ORIGINAL PAPER

PSEUDO-SLANT SUBMANIFOLDS OF AN R-SASAKIAN MANIFOLD AND THEIR PROPERTIES

PRASHANT KUMAR SHUKLA[1], GEETA VERMA[1], VIRENDRA NATH PATHAK[2]

Manuscript received: 22.12.2022; Accepted paper: 04.08.2023;
Published online: 30.09.2023.

Abstract. *The present paper aims to study pseudo-slant submanifolds of an r-Sasakian manifold and find few results. The integrability conditions of distributions that involve in the definition of pseudo-slant submanifolds of an r-Sasakian manifold are investigated. Finally, the necessary & sufficient condition for a pseudo-slant submanifold of an r-Sasakian manifold to be the pseudo-slant product is obtained successfully.*
Keywords: *Slant submanifold; Pseudo-slant submanifold; r-Sasakian manifold; totally geodesic.*

1. INTRODUCTION

In [1], D. Blair investigated r-contact manifolds in Riemannian geometry. A. Lotta defined and studied slant submanifolds of an almost r-contact metric manifold in [2]. Later, submanifolds in Sasakian manifold were investigated by A.Carriazo [3] and J. L. Cabrerizo et al. [4]. Since B.Y. Chen introduced slant submanifolds in complex manifolds as a natural generalization of both invariant and anti-invariant submanifolds [5, 6] the differential geometry of slant submanifolds has exhibited growing progress. The existence of these submanifolds in many known spaces has since been the subject of numerous research articles. As a generalisation of slant submanifolds, N. Papaghuice [7] described semi-slant submanifolds of the Keahler manifold. An almost Hermitian manifold included the introduction of bi-slant submanifolds. In an almost Hermitian manifold, Carriazo recently defined and investigated bi-slant submanifolds and introduced the concept of pseudo-slant submanifold. In an r-Sasakian manifold, the contact version of the pseudo-slant submanifold was defined and investigated by V.A. Khan and M.A. Khan [8]. The pseudo-slant submanifolds of trans-r-Sasakian manifolds were also explored by U.C. De and Avijit Sarkar in [9]. M.A. Khan reported a number of findings in [10] on totally umbilical hemi-slant submanifolds of Cosymplectic manifolds. Recently, M. Atceken [11] explored the geometry of pseudo-slant submanifolds of a Kenmotsu manifold in [12, 13] for approximately Cosymplectic manifolds, as well as slant and pseudo-slant submanifolds in (LCS)n-manifolds and CR-submanifolds of Kenmotsu manifolds in [14].

S. Uddin et al. researched the warped product pseudo-slant submanifolds of a nearly Cosympletic manifold in [15]. S.K. Srivastava et al have found Characterizations of PR-Pseudo-Slant Warped Product Submanifold of Para-Kenmotsu Manifold with Slant Base in [16]. F. Alghamdi, B.Y. Chen & S. Uddin studied S. Geometry of Pointwise Semi-slant

[1] Shri Ramswaroop Memorial College of Engineering and Management, Department of Mathematics, Lucknow, India. E-mail: pshukla7@yahoo.com; geeta.verma15@gmail.com.
[2] Shri Ramswaroop Memorial University, Faculty of Mathematical and Statistical Sciences, Lucknow, India. E-mail: pathak_virendra@rediffmail.com.

https://doi.org/10.46939/J.Sci.Arts-23.3-a16

Warped Products in Locally Conformal Kaehler Manifolds in [17]. For nearly trans-r-Sasakian manifolds in [18] and for nearly Kenmotsu manifolds in [15], S. Uddin and M.A. Khan discovered a classification on totally umbilical proper slant and hemi-slant submanifolds. We find some intriguing results on pseudo-slant submanifolds of an r-Sasakian manifold as a result of the aforementioned investigation. For the recent studies on pseudo-slant submanifolds we refer to [2, 7, 8, 15, 16, 19-29] and many more.

The pseudo-slant submanifolds of an r-Sasakian manifold have certain characteristics, which we uncover in this study. We provide a basic overview of an r-Sasakian manifold and their submanifolds, along with some formulas. Additionally, we provide some fundamental definitions and results for a pseudo-slant submanifold of almost r-contact metric manifolds. In the context of an r-Sasakian manifold, we obtain the integrability conditions of distributions on the pseudo-slant submanifolds and then obtain comparable findings for these submanifolds. A pseudo-slant submanifold of an r-Sasakian manifold must satisfy both a necessary and sufficient condition in order to be a pseudo-slant product, which we finally obtain.

2. PRELIMINARIES

Let $\bar{M}$ be an odd dimensional C^{∞}-differentiable manifold with the almost r-contact metric structure (J, ξ, η, g), where J is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\bar{M}$, satisfying

$$J^2X = -X + \eta^p(X)\,\xi_p, \tag{1}$$

$$J\,\xi_p = 0,\ \eta^p\ o\ \phi = 0, \eta^p(\,\xi_p) = 1, g(X,\ \xi_p) = \eta^p(X), \tag{2}$$

and

$$g(JX,\ JY) = g(X,Y) - \eta^p(X)\,\eta^p(Y),\ \ g(JX,Y) = -g(X,\ JY), \tag{3}$$

for any vector fields $X, Y \,\epsilon\, \Gamma(T\,\bar{M})$ An almost r-contact structure $(\,J,\ \xi_p, \eta^p, g\,)$ is said to be normal if the almost complex structure ϕ on the product manifold $\bar{M}\ \times$ R given by

$$\phi\left(X, f\frac{d}{dt}\right) = \left(J\,X - f\xi_p\,,\ \eta^p(X)\frac{d}{dt}\right),$$

where f is the C^{∞}- function on $\bar{M}\ \times$R. The condition for normality in terms of J, ξ_p and η^p is $[J,\ J] + 2d\eta^p \otimes \xi_p = 0$ on $\bar{M}$, where $[J,\ J](X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$ is the Nijenhuis tensor of J. Finally, the fundamental 2-form $\boldsymbol{\phi}$ is defined by

$$\boldsymbol{\phi}(X,Y) = g(X,\phi Y).$$

A normal almost r-contact metric structure is called an r-Sasakian structure, which satisfies

$$(\nabla_X J) = g(X,Y)\xi_p - \eta^p(Y)X \tag{4}$$

and

$$\left(\nabla_X \xi_p\right) = -JX \tag{5}$$

For any vector fields $X, Y \in \Gamma(T\bar{M})$. Then an almost r-contact metric structure $(\bar{M}, J, \xi_p\,,\ \eta^p, g)$ is called an r-Sasakian manifold.

Now, let M be a submanifold of an r-contact metric manifold $\bar{M}$ with induced metric g. Also let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Wiengarten formulas are, respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp} V, \tag{7}$$

where h and A_V are the second fundamental form and the shape operator corresponding to the normal vector field V, respectively, for the immersion of M into $\bar{M}$.

The second fundamental form and shape operator are related by formula

$$g(h(X,Y),V) = g(A_V X, Y) \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. M is said to be totally geodesic submanifold if $h(X,Y) = 0$ for each $X, Y \in \Gamma(TM)$.

Example 1. We consider R^{2n+1} with Cartesian coordinates $(x_i, y_i, z_i)(i = 1, \ldots\ldots, n)$ and its usual contact form

$$\eta^p = \frac{1}{2}(dz - \sum y_i dx_i).$$

The characteristic vector field ξ_p is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric g and its tensor field J are given by

$$g = \eta^p \otimes \eta^p + \frac{1}{4}(\sum((dx_i)^2 + (dy_i)^2), \qquad J = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, i = 1, \ldots\ldots, n$$

This gives an r-contact structure on R^{2n+1}. The vector fields $E_i = 2\frac{\partial}{\partial y_i}, E_{n+i} = 2\left(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}\right)$, ξ_p form a $J-$ basis for the r-contact metric structure. On the other hand, it can be shown that $R^{2n+1}(J, \xi_p, \eta^p, g)$ is a r-Sasakian manifold.

3. PSEUDO-SLANT SUBMANIFOLDS OF AN R-SASAKIAN MANIFOLD

We obtain the integrability conditions of the distributions of pseudo-slant submanifolds of an r-Sasakian manifold. At last, we will get necessary and sufficient conditions for a pseudo-slant submanifold to be pseudo-slant product. In contact geometry A. Lotta introduced slant submanifold as follows [2]:

Definition 1. A submanifold M of an almost r-contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $p \in M$ and $X \in T_pM - \{\xi_p\}$, the angle between JX and T_pM is constant. The constant angle $\theta X \epsilon [0, \frac{\pi}{2}]$ is called slant angle of M in $\bar{M}$.

(1) If $\theta = 0$ the submanifold is invariant submanifold.
(2) If $\theta = \frac{\pi}{2}$ then it is anti-invariant submanifold.
(3) If $\theta \neq 0, \frac{\pi}{2}$ then it is proper slant submanifold.

The tangent bundle TM of M is decomposed as $\mathrm{TM} = \mathrm{D} \oplus < \xi_p >$, where the orthogonal complementary distribution D of $< \xi_p >$ is known as the slant distribution on M.

Definition 2. Let M be a submanifold of an almost r-contact metric manifold $\overline{\mathrm{M}}$. M is said to be pseudo-slant of $\overline{\mathrm{M}}$ if there exist two orthogonal distributions D_θ and $\mathrm{D}^\perp$ on M such that:

(1) TM has the orthogonal direct decomposition $\mathrm{TM} = \mathrm{D}^\perp \oplus \mathrm{D}_\theta \oplus < \xi_p >$.

(2) The distribution $\mathrm{D}^\perp$ is an anti-invariant submanifold.

(3) The distribution D_θ is a slant, which is the slant angle between of D_θ and $\mathrm{J}\,\mathrm{D}_\theta$ is constant.

Let $\mathrm{m} = \dim(\mathrm{D}^\perp)$ and $\mathrm{n} = \dim(\mathrm{D}_\theta)$. We distinguish the following five cases.

(1) If $\mathrm{n} = 0$ or $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.

(2) If $\mathrm{m} = 0$ and $\theta = 0$, then M is invariant submanifold.

(3) If $\mathrm{m} = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is a proper slant submanifold.

(4) If $\mathrm{m}, \mathrm{n} \neq 0$ and $\theta = 0$, then M is semi-invariant submanifold.

(5) If $\mathrm{m}, \mathrm{n} \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$, then M is pseudo-slant submanifold [8].

Now we give the following results in the setting of almost r-contact manifolds given by Cabrerizo et.al [19].

Theorem 1. Let M be a slant submanifold of an almost r-contact metric manifold $\overline{M}$ such that $\xi_p \in \Gamma(TM)$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that.

Now, let M be a submanifold of an almost r-contact metric manifold $\overline{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$J\,X = \phi X + \omega X, \tag{9}$$

where ϕX and λX are the tangential and normal component of JX respectively.

Similarly, for $\mathrm{V} \in \Gamma(T^\perp M)$, we have

$$JV = BV + CV \tag{10}$$

where BV and CV are the tangential and normal component of JV. Then, using (1), (9) and (10), we have

$$\phi^2 = -I + \eta^p \otimes \xi_p - B\omega,\ \ \omega\phi + C\omega = 0, \tag{11}$$

and

$$\phi B + BC = 0, \omega B + C^2 = -I \tag{12}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(\phi X, Y) = -g(X, \phi Y)$ and $U, V \in \Gamma(T^\perp M)$, we get $g(U, CV) = -g(U, CV)$. These show that ϕ and C are skew symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we get

$$g(\omega X, V) = -g(X, BV) \tag{13}$$

which gives the relation between ω and B.

Furthermore, the covariant derivatives of the tensor field ϕ, ω, B and C are respectively defined by

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{14}$$

$$(\nabla_X \omega)Y = \nabla_X^{\perp} \omega Y - \phi \omega_X Y, \tag{15}$$

$$(\nabla_X B)Y = \nabla_X BY - B\nabla_X^{\perp} Y, \tag{16}$$

$$(\nabla_X C)Y = \nabla_X^{\perp} CY - C\nabla_X^{\perp} Y. \tag{17}$$

A submanifold M is said to be invariant if ω is identically zero, that is $J\, X \in \Gamma(TM)$ for all $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if ϕ is identically zero, that is $J\, X \in \Gamma(T^{\perp}M)$ for all $X \in \Gamma(TM)$. Now, we get easily

$$(\nabla_X \phi)Y = \mathrm{A}_{\omega Y} X + Bh(X, Y), \tag{18}$$

and

$$(\nabla_X \omega)Y = \mathrm{Ch(X, Y)} - h(X, \phi Y), \tag{19}$$

Similarly, for any $V \in \Gamma(T^2M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_X B)Y = \mathrm{A}_{CY} X + \phi \mathrm{A}_V X, \tag{20}$$

and

$$(\nabla_X C)Y = -h(\mathrm{BV, X}) - \omega \mathrm{A}_V X. \tag{21}$$

since M is tangent to ξ_p then using (5), (6), (8) and (9)

$$\nabla_{\xi_p} \xi_p = 0,\ h(\xi_p, \xi_p) = 0,\ \mathrm{A}_V \xi_p = 0 \tag{22}$$

for all $V \in \Gamma(T^{\perp}M)$ and $\xi_p \in \Gamma(TM)$.

Now, we have the following result of an almost r-contact manifold given by Cabrerizo et al. [19].

Theorem 2. Let M be a slant submanifold of an almost r-contact manifold of $\overline{\mathrm{M}}$ such that $\xi_p \in \Gamma(\mathrm{TM})$. Then, Mis slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$\phi^2 = -\lambda(\mathrm{I} - \eta^p \otimes \xi_p) \tag{23}$$

furthermore, in this case, if θ is slant angle of M, then $\lambda = \cos^2\theta$ [19].

Corollary 1. Let M be a slant submanifold of an almost r-contact manifold of $\overline{\mathrm{M}}$ with slant angle θ then for any $\mathrm{X, Y} \in \Gamma(\mathrm{TM})$, we have

$$g(\phi X, \phi Y) = \cos^2\theta\left(g(X, Y) - \eta^p(X)\eta^p(Y)\right), \tag{24}$$

and

$$g(\omega X, \omega Y) = \sin^2\theta\left(g(X, Y) - \eta^p(X)\eta^p(Y)\right). \tag{25}$$

By using (18) and (22), we get

$$\eta^p\left((\nabla_X T)Y\right) = g(X, Y) - \eta^p(X)\eta^p(Y) \tag{26}$$

for $X, Y \in \Gamma(D^{\theta})$.

If we denote the projection on $D^{\perp}$ and D^{θ} by P and Q respectively then for any vector field $X \in \Gamma(TM)$, We can write

$$X = P X + QX + \eta^{p}(X)\xi_{p} \tag{27}$$

Now operating J on both sides of equation (27), we get

$$JX = JPX + JQX$$

and

$$\phi X + \omega X = \omega PX + \phi QX + \omega QX.$$

we can easily see that

$$\phi X = \phi QX, \omega X = \omega PX + \omega QX,$$

and

$$JPX = \omega PX, \phi PX = 0, \; JQX = \phi QX + \omega QX, \phi QX \in \Gamma(D_{\theta}).$$

If we denote the orthogonal complementary of $J(TM)$ in $T^{\perp}M$ by μ, then the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \omega(D^{\perp}) \oplus \omega(D_{\theta}) \oplus \mu. \tag{28}$$

We can easily see that bundle μ is an invariant sub bundle with respect to J. Since $D^{\perp}$ and D_{θ} are orthogonal distributions on $M, g(Z, X) = 0$ for each $Z \in (D^{\perp})$ and $X \in \Gamma(D_{\theta})$. Thus, by equation (3) and (9), we can write

$$g(\omega Z, \omega X) = g(JZ, JX) = g(Z, X) = 0,$$

that is distributions $\omega(D^{\perp})$ and $\omega(D_{\theta})$ are also mutually perpendicular. In fact, decomposition (28) is an orthogonal direct decomposition.

Theorem 3. Let M be a submanifold of an almost r-contact metric manifold of $\overline{M}$. Then D^{θ} is slant distribution if and only if there is a constant $\lambda \in [0, 1]$ such that

$$(\phi Q)^{2}X = -\lambda X. \tag{29}$$

for any $X \in \Gamma(D_{\theta})$. In this case, the slant angle θ satisfies $\lambda = \cos^{2}\theta$.

Moreover, for any $Z, W \in \Gamma(D^{\perp})$ and $U \in \Gamma(TM)$, also by using (4), (7) and (8), we get

$$\begin{aligned} g(A_{\omega Z}W - A_{\omega WZ}, U) &= g(h(W, U), \omega Z) - g(h(Z, U), \omega W) \\ &= g(\overline{\nabla}_{U}W, JZ) - g(\overline{\nabla}_{U}Z, JW) \\ &= -g(J\overline{\nabla}_{U}W, Z) + g(J\overline{\nabla}_{U}Z, W) \\ &= g(\overline{\nabla}_{U}JZ - (\overline{\nabla}_{U}J)Z, W) + g((\overline{\nabla}_{U}J)W - \overline{\nabla}_{U}JW, Z) \\ &= g(\overline{\nabla}_{U}JZ, W) - g(\overline{\nabla}_{U}JW, Z) \\ &= -g(A_{\omega Z}U, W) + g(A_{\omega W}U, Z) \\ &= g(A_{\omega W}Z - A_{\omega Z}W, U). \end{aligned}$$

It follows that

$$(A_{\omega W}Z = A_{\omega Z}\, W). \tag{30}$$

Theorem 4. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\overline{M}$, then

$$\nabla^{\perp}_{W}\omega Z - \nabla^{\perp}_{Z}\omega W \in (D^{\perp})$$

for any $Z, W \in \Gamma(D^{\perp})$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $V \in \mu$ and using (4), (30), we obtain

$$\begin{aligned} g(\nabla^{\perp}_{W}\omega Z - \nabla^{\perp}_{Z}\omega W, V) &= g(\overline{\nabla}_{W}JZ + A_{JZ}W - \overline{\nabla}_{Z}JW + A_{JW}Z, V) \\ &= g(\overline{\nabla}_{W}JZ - \overline{\nabla}_{Z}JW, V) \\ &= g((\overline{\nabla}_{W}J)Z + J\overline{\nabla}_{W}Z, V) - g((\overline{\nabla}_{Z}J)W + J\overline{\nabla}_{Z}W, V) \\ &= g(J\overline{\nabla}_{W}Z, V) - g(J\overline{\nabla}_{Z}W, V) \\ &= g(\overline{\nabla}_{W}Z, JV) - g(\overline{\nabla}_{Z}W, JV) \\ &= g(\nabla_{W}Z, V) - g(\nabla_{Z}W, V) + g(h(Z, W), JV) - g(h(W, Z), JV) \\ &= 0 \end{aligned}$$

Theorem 5. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\overline{M}$. Then the anti-invariant distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of $\overline{M}$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_{\theta})$, by using (4), (6), (7) and (8), we get

$$\begin{aligned} g([Z, W], X) &= g(\overline{\nabla}_{Z}W - \overline{\nabla}_{Z}W, X) \\ &= g(\overline{\nabla}_{W}X, Z) - g(\overline{\nabla}_{Z}X, W) \\ &= g(J\overline{\nabla}_{W}X, JZ) - g(J\overline{\nabla}_{Z}X, JW) \\ &= g(\overline{\nabla}_{W}JX, JZ) - g(\overline{\nabla}_{Z}JX, JW) - g((\overline{\nabla}_{W}J)X, JZ) + g((\overline{\nabla}_{Z}J)X, JW) \\ &= g(\overline{\nabla}_{W}\phi X + \overline{\nabla}_{W}\omega X, \omega Z) - g(\overline{\nabla}_{Z}\phi X + \overline{\nabla}_{Z}\omega X, \omega W) \\ &= g(h(\phi X, W), \omega Z) - g(h(\phi X, Z), \omega W) + g(\nabla^{\perp}_{W}\omega X, \omega Z) - g(\nabla^{\perp}_{Z}\omega X, \omega W) \\ &= g(A_{\omega Z}W - A_{\omega W}Z, \phi X) + g(\nabla^{\perp}_{W}\omega X, \omega Z) - g(\nabla^{\perp}_{Z}\omega X, \omega W), \end{aligned}$$

by using (15), (19) and (30), we obtain

$$\begin{aligned} g([Z, W], X) &= g(\nabla^{\perp}_{W}\omega X, \omega Z) - g(\nabla^{\perp}_{Z}\omega X, \omega W) \\ &= g((\nabla_{W}\omega)X + \omega\nabla_{W}X, \omega Z) - g((\nabla_{Z}\omega)X + \omega\nabla_{Z}X, \omega W) \\ &= g(Ch(W, X) - h(W, \phi X), \omega Z) - g(Ch(Z, X) - h(Z, \phi X), \omega W) \\ &\quad + g(\omega\nabla_{W}X, \omega Z) - g(\omega\nabla_{Z}X, \omega W) \\ &= -g(h(W, \phi X), \omega Z) + g(h(Z, \phi X), \omega W) + g(\omega\nabla_{W}X, \omega Z) - g(\omega\nabla_{Z}X, \omega W) \end{aligned}$$

by using (25), we have

$$\begin{aligned} g([Z, W], X) &= \sin\theta g(\nabla_{W}X, Z) - \sin^{\theta} g(\nabla_{Z}X, W) \\ &= \mathit{sin}\,\theta g(\nabla_{Z}W, X) - \sin^{\theta} g(\nabla_{W}Z, X) \\ &= \sin^{2}\theta g([Z, W], X), \end{aligned}$$

hence

$$\cos^{2}\theta g([Z, W], X) = 0.$$

Thus $[Z,W] \in \Gamma(D^{\perp})$, that is, anti-invariant distribution $D^{\perp}$ is always integrable and its integral submanifold is an anti-invariant submanifold of $\overline{M}$.

Thus, the proof is complete.

Now, by using (4), we get

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y = g(X,Y)\xi_p - \eta^p(Y)X.$$

Hence by using (6), (7), (9) and (10), we have

$$-A_{\omega Y}X + \nabla^{\perp}_X \omega Y - \phi\nabla_X Y - Bh(X,Y) - Ch(X,Y) = g(X,Y)\xi_p - \eta^p(Y)X,$$

for any $X, Y \in \Gamma(D^{\perp})$. From the tangent component of this last equation, we have

$$A_{\omega Y}X + \phi\nabla_X Y + Bh(X,Y) + g(X,Y)\xi_p = 0. \tag{31}$$

By interchanging roles of X and Y in (31), we have

$$A_{\omega X}Y + \phi\nabla_Y X + Bh(Y,X) + g(Y,X)\xi_p = 0, \tag{32}$$

which is equivalent to

$$T[X,Y] = A_{\omega X}Y - A_{\omega Y}X.$$

From (30), we can easily see that the anti-invariant distribution $D^{\perp}$ is always integrable. Since the ambient manifold $\overline{M}$ is Sasakian, for any $Z, W \in \Gamma(D^{\perp})$

$$(\overline{\nabla}_Z J)W = g(Z,W)\xi_p - \eta^p(W)Z,$$

which implies that

$$\overline{\nabla}_Z JW - J\overline{\nabla}_Z W = \overline{\nabla}_Z \omega W - J(\nabla_Z W + h(W,Z)) - g(Z,W)\xi_p$$

So, we have

$$-A_{\omega W}Z + \nabla^{\perp}_Z \omega W - \phi\nabla_Z W - \omega\nabla_Z W - Bh(W,Z) - Ch(W,Z) - g(Z,W)\xi_p = 0.$$

From the tangential components of the last equation, we have

$$A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z) + g(Z,W)\xi_p\,.$$

From the above equation, we obtain

$$T[W,Z] = A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z)$$

The anti-invariant distribution $D^{\perp}$ is integrable, $J[Z,W] = \omega[Z,W]$ because tangential component of $J[Z,W]$ is zero. So, we have

$$A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z) = 0. \tag{33}$$

Similarly, we get,

$$A_{\omega Z}W + \phi\nabla_W Z + Ch(Z,W) = 0. \tag{34}$$

Here, by using (30), (33) and (34), we have

$$(\nabla_Z\phi)W = (\nabla_W\phi)Z.$$

Lemma 1. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\bar{M}$, Then we get

$$(\nabla_Z\phi)W = (\nabla_W\phi)Z, \tag{35}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Theorem 6. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if

$$P_1\{\nabla_X\phi Y - \phi\nabla_X Y - A_{\omega Y}X - Bh(X,Y) + g(X,Y)\xi_p - \eta^p(Y)X\} = 0, \tag{36}$$

for any $X, Y \in \Gamma(D_\theta)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$, by using (4) and considering the tangential component, we have

$$T[X,Y] = \nabla_X\phi Y - \phi\nabla_Y X - A_{\omega Y}X - Bh(X,Y) + g(X,Y)\xi_p - \eta^p(Y)X. \tag{37}$$

Applying P_1 to (37), we have (36)

Theorem 7. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if

$$\nabla^{\perp}_Z\omega W - \nabla^{\perp}_W\omega Z + h(Z,\phi W) - h(W,\phi Z) \in \mu \oplus \omega(D_\theta),$$

for any $Z, W \in \Gamma(D_\theta)$.

Proof. For any $Z, W \in \Gamma(D_\theta)$ and $X \in \Gamma(D^{\perp})$, by using (3), we obtain

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z W, X) - g(\bar{\nabla}_W Z, X) \\ &= g(J\bar{\nabla}_Z W, JX) + \eta^p(\bar{\nabla}_Z W)\eta^p(X) - g(J\bar{\nabla}_W Z, JX) - \eta^p(\bar{\nabla}_W Z)\eta^p(X). \end{aligned}$$

Thus, we have

$$g([Z,W],X) = g(\bar{\nabla}_Z JW, \omega X) - g((\bar{\nabla}_Z J)W, \omega X) - g(\bar{\nabla}_W JZ, \omega X) + g((\bar{\nabla}_W J)Z, \omega X).$$

Taking into account (4) and (9), we get

$$g([Z,W],X) = g(\bar{\nabla}_Z(\phi W + \omega W), \omega X) - g(\bar{\nabla}_W(\phi Z + \omega Z), \omega X).$$

Then from the Gauss and Weingarten formulas the above equation takes the form, we obtain

$$g([Z,W],X) = g(h(Z,\phi W), \omega X) + g(\nabla^{\perp}_Z\omega W, \omega X) - g(h(W,\phi Z), \omega X) + g(\nabla^{\perp}_W\omega Z, \omega X).$$

Since, we have $\omega X \in (D^{\perp}) \subseteq (T^{\perp}M)$, we conclude

$$\nabla^{\perp}_{Z}\omega W - \nabla^{\perp}_{W}\omega Z + h(Z, \phi W) - h(W, \phi Z) \in \mu \oplus \omega(D_{\theta}).$$

Theorem 8. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\overline{M}$. Then the slant distribution D_{θ} is integrable if and only if

$$\phi A_{\omega U}X + A_{\omega U}\phi X = 0,$$

for any $U \in (D^{\perp})$ and $X \in \Gamma(D_{\theta})$.

Proof. For any $U \in (D^{\perp})$ and $X \in \Gamma(D_{\theta})$, by direct calculation, we get

$$\begin{aligned} g([X, Y], U) &= g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, U) \\ &= g(J\overline{\nabla}_X Y, JU) - g(J\overline{\nabla}_Y X, JU) \\ &= g(J\overline{\nabla}_X Y, \omega U) - g(J\overline{\nabla}_Y X, \omega U) \\ &= g(\overline{\nabla}_X JY, \omega U) - g(\overline{\nabla}_Y JX, \omega U) - g\big((\overline{\nabla}_X J)Y, \omega U\big) + g\big((\overline{\nabla}_Y J)X, \omega U\big) \end{aligned}$$

Hence, by using (4) and (9), we get

$$\begin{aligned} g([X, Y], U) &= g(\overline{\nabla}_Y \omega U, JX) - g(\overline{\nabla}_X \omega U, JY) \\ &= g(\overline{\nabla}_Y \omega U, \phi X) + g(\overline{\nabla}_Y \omega U, \omega X) - g(\overline{\nabla}_X \omega U, \phi Y) - g(\overline{\nabla}_X \omega U, \omega Y) \end{aligned}$$

On the other hand, using (4), (6) and (7), we obtain

$$(\overline{\nabla}_X J)U = \overline{\nabla}_X JU - J\overline{\nabla}_X U$$
$$g(X, U)\xi_p - \eta^p(U)X = \overline{\nabla}_X \omega U - \phi\nabla_X U - \omega\nabla_X U - Bh(X, U) - Ch(X, U)$$
$$0 = \overline{\nabla}_X \omega U - \phi\nabla_X U - \omega\nabla_X U - Bh(X, U) - Ch(X, U)$$

that is,

$$-A_{\omega U}X + \nabla^{\perp}_{X}\omega U = \phi\nabla_X U + \omega\nabla_X U + Bh(X, U) + Ch(X, U),$$

From the tangential components, we have

$$\begin{aligned} -A_{\omega U}X &= \phi\nabla_X U + Bh(X, U) \\ (\nabla_X \omega)U &= Ch(X, U). \end{aligned} \tag{38}$$

Also, by using (15) and (38), we obtain

$$\begin{aligned} g([X, Y], U) &= g(A_{\omega U}X, \phi Y) - g(A_{\omega U}Y, \phi X) + g(\nabla^{\perp}_{Y}\omega U, \omega X) - g(\nabla^{\perp}_{X}\omega U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g((\nabla_Y \omega)U) + \omega\nabla_Y U, \omega X) \\ &\quad -g((\nabla_X \omega)U + \omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g(Ch(Y, U), \omega X) + g(C\nabla_Y U, \omega X) \\ &\quad -g(Ch(X, U), \omega Y) - g(\omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g(\omega\nabla_Y U, \omega X) - g(\omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + \sin^2\theta\{g(\nabla_Y U, X) - g(\nabla_X U, Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + \sin^2\theta\{g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + \sin^2\theta\{g([X, Y], U)\}. \end{aligned}$$

So, we have

$$\cos^2\theta\{g([X,Y],U)\} = -g(\phi A_{\omega U}X,Y) - g(A_{\omega U}\phi X,Y)$$

For a pseudo-slant submanifold M of $\overline{M}$, the slant and anti-invariant distributions are totally geodesic in M, then M is called pseudo-slant product.

The following theorem characterizes the pseudo-slant product in an r-Sasakian manifold.

Theorem 9. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\overline{M}$. Then M is a pseudo-slant product if and only if the second fundamental form h satisfies

$$Bh(X,Z) = 0, \tag{39}$$

for all $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(TM)$.

Proof: For all $X, Y \in \Gamma(D_\theta)$ and $U, V \in \Gamma(TM)$, we get

$$\begin{aligned} g(\nabla_X Y,U) &= -g(\nabla_X U,Y) = -g(\overline{\nabla}_X U,Y) \\ &= -g(J\overline{\nabla}_X U,JY) - \eta^p(\overline{\nabla}_X U)\eta^p(Y) \\ &= -g((\overline{\nabla}_X J)U - \overline{\nabla}_X JU,JY) - g(\nabla_X U + h(X,U),\xi_p)\,\eta^p(Y) \\ &= -g(\overline{\nabla}_X JU,JY) - g(\nabla_X U,\xi_p)\,\eta^p(Y) \\ &= -g(\overline{\nabla}_X JU,JY) + g(\nabla_X \xi_p,U)\,\eta^p(Y) \\ &= -g(\overline{\nabla}_X JU,\phi Y) - g(\overline{\nabla}_X JU,\omega Y). \end{aligned}$$

Now, put $JU = \omega U$ and using (22), we obtain

$$g(\nabla_X Y,U) = -g(\overline{\nabla}_X \omega U,\phi Y) - g(\overline{\nabla}_X \omega U,\omega Y).$$

Using (6) and (7), we get

$$\begin{aligned} g(\nabla_X Y,U) &= g(A_{\omega U}X - \nabla^\perp_X \omega U,\phi Y) + g(A_{\omega U}X - \nabla^\perp_X \omega U,\omega Y) \\ &= (A_{\omega U}X,\phi Y) - g\big((\nabla_X \omega)U,\omega Y\big) - g(\omega\nabla_X U,\omega Y) \\ &= (A_{\omega U}X,\phi Y) - g(\omega\nabla_X U,\omega Y) - g(Ch(X,U),\omega Y), \end{aligned}$$

Hence using (22) and (25), we have

$$\begin{aligned} g(\nabla_X Y,U) &= g(A_{\omega U}X,\phi Y) - g(\omega\nabla_X U,\omega Y) \\ &= g(A_{\omega U}X,\phi Y) - \sin^2\theta\{g(\nabla_X U,Y) - \eta^p(\nabla_X U)\eta^p(Y)\} \\ &= g(h(X,\phi Y),\omega U) - \sin^2\theta g(\nabla_X U,Y) + \sin^2\theta g\big(\nabla_X U,\xi_p\big)\eta^p(Y)\} \\ &= g(h(X,\phi Y),\omega U) - \sin^2\theta g(\nabla_X Y,U) - \sin^2\theta g\big(\nabla_X \xi_p,U\big)\eta^p(Y)\} \\ &= g(h(X,\phi Y),\omega U) - \sin^2\theta g(\nabla_X Y,U) \end{aligned}$$

that is

$$\cos^2\theta g(\nabla_X Y,U) = g(h(X,\phi Y),\omega U) = -g(Bh(X,\phi Y),U). \tag{40}$$

In the same way, we can obtain

$$\begin{aligned} g(\nabla_V U, X) &= g(\overline{\nabla}_V U, X) = -g(\overline{\nabla}_V X, U) \\ &= -g(J\overline{\nabla}_V X, JU) - \eta^p(\overline{\nabla}_V X)\eta^p(U) \\ &= g((\overline{\nabla}_V J)X, JU) - g(\overline{\nabla}_V JX, JU) \end{aligned}$$

For $U, V \in \Gamma(D^\perp)$, since the tangent component of JU and ϕU are zero, we get

$$\begin{aligned} g(\nabla_V U, X) &= g((\overline{\nabla}_V J)X, \omega U) - g(\overline{\nabla}_V JX, \omega U) \\ &= g(\overline{\nabla}_V JX, \omega U) = -g(\overline{\nabla}_V \phi X, \omega U) - g(\overline{\nabla}_V \omega X, \omega U) \\ &= -g(\overline{\nabla}_V \phi X + h(\phi X, V), \omega U) + g(A_{\omega X} V - \nabla^\perp_V \omega X, \omega U) \\ &= -g(h(\phi X, V), \omega U) - g(\nabla^\perp_V \omega X, \omega U) \\ &= -g(h(\phi X, V), \omega U) - g\big((\nabla_V \omega)X + \omega \nabla_V X, \omega U\big), \end{aligned}$$

Hence using (22) we have

$$\begin{aligned} g(\nabla_V U, X) &= -g(h(V, \phi X), \omega U) - g(\omega \nabla_V X, \omega U) + g(h(V, \phi X), \omega U) - g(Ch(V, X), \omega U) \\ &= -g(\omega \nabla_V X, \omega U) - g(Ch(V, X), \omega U) \\ &= g(Ch(V, X), \omega U) + \sin^2\theta g(\nabla_V U, X), \end{aligned}$$

that is

$$\cos^2\theta g(\nabla_V U, X) = -g(Ch(V, X), \omega U) = g(Bh(V, X), U). \tag{41}$$

From equations (40) and (41). Thus D_θ and $D^\perp$ are totally geodesic in M if and only if (39) is satisfied.

Theorem 10. Let M be a pseudo-slant submanifold of an r-Sasakian manifold $\overline{M}$. If ω is parallel on D_θ, then either M is a D_θ-geodesic submanifold or $h(X, Y)$ is an eigen vector of C^2 with eigen values $-\cos^2\theta$, for any $X, Y \in \Gamma(D_\theta)$.

Proof: For any $X, Y \in \Gamma(D_\theta)$, from (19), we have

$$Ch(X, Y) - h(X, \phi Y) = 0 \tag{42}$$

Since D_θ is a slant distribution, we have

$$\begin{aligned} 0 &= Ch\big(X, Y - \eta^p(Y)\xi_p\big) - h(X, \phi(Y - \eta^p(Y)\xi_p)) \\ &= Ch\big(X, Y - \eta^p(Y)\xi_p\big) - h(X, \phi Y), \end{aligned}$$

that is

$$Ch\big(X, Y - \eta^p(Y)\xi_p\big) = h(X, \phi Y). \tag{43}$$

Now, applying C to (43), we obtain

$$C^2 h\big(X, Y - \eta^p(Y)\xi_p\big) = Ch(X, \phi Y).$$

by interchanging of Y and ϕY in (3.34), we get

$$h(X, \phi^2 Y) = Ch(X, \phi Y).$$

Hence, using (23), we obtain

$$C^2h\big(X, Y - \eta^p(Y)\xi_p\big) = Ch(X, \phi Y) = h(X, \phi^2 Y) = -\cos^2\theta\, h(X, Y - \eta^p(Y)\xi_p)$$

This implies that either h vanishes on D_θ or h is eigen vector of C^2 with eigen values $-\cos^2\theta$.

4. CONCLUSION

We considered pseudo-slant submanifolds of an r-Sasakian manifold and obtained basic results. The necessary & sufficient condition for a pseudo-slant submanifold of an r-Sasakian manifold to be the pseudo-slant product have been determined. Future studies could fruitfully explore this issue further by considering the para Sasakian manifold, Trans Sasakian manifold, etc.

REFERENCES

[1] Blair, D.E, *Lecture Notes in Mathematics*, 509, Springer-Verlag, New York, 1976.
[2] Khan, M.N.I, *Tensor N.S.*, **68**(2), 154, 2007.
[3] Carriazo, A., *New Developments in Slant Submanifolds Theory*, Narosa Publishing House, New Delhi, India, 2002.
[4] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., Fernandez, M., *Geomeatriae Dedicate*, **78** , 183, 1999.
[5] Chen, B.Y., *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, View at Zentralblatt Math., 1990.
[6] Chen, B.Y., *Bulletin of the Australian Mathematical Society*, **41**, 135, 1990.
[7] Khan, M.N.I., De, U. C., *AIMS Mathematics*, **7**(5), 7888, 2022.
[8] Khan, V.A., Khan, M.A., *Indian Journal of Pure and Applied Mathematics*, **38** , 31, 2007.
[9] De Chand, U., Sarkar, A., *Proceedings of the Estonian Academy of Science*, **60**(1), 2011.
[10] Khan, M.A., *Mathematica Aeterna*, **3**(8), 845, 2013.
[11] Atceken, M., Hui, S.K., *Czechoslovak Mathematical Journal*, **63** , 177, 2013.
[12] Atceken, M., Dirik, S., *Gulf journal of Mathematics*, **2** , 51, 2014.
[13] Dirik, S., Atceken, M., *Turkish Journal of Mathematics and Computer Sciences*, **2**, 1, 2014.
[14] Atceken, M., Dirik, S., *Acta Universitatis Sapientiae*, **4**, 182, 2012.
[15] Uddin, S., Ozel, C., Khan, M.A., Singh, K., *International Journal of Physical Sciences*, **7**, 5538, 2012.
[16] Srivastava, S.K., Mofarreh, F., Kumar, A., Ali, A., *Symmetry*, **14**, 1001, 2022.
[17] Alghamdi, F., Chen, B.Y., Uddin, S., *Results Math*, **76**, 204, 2021.
[18] Khan, M.A., Uddin, S., Singh, K., *Differential Geometry-dynamical System*, **13**, 117, 2011.
[19] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., Fernandez, M., *Glasgow Mathematical Journal*, **4**, 125, 2000.
[20] Das, L.S., Khan, M.N.I., *Mathematical Sciences Research Journal*, **17**(10), 263, 2013.

[21] Khan, M.N.I, Mofarreh, F., Haseeb, A., *Symmetry*, **15**(3), 753, 2023.
[22] Khan, M.N.I, Mofarreh, F., Haseeb, A., Saxena, M., *Symmetry*, **15**(8), 1553, 2023.
[23] Kumar, R., Colney, L., Khan, M.N.I., *Results in Nonlinear Analysis*, **6**(3), 50, 2023.
[24] Khan, M.N.I., *International Journal of Mathematics and Computer Science*, **17**(1), 909, 2022.
[25] Lotta, A., *Bulletin Mathematical Society*, Roumanie, **39**, 183, 1996.
[26] Khan, M.N.I., *Journal of Science and Arts*, **53** (4), 801, 2019.
[27] Papaghiuc, N., *Ann. St. Univ. Iasi*, tom. XL, S.I. **9**(f1), 55, 1994.
[28] Sardar, A., Khan, M.N.I., De Chand, U, h-∗-Ricci Solitons and Almost co-Kähler Manifolds, *Mathematics*, **9**(24), 3200, 2021.
[29] Uddin, S., Bernardine, W.R., Mustafa, A.A., *Abstract and Applied Analysis*, **2012**, 420890, 2012.