ORIGINAL PAPER

NUMERICAL RESOLUTION OF NON-LINEAR EQUATIONS

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Abstract. *In this study, we have employed the highly significant hyperbolic tangent (tanh) method to conduct an in-depth analysis of nonlinear coupled KdV systems of partial differential equations. In comparison to existing sophisticated approaches, this proposed method yields more comprehensive exact solutions for traveling waves without requiring excessive additional effort. We have successfully applied this method to two examples drawn from the literature of nonlinear partial differential equation systems.*

Keywords: *method; non-linear system; exact solutions; Boussinesq equations, nonlinear partial differential; non-linear waves.*

1. INTRODUCTION

The hyperbolic tangent (tanh) method stands as a robust technique for symbolically computing traveling wave solutions of nonlinear wave and evolution equations. It particularly excels in scenarios where dispersion, convection, and reaction-diffusion phenomena assume significance [1].Nonlinear coupled partial differential equations hold considerable importance across various scientific domains, notably in fluid mechanics, solid-state physics, plasma physics, plasma waves, capillary-gravity waves, and chemical physics. The nonlinear wave phenomena observed in these scientific realms are frequently described by bell-shaped sech, solutions and kink-shaped tanh solutions. The availability of these exact solutions, significantly aids in verifying the stability analysis of numerical solvers [2-3]. In this study, we delve into the realm of two coupled KdV equations.Various methods, such as the Adomian decomposition method [4], Backlund and Darboux transformation [5], inverse Scattering method [6], and Hirota's bilinear method [7], are deployed to obtain both exact and numerical solutions. Within this investigation, we explore traveling wave solutions for the KdV equations in the form of $u(x,t)=u(\xi)$, $\xi=k(x-\lambda t)$, where λ signifies the wave speed (see [8]). It's worth noting that this technique is primarily tailored for the pursuit of traveling waves. Therefore, our focus narrows down to one-dimensional shock waves (of the kink type) and solitary-wave solutions (of the pulse type) in a moving reference frame. Building upon the tanh method and its extensions, several symbolic software programs have been developed to facilitate the discovery of exact traveling wave solutions [9].

2. EXPLANATION OF THE TANH METHOD

The Tanh method will be introduced as presentedby Malfliet [10] and by Wazwaz [11-13]. The tanh method is based on a priori assumption that the travelingwave solutions can be expressed in terms of the tanh function to solve the coupled KdVequations.

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The tanh method is developed by Malfliet [10].The method is applied to find out an exact solution of a nonlinear ordinary differential equation

$$P(u,u_x,u_t,u_{xx},u_{xxx},........)=0 \tag{2.1}$$

where P is a polynomial of the variable u and its derivatives. If we consider $u(x,t)=u(\xi),\ \xi=k(x-\lambda t)$, so that $u(x,t)=U(\xi),$ we can use the following changes:

$$\frac{\partial}{\partial t}=-k\lambda\frac{d}{d\xi},\ \frac{\partial}{\partial x}=k\frac{d}{d\xi},\ \frac{\partial^2}{\partial x^2}=k^2\frac{d^2}{d\xi^2},\ \frac{\partial^3}{\partial x^3}=k^3\frac{d^3}{d\xi^3},$$

and so on, then Eq. (2.1) becomes an ordinary differential equation

$$Q(U,U^{/},U^{//},U^{///},.........)=0 \tag{2.2}$$

with Q being another polynomial form of its argument, which will be called the reduced ordinary differentialequation of Eq. (2.2). Integrating Eq. (2.2) as long as all terms contain derivatives, the integrationconstants are considered to be zeros in view of the localized solutions. However, the nonzero constantscan be used and handled as well [13]. Now finding the traveling wave solutions to Eq. (2.1) is equivalentto obtaining the solution to the reduced ordinary differential equation (2.2). For the tanh method, we introduce the new independent variable [14]:

$$Y(x,t)=\tanh(\xi) \tag{2.3}$$

that leads to the change of variables:

$$\begin{aligned}
&\frac{d}{d\xi}=(1-Y^2)\frac{d}{dY}\\
&\frac{d^2}{d\xi^2}=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2}\\
&\frac{d^3}{d\xi^3}=2(1-Y^2)(3Y^2-1)\frac{d}{dY}-6Y(1-Y^2)^2\frac{d^2}{dY^2}+(1-Y^2)^3\frac{d^3}{dY^3}
\end{aligned} \tag{2.4}$$

The next crucial step is that the solution we are looking for is expressed in the form

$$u(x,t)=U(\xi)=\sum_{i=1}^{m}a_iY^i \tag{2.5}$$

where the parameter m can be found by balancing the highest-order linear term with the nonlinear terms in Eq. (2.2), and $k,\lambda,a_0,a_1,....,a_m$ are to be determined. Substituting (2.5) into (2.2) will yield a set of algebraicequations for $k,\lambda,a_0,a_1,....,a_m$ because all coefficients of Y^i have to vanish. From these relations, $k, \lambda, a_0, a_1, ..., a_m$ can be obtained. Having determined these parameters, knowing that m is a positive integerin most cases, and using (2.5) we obtain an analytic solution $u(x,t)$ in a closed form [13]. The tanh methodseems to be powerful tool in dealing with coupled nonlinear physical models. For a coupled system of nonlineardifferential equations with two unknowns:

$$\begin{aligned} P_1(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{vv}, ...) = 0 \\ P_2(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{vv}, ...) = 0 \end{aligned} \tag{2.6}$$

As for the traveling wave solutions to (2.6) concerned, we have to solve its corresponding reduced ordinarydifferential equations

$$\begin{aligned} Q_1(u, v, u^{/}, v^{/}, u^{//}, v^{//}, ...) = 0 \\ Q_2(u, v, u^{/}, v^{/}, u^{//}, v^{//}, ...) = 0 \end{aligned} \tag{2.7}$$

In most cases, the exact solvability of (2.7) depends on a delicate explicit assumption between the twounknowns or their derivatives, for more details see [13].

3. NUMERICAL APPLICATIONS

The tanh method is generalized on two examples including systems of coupled KdV equations.These systems were studied from Sayed Tauseef [15] by applying the variational iteration method.

Example 1. Considerthe following (1+1) - dimensional nonlinear Boussinesq equations [14]:

$$\begin{aligned} u_t + v_x + u.u_x = 0 \\ v_t + (vu)_x + u_{xxx} = 0 \end{aligned} \tag{3.1}$$

Using the traveling wave transformations

$$u(x,t) = U(\xi) = \sum_{i=1}^{m} a_i Y^i \tag{3.2}$$

$$v(x,t) = V(\xi) = \sum_{i=1}^{n} b_i Y^i \tag{3.3}$$

where

$$\xi = k(x - \lambda t) \tag{3.4}$$

The nonlinear system of partial differential equations (3.1) is carried to a system of ordinary differentialequations

$$-\lambda k\frac{dU}{d\xi}+k\frac{dV}{d\xi}+kU\frac{dU}{d\xi}=0$$
$$-\lambda k\frac{dV}{d\xi}+kV\frac{dU}{d\xi}+kU\frac{dV}{d\xi}+k^3\frac{d^3U}{d\xi^3}=0 \tag{3.5}$$

we postulate the following tanh series in Eq. (3.2), Eq. (3.3), Eq. (2.3) and the transformation given in (2.4), the first equation in (3.5) reduces to

$$-\lambda k(1-Y^2)\frac{dU}{dY}+k(1-Y^2)\frac{dV}{dY}+kU(1-Y^2)\frac{dU}{dY}=0 \tag{3.6}$$

the second equation in (3.5) reduces to

$$-\lambda k(1-Y^2)\frac{dV}{dY}+kV(1-Y^2)\frac{dU}{dY}+kU(1-Y^2)\frac{dV}{dY}+2k^3(1-Y^2)(3Y^2-1)\frac{dU}{dY}$$
$$-6k^3Y(1-Y^2)^2\frac{d^2U}{dY^2}+k^3(1-Y^2)^3\frac{d^3U}{dY^3}=0 \tag{3.7}$$

Now, to determine the parameters m and n, we balance the linear term of highest-order with the highestordernonlinear terms. So, in Eq. (3.6) we balance $V^{/}$ with $UU^{/}$, to obtain

$2 + n - 1 = 2 + m + m - 1$, then $n = 2m$.

While in Eq. (3.7) we balance $U^{///}$ with $UV^{/}$, to obtain

$6 + m - 3 = 2 + m + n - 1$ then $n = 2$, $m = 1$.

The tanh method admits the use of the finite expansion for both:

$$u(x,t)=U(Y)=a_0+a_1Y, \quad a_1\neq 0 \tag{3.8}$$

and

$$v(x,t)=V(Y)=b_0+b_1Y+b_2Y^2, \quad b_2\neq 0 \tag{3.9}$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}$ from Eq. (3.8) and Eq. (3.9) respectively into Eq. (3.6), then equating the coefficient of Y^i, i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$Y^0 : -\lambda a_1 + a_1 a_0 +_1 = 0$$
$$Y^1 : 2b_2 + a_1^2 = 0 \tag{3.10}$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}$ from Eq. (3.8) and Eq. (3.9) respectively into Eq. (3.7), then equating the coefficient of Y^{i} ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned} &Y^{0} : -\lambda b_1 + a_1 b_0 + a_0 b_1 - 2k^2 a_1 = 0 \\ &Y^{1} : -\lambda b_2 + a_1 b_1 + a_0 b_2 = 0 \\ &Y^{2} : a_1 b_2 + 2c^2 a_1 = 0 \end{aligned} \tag{3.11}$$

Solving the nonlinear systems of equations (3.12) and (3.13) with help of Mathematica we can get:

$$a_0 = \lambda \ , \ a_1 = 2k \, , \ b_0 = 2k^2 \ \ , \ b_1 = 0 \ \ , \ b_2 = -2k^2$$

Then:

$$u(x, t) = \lambda + 2k \tanh(k(x - \lambda t)) \tag{3.12}$$

and

$$v(x, t) = 2 k^2 \operatorname{sech}^2(k(x - \lambda t)) \tag{3.13}$$

The solitary wave and behavior of the solutions $u(x,t)$ and $v(x,t)$ are shown in Figs. 3.1 and 3.2 respectively for some fixed values of the parameters ($\lambda = 0.5$, $k = 0.5$)

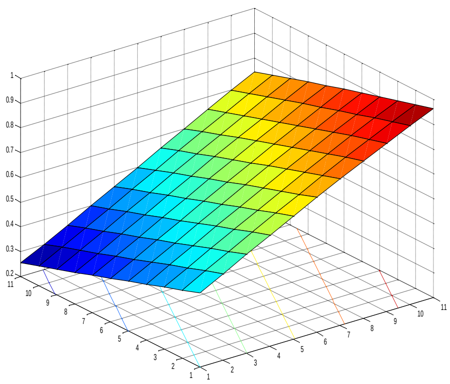

Figure 3.1. The solitary wave *u*(*x*,*t*)

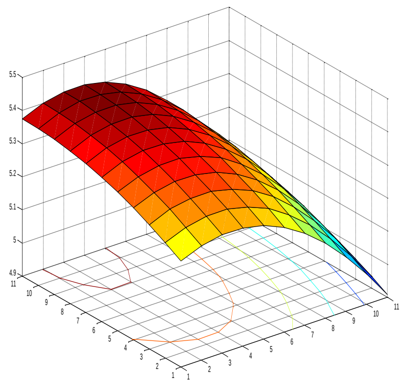

Figure 3.2. The solitary wave *v*(*x*,*t*)

Example 2. Considerthe following (1+1)- dimensional new coupled modified KdV nonlinear equations [14]:

$$\begin{aligned} &u_t - \frac{1}{2} u_{xxx} + 3u^2 u_x - \frac{3}{2} v_{xx} - 3(uv)_x + 3\alpha . u_x = 0 \\ &v_t + v_{xxx} + 3vu_x - 3u_x v_x - 3u^2 v_x - 3\alpha . u_x = 0 \end{aligned} \tag{3.14}$$

Using the traveling wave transformations

$$u(x,t) = U(\xi) = \sum_{i=1}^{m} a_i Y^{i} \tag{3.15}$$

$$v(x,t) = V(\xi) = \sum_{i=1}^{n} b_i Y^{i} \tag{3.16}$$

where

$$\xi = k(x - \lambda t) \tag{3.17}$$

The nonlinear system of partial differential equations (3.14) is carried to a system of ordinary differentialequations

$$-\lambda k \frac{dU}{d\xi} - \frac{1}{2}k^3 \frac{d^3U}{d\xi^3} + 3kU^2 \frac{dU}{d\xi} - \frac{3}{2}k^2 \frac{d^2V}{d\xi^2} - 3kU \frac{dV}{d\xi}$$
$$-3kV \frac{dU}{d\xi} + 3\alpha k \frac{dU}{d\xi} = 0$$
$$-\lambda k \frac{dV}{d\xi} + k^3 \frac{d^3V}{d\xi^3} + 3kV \frac{dU}{d\xi} - 3k^2 \frac{dU}{d\xi}\frac{dV}{d\xi} - 3kU^2 \frac{dV}{d\xi}$$
$$-3\alpha k \frac{dU}{d\xi} = 0 \tag{3.18}$$

We postulate the following tanh series in Eq. (3.2), Eq. (3.3), Eq. (2.3) and the transformation given in (2.4), the first equation in (3.18) reduces to

$$-\lambda k(1-Y^2)\frac{dU}{dY} - \frac{1}{2}k^3[2(1-Y^2)(3Y^2-1)\frac{dU}{dY} - 6Y(1-Y^2)^2\frac{d^2U}{dY^2} + (1-Y^2)^3\frac{d^3U}{dY^3}]$$
$$+3kU^2(1-Y^2)\frac{dU}{dY} - \frac{3}{2}k^2(1-Y^2)[(-2Y\frac{dV}{dY} + (1-Y^2)\frac{d^2V}{dY^2})] - 3kU(1-Y^2)\frac{dV}{dY} \tag{3.19}$$
$$-3kV(1-Y^2)\frac{dU}{dY} + 3\alpha k(1-Y^2)\frac{dU}{dY} = 0$$

the second equation in (3.18) reduces to

$$-\lambda k(1-Y^2)\frac{dV}{dY} + k^3[2(1-Y^2)(3Y^2-1)\frac{dV}{dY} - 6Y(1-Y^2)^2\frac{d^2V}{dY^2} + (1-Y^2)^3\frac{d^3V}{dY^3}]$$
$$+3kV(1-Y^2)\frac{dU}{dY} - 3k^2(1-Y^2)^2\frac{dV}{dY}\frac{dU}{dY} - 3kU^2(1-Y^2)\frac{dV}{dY} - 3\alpha k(1-Y^2)\frac{dU}{dY} = 0 \tag{3.20}$$

Now, to determine the parameters m and n, we balance the linear term of highest-order with the highestordernonlinear terms. So, in Eq. (3.19) we balance $U^{///}$ with $UV^{/}$, to obtain

6 + *m* – 3 = 2 + *m* + *n* – 1, then *n* = 2.

while in Eq. (3.20) we balance $V^{///}$ with $UV^{/}$, to obtain

6 + *n* – 3 = 4 + *n* – 1 + *m* – 1, then *m* = 1.

The tanh method admits the use of the finite expansion for both

$$u(x,t) = U(Y) = a_0 + a_1 Y \tag{3.21}$$

and

$$v(x,t) = V(Y) = b_0 + b_1 Y + b_2 Y^2 \tag{3.22}$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}, V^{//}, V^{///}$ from Eq. (3.21) and Eq. (3.22) respectively into Eq. (3.19), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned} &Y^0 : -\lambda a_1 + k^2 a_1 + 3a_1 a_0^2 - 3kb_2 - 3a_0 b_1 - 3b_0 a_1 + 3\alpha a_1 = 0 \\ &Y^1 : 2a_0 a_1^2 + b_1 k - 2a_0 b_2 - 2a_1 b_1 = 0 \end{aligned} \tag{3.23}$$

$$Y^2 : -k^2 a_1 + a_1^3 + 3kb_2 - 3a_1 b_2 = 0 \tag{3.23}$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}, V^{//}, V^{///}$ from Eq. (3.21) and Eq. (3.22) respectively into Eq. (3.20), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$\begin{aligned} &Y^0 : -\lambda b_1 - 2k^2 b_1 + 3a_1 b_0 - 3ka_1 b_1 - 3b_1 a_0^2 - 3\alpha a_1 = 0 \\ &Y^1 : -2\lambda b_2 - 4k^2 b_2 - 12k^2 b_2 + 3a_1 b_1 - 6ka_1 b_2 - 6b_1 a_0 a_1 - 6b_2 a_0^2 = 0 \\ &Y^2 : 2k^2 b_1 + a_1 b_2 + ka_1 b_1 - b_1 a_1^2 - 4b_2 a_0 a_1 = 0 \\ &Y^3 : (4k^2 + ka_1 - a_1^2) b_2 = 0 \end{aligned} \tag{3.24}$$

Solving the nonlinear systems of equations (3.23) and (3.24) with help of Mathematica we can get:

$$a_0 = \frac{1}{4} \text{ , } a_1 = \frac{k}{2}(1 \pm \sqrt{17}), \ b_0 = \alpha \ , \ b_1 = 0, \ b_2 = \frac{k^2}{2}(9 \pm \sqrt{17}),$$

$$\lambda = \frac{-k^2}{2}(1 \pm 3\sqrt{17}) + \frac{3}{16}.$$

Then:

$$\mathrm{u(x,t)} = \frac{1}{4} + \frac{\mathrm{k}}{2}(1 \pm \sqrt{17}) \tanh(\mathrm{k}\,(\mathrm{x} - \lambda \mathrm{t})) \tag{3.25}$$

and

$$\mathrm{v(x,t)} = \alpha + \frac{\mathrm{k}^2}{2}(9 \pm \sqrt{17}) \tanh^2(\mathrm{k}\,(\mathrm{x} - \lambda \mathrm{t})) \tag{3.26}$$

The solitary wave and behavior of the solutions $u(x,t)$ and $v(x,t)$ are shown in Figs. 3.3 and 3.4 for some fixed values of the parameters ($\alpha = 1.0$, $k = 1.0$).

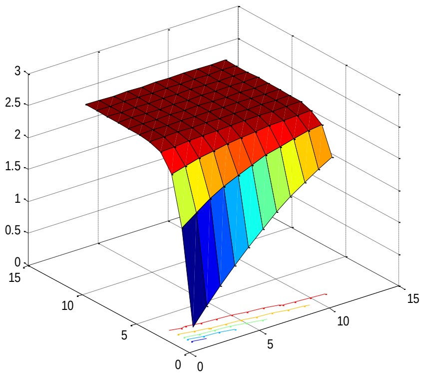

Figure 3.3. The solitary wave of $u(x,t)$

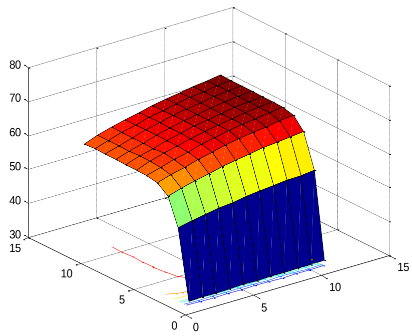

Fig. 3.4. The solitary wave of $v(x,t)$

4. CONCLUSION

We harnessed the formidable tanh method for the analytical treatment of nonlinear coupled partial differential equations. Utilizing transformation formulas, we derived solutions for traveling waves and kinks. Notably, our solutions for coupled KdV systems of PDEs (3.1) and (3.14) are in concordance with the findings of Sayed Tauseef [15], who employed the variational iteration method. This alignment underscores the reliability and effectiveness of the tanh method in our study.

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