$\underline{\hat{y}}_{\alpha}(t)=\left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)\right)\left(\frac{cos\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{(2-\alpha)^{1/4}(4-\alpha)}{\alpha^{3/4}}\right)\left(\frac{sin\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)+\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{(2-\alpha)^{1/4}(4-\alpha)}{\alpha^{3/4}}\right)\left(\frac{e^{\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right)$$
$$+\left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)-\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}-\frac{(2-\alpha)^{1/4}(4-\alpha)}{\alpha^{3/4}}\right)\left(\frac{e^{-\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right),$$

$$\overline{\hat{y}}_{\alpha}(t)=\left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)\right)\left(\frac{cos\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{\alpha^{1/4}(2+\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{sin\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)+\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{\alpha^{1/4}(2+\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{e^{\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right)$$
$$+\left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)-\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}-\frac{\alpha^{1/4}(2+\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{e^{-\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right),$$

$$[\hat{y}(t)]^{\alpha}=\left[\underline{\hat{y}}_{\alpha}(t),\overline{\hat{y}}_{\alpha}(t)\right].$$

(1,2)-solution is

$$\begin{cases} \underline{\hat{y}}_{\alpha}(t)=\frac{1}{2}\left(\left(1+\alpha+\frac{2+\alpha}{\alpha^{1/2}}\right)e^{\alpha^{1/2}t}+\left(1+\alpha-\left(\frac{2+\alpha}{\alpha^{1/2}}\right)\right)e^{-\alpha^{1/2}t}\right) \\ \overline{\hat{y}}_{\alpha}(t)=\frac{1}{2}\left(\left(3-\alpha+\frac{4-\alpha}{(2-\alpha)^{1/2}}\right)e^{(2-\alpha)^{1/2}t}+\left(3-\alpha-\left(\frac{4-\alpha}{(2-\alpha)^{1/2}}\right)\right)e^{-(2-\alpha)^{1/2}t}\right) \end{cases},$$

$$[\hat{y}(t)]^{\alpha}=\left[\underline{\hat{y}}_{\alpha}(t),\overline{\hat{y}}_{\alpha}(t)\right].$$

(2,1)-solution is

$$\begin{cases} \underline{\hat{y}}_{\alpha}(t) = \frac{1}{2}\left(\left(1+\alpha+\frac{4-\alpha}{\alpha^{1/2}}\right)e^{\alpha^{1/2}t} + \left(1+\alpha-\left(\frac{4-\alpha}{\alpha^{1/2}}\right)\right)e^{-\alpha^{1/2}t}\right) \\ \overline{\hat{y}}_{\alpha}(t) = \frac{1}{2}\left(\left(3-\alpha+\frac{2+\alpha}{(2-\alpha)^{1/2}}\right)e^{(2-\alpha)^{1/2}t} + \left(3-\alpha-\left(\frac{2+\alpha}{(2-\alpha)^{1/2}}\right)\right)e^{-(2-\alpha)^{1/2}t}\right) \end{cases},$$

$$[\hat{y}(t)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(t), \overline{\hat{y}}_{\alpha}(t)\right].$$

(2,2)-solution is

$$\underline{\hat{y}}_{\alpha}(t) = \left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)\right)\left(\frac{\cos\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{(2-\alpha)^{1/4}(2+\alpha)}{\alpha^{3/4}}\right)\left(\frac{\sin\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)+\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{(2-\alpha)^{1/4}(2+\alpha)}{\alpha^{3/4}}\right)\left(\frac{e^{\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right)$$
$$+\left(1+\alpha+\left(\frac{2-\alpha}{\alpha}\right)^{1/2}(3-\alpha)-\frac{4-\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}-\frac{(2-\alpha)^{1/4}(2+\alpha)}{\alpha^{3/4}}\right)\left(\frac{e^{-\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right),$$

$$\overline{\hat{y}}_{\alpha}(t) = \left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)\right)\left(\frac{\cos\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{\alpha^{1/4}(4-\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{\sin\left(\left(\alpha(2-\alpha)\right)^{1/4}t\right)}{2}\right)$$
$$+\left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)+\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}+\frac{\alpha^{1/4}(4-\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{e^{\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right)$$
$$+\left(3-\alpha+\left(\frac{\alpha}{2-\alpha}\right)^{1/2}(1+\alpha)-\frac{2+\alpha}{\left(\alpha(2-\alpha)\right)^{1/4}}-\frac{\alpha^{1/4}(4-\alpha)}{(2-\alpha)^{3/4}}\right)\left(\frac{e^{-\left(\alpha(2-\alpha)\right)^{1/4}t}}{4}\right),$$

$$[\hat{y}(t)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(t), \overline{\hat{y}}_{\alpha}(t)\right].$$

From definition 5 and Figs. 1-8, solutions (1,2) and (2,1) are valid alfa-level sets, while solutions (1,1) and (2,2) are not valid alfa-level sets.

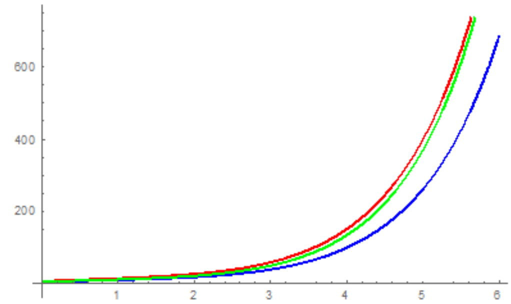

Figure 1. (1,1)-solution for α=0.6

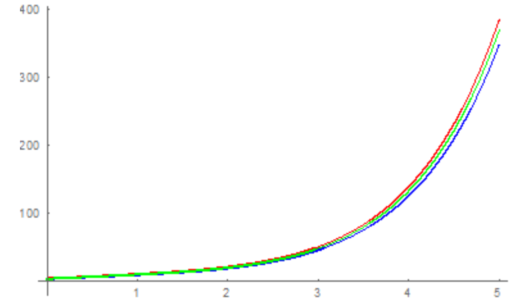

Figure 2. (1,1)-solution for α=0.9

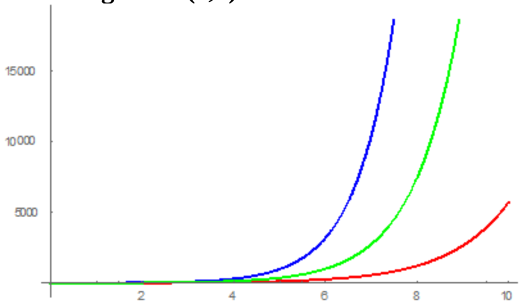

Figure 3. (1,2)-solution for α=0.6

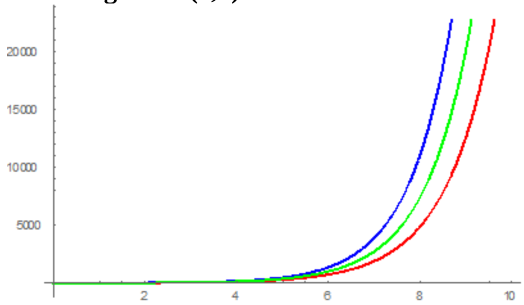

Figure 4. (1,2)-solution for α=0.9

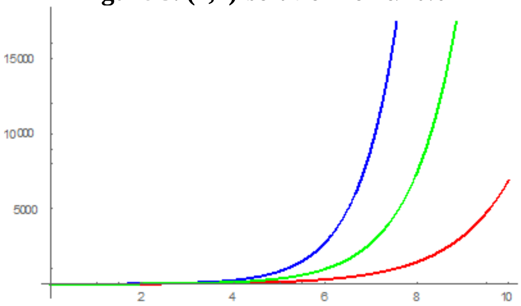

Figure 5. (2,1)-solution for α=0.6

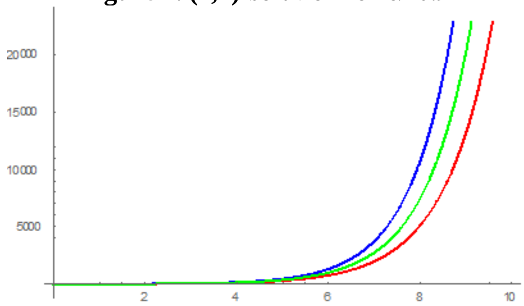

Figure 6. (2,1)-solution for α=0.9

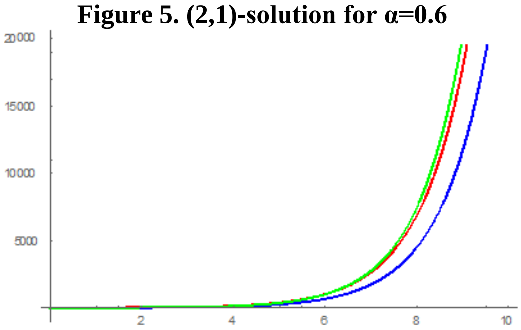

Figure 7. (2,2)-solution for α=0.6

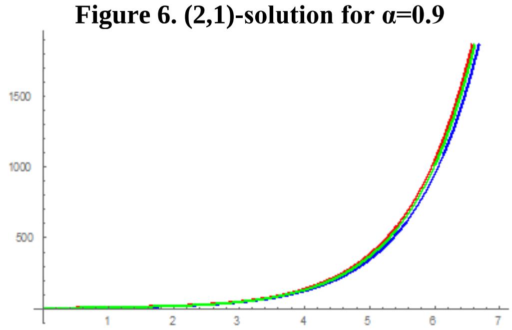

Figure 8. (2,2)-solution for α=0.9

$-\underline{y}_{\alpha}(x),\ -\overline{y}_{\alpha}(x),\ -\underline{y}_{1}(x) = \overline{y}_{1}(x)$

4. CONCLUSION

In this study, we examine the problem with negative triangular fuzzy coefficient. Solutions are found via fuzzy Laplace transform method under the strong generalized differentiability. A numerical example is given to illustrate the problem. For alfa level sets, we draw the graphics of the solutions using the mathematica program. It is seen that the solutions (1,2) and (2,1) of the problem are valid alfa-level sets, while the solutions (1,1) and (2,2) of the problem are not valid alfa-level sets.

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