ORIGINAL PAPER

ON TWO NEW IDENTITIES CONCERNING THE Q-INTEGERS AND SEVERAL DIVISIBILITY PROPERTIES FOR THESE INTEGERS

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Abstract. *In this paper, we first prove two new identities concerning the q-integers and then for these integers we derive several divisibility properties based on these identities and on properties of the q-integers.*

Keywords: *Q-calculus; Q-integers; divisibility; Euclidean algorithm; greatest common divisors.*

1. INTRODUCTION

Quantum calculus or q-calculus receiving significant attention from researchers play an important role in various applications in sciences and technology (see, [1-6]). Several books already present general survey for quantum calculus as [7-10] for the more recent ones.

If $0 < m \in \mathbb{Z}$ and $0 < q \in \mathbb{Z}$, the q-integer $[m]_q$, which is the q-analog of m, is defined by

$$[m]_q = \sum_{i=0}^{m-1} q^i.$$

By convention, the empty sum is zero and therefore, we have

$$[0]_q = 0, \quad [1]_q = 1, \quad [2]_q = 1+q, \quad \ldots, \quad [m]_q = 1+q+\cdots+q^{m-1}, \quad \ldots.$$

If $m \neq 0$, the q-analog of $-m$ is defined as

$$[-m]_q = -\sum_{i=1}^{m} q^{-i}$$

and therefore, we have

$$[-1]_q = -\frac{1}{q}, \quad [-2]_q = -\frac{1}{q} - \frac{1}{q^2} = -\frac{1+q}{q^2}, \quad \ldots,$$

and

$$[-m]_q = -\frac{1}{q} - \cdots - \frac{1}{q^m} = -\frac{1+q+\cdots+q^{m-1}}{q^m}, \quad \ldots.$$

In the following we have a short review about some applications of the q-integers. In Section 1 of [3], Bernard and Guirós study how the choice of the data will affect the behavior of the quantum integers. The q-integers are used to develop q-combinatorics. In Section 2 of

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[3], Bernard and Quirós state and prove some classical results on binomial coefficients with a special emphasis on Lucas formula. In the book by Kac and Cheung is shown that the q-analog of a binomial coefficient will count the number of rational points of the corresponding Grassmannian over a finite field with q elements [9, Theorem 7.1]. Also, in Section 1.8 in [11] and Chapter 3 of [12], we can see that the q-analogs of the binomial coefficients are given by the Gaussian polynomials.

The study of the various properties of the natural integers is often based on their arithmetic properties such as periodicity, congruences, and divisibility. Since we will be dealing with divisibility, let us remind ourselves of a few basic facts, all of which can be found in [13]:

i. Let a be any integer and b be a positive integer. Then there exist unique integers q and r such that $a = b \cdot q + r$ where $0 \le r < b$ [13, p.69].

ii. If $a = bq$, we say that b divides a and write $b | a$. If b is not a factor of a, we write $b \nmid a$ [13, p.74].

iii. Let a and b be positive integers such that $a | b$ and $b | a$. Then $a = b$ [13, p.75].

iv. Let a, b, c, α and β be any integers. If $a | b$ and $b | c$, then $a | c$. If $a | b$ and $a | c$, then $a | \left(ab + \beta c\right)$. If $a | b$, then $a | bc$ [13, p.75].

v. A positive integer d is the gcd of a and b if $d | a$ and $d | b$; and if $d' | a$ and $d' | b$, then $d' | d$, where d' is a positive integer [13, p.156].

vi. If $d = \left(a, b\right)$ and d' is any common divisor of a and b, then $d' | d$ [13, p.160].

In this paper, we first introduce new identities for the q-integers and then for these integers we derive several divisibility properties based on these identities and on properties of the q-integers.

2. MAIN RESULTS

The purpose of this section is to present two new identities concerning the q-integers and many divisibility properties of the q-integers. First we give an identity theorem for the q-integers.

Theorem 2.1. For $m, n \ge 0$ and $q > 1$, we have

$$[m+n+1]_q = [m+1]_q [n+1]_q - q[m]_q [n]_q. \tag{1}$$

Proof: Let *LHS* and *RHS* be $LHS = [m+n+1]_q$ and $RHS = \left[m+1\right]_q \left[n+1\right]_q - q\left[m\right]_q \left[n\right]_q$. We simplify the *RHS* and show that it equals the *LHS*. From the definition of q-integer, we get the following result for the *RHS*:

$$\begin{aligned} RHS &= \frac{q^{m+1}-1}{q-1}\frac{q^{n+1}-1}{q-1} - q\frac{q^m-1}{q-1}\frac{q^n-1}{q-1} \\ &= \frac{1}{\left(q-1\right)^2}\left[q^{m+n+2} - q^{m+1} - q^{n+1} + 1 - q\left(q^{m+n} - q^m - q^n + 1\right)\right] \end{aligned}$$

$$= \frac{1}{(q-1)^2}\left[q^{m+n+1}(q-1)-(q-1)\right]$$
$$= \frac{q^{m+n+1}-1}{q-1}$$
$$= [m+n+1]_q$$
$$= LHS.$$

This proves (1), which concludes the proof of this theorem. In order to establish the gcd Property $([n]_q,[n+1]_q)=1$ for the q-integers, we need the following corollary and lemma. The next corollary follows by Theorem 2.1.

Corollary 2.1. For $k \geq 1$ and $q > 1$, we have

$$[k+1]_q = (1+q)[k]_q - q[k-1]_q. \tag{2}$$

Proof: Suppose we let $m = k-1$ and $n = 1$ in (1). Then formula (1) yields

$$[k+1]_q = (1+q)[k]_q - q[k-1]_q.$$

as desired.

Lemma 2.1. For $n \geq 1$ and $q > 1$, we have $\left(-q,[n]_q\right)=1$.

Proof: The assertion is clearly true for $n=1$ since $\left(-q,[1]_q\right)=\left(-q,1\right)=1$. Assume that it is true for any fixed integer $k \geq 1$. This gives us $\left(-q,[k]_q\right)=1$. Turning to the case where $n=k+1$, then for $d=\left(-q,[k+1]_q\right)$ we have $d \mid -q$ and $d \mid [k+1]_q$. Since $d \mid -q$, we have $d \mid -q^k$. On the other hand, we know that

$$[k+1]_q = 1+q+q^2+\ldots+q^k.$$

Thus, if $d \mid -q^k$ and $d \mid [k+1]_q$, it follows that

$$d \mid -q^k + [k+1]_q = 1+q+q^2+\ldots+q^{k-1}$$

and therefore

$$d \mid [k]_q.$$

Then, $d \mid -q$ and $d \mid [k]_q$, it follows that $d \mid 1$. Hence, since d is positive we conclude that $d=1$. This proves the inductive step and so completes the proof.

Theorem 2.2. For $n \geq 1$ and $q > 1$, we have $\left([n]_q,[n+1]_q\right)=1$.

Proof: The result is trivially true for $n=1$ and $n=2$ since

$$\left([1]_q,[2]_q\right)=\left(1,1+q\right)=1$$

and

$$\left([2]_q,[3]_q\right)=\left(1+q,1+q+q^2\right)=1.$$

We now assume that the result is true for $n=k-1$ where k is any fixed integer and $k\geq 2$. This gives us $\left([k-1]_q,[k]_q\right)=1$. Let $d=\left([k]_q,[k+1]_q\right)$. Then $d\mid [k]_q$ and $d\mid [k+1]_q$. Since $d\mid [k]_q$, it follows from (2) that $d\mid -q[k-1]_q$. On the other hand, by Lemma 2.1, we obtain

$$\begin{aligned}\left(-q,d\right)&=\left(-q,\left([k]_q,[k+1]_q\right)\right)\\&=\left(\left(-q,[k]_q\right),[k+1]_q\right)\\&=\left(1,[k+1]_q\right)\\&=1.\end{aligned}$$

Since $d\mid -q[k-1]_q$ and $\left(-q,d\right)=1$, it follows that $d\mid [k-1]_q$. Since $d\mid [k-1]_q$ and $d\mid [k]_q$ by the inductive hypothesis, this clearly implies that $d\mid 1$. Since d is positive, we reach $d=1$. Consequently, we see that $\left([n]_q,[n+1]_q\right)=1$ for all $n\geq 1$. This completes the proof.

The following result addresses the divisibility of a q-integer by another q-integer.

Theorem 2.3. For $n\geq 1$ and $q>1$, then $[m]_q\mid [n]_q$ if and only if $m\mid n$.

Proof: The proof consists of two parts:

i. If $m\mid n$, then $[m]_q\mid [n]_q$.

ii. If $[m]_q\mid [n]_q$, then $m\mid n$.

We will prove each part one by one in that order.

i. To prove that if $m\mid n$, then $[m]_q\mid [n]_q$: Suppose $m\mid n$. Then $n=mt$ for some integer t. We will carry out the proof by induction on t. Since $[m]_q\mid [m]_q$, the given statement is clearly true when $t=1$. Now assume it is true for every positive integer $t\leq k$, where $k\geq 1$. Consider the integer $k+1$. Then, from the identity (1) we see that

$$\begin{aligned}[(k+1)m]_q&=[km+m]_q\\&=[km-1+m+1]_q\\&=[km]_q[m+1]_q-q[km-1]_q[m]_q.\end{aligned}\tag{3}$$

On the other hand, by the inductive hypothesis, we know that $[m]_q\mid [km]_q$. Since $[m]_q\mid [km]_q$ and $[m]_q\mid [m]_q$, we conclude that

$$[m]_q\mid\left([km]_q[m+1]_q-q[km-1]_q[m]_q\right)$$

and from (3)

$$[m]_q\mid [(k+1)m]_q.$$

Thus, we reach that if $m\mid n$, then $[m]_q\mid [n]_q$.

ii. To prove that if $[m]_q \mid [n]_q$, then $m \mid n$: Suppose $[m]_q \mid [n]_q$. Then there exist unique integers b and r such that $n = mb + r$, where $0 \le r < m$. Then, from the identity (1) we obtain that

$$\begin{aligned} [n]_q &= [mb+r]_q \\ &= [mb+r-1+1]_q \\ &= [mb+1]_q [r]_q - q[mb]_q [r-1]_q . \end{aligned} \tag{4}$$

Since $m \mid mb$, by the first part of the proof, this implies that $[m]_q \mid [mb]_q$. Since $[m]_q \mid [n]_q$ and $[m]_q \mid [mb]_q$, we get

$$[m]_q \mid \left([n]_q + q[mb]_q [r-1]_q\right)$$

and from (4)

$$[m]_q \mid [mb+1]_q [r]_q .$$

Also, $\left([m]_q, [mb+1]_q\right) = 1$ follows from the fact that $[m]_q \mid [mb]_q$ and $\left([mb]_q, [mb+1]_q\right) = 1$ by Theorem 2.2. The fact that $[m]_q \mid [mb+1]_q [r]_q$ and $\left([m]_q, [mb+1]_q\right) = 1$ tells us that $[m]_q \mid [r]_q$. Thus, $[m]_q$ is smaller than $[r]_q$. This contradicts our choice of r, so $r < m$. Therefore, we obtain $r = 0$ and $m \mid n$. This concludes the proof of Theorem 2.3.

Theorem 2.4. For $m, n \ge 1$ and $q > 1$, we have $\left([m]_q, [n]_q\right) = \left[(m,n)\right]_q$.

Proof: Let $d_1 = \left([m]_q, [n]_q\right)$ and $d_2 = \left[(m,n)\right]_q$. Since $(m,n) \mid m$ and $(m,n) \mid n$, this implies by Theorem 2.3 that $\left[(m,n)\right]_q \mid [m]_q$ and $\left[(m,n)\right]_q \mid [n]_q$, respectively. Therefore, we obtain $\left[(m,n)\right]_q \mid \left([m]_q, [n]_q\right)$ and $d_2 \mid d_1$. Since the gcd of the positive integers m and n is a linear combination of m and n, there exist integers a and b such that $(m,n) = ma + nb$. Then, we get

$$\begin{aligned} [ma]_q &= [(m,n) - nb]_q \\ &= [(m,n) - 1 - nb + 1]_q . \end{aligned} \tag{5}$$

Thus, it follows from (1) and (5) that

$$[ma]_q = [(m,n)]_q [-nb+1]_q - q[(m,n)-1]_q [-nb]_q . \tag{6}$$

Since $m \mid ma$ and $n \mid -nb$, we obtain by Theorem 2.3 $[m]_q \mid [ma]_q$ and $[n]_q \mid [-nb]_q$, respectively. Since $d_1 \mid [m]_q$ and $[m]_q \mid [ma]_q$, this gives us $d_1 \mid [ma]_q$. Similarly, since $d_1 \mid [n]_q$ and $[n]_q \mid [-nb]_q$, this gives us $d_1 \mid [-nb]_q$. Therefore, from (6) we reach

$$d_1 \mid [ma]_q + q[(m,n)-1]_q [-nb]_q$$

and hence

$$d_1 \mid [(m,n)]_q[-nb+1]_q.$$

Since $d_1 \mid [-nb]_q$ and $\left([-nb]_q,[-nb+1]_q\right)=1$ by Theorem 2.2, this means that $\left(d_1,[-nb+1]_q\right)=1$. Thus since $\left(d_1,[-nb+1]_q\right)=1$ and $d_1 \mid [(m,n)]_q[-nb+1]_q$, we have $d_1 \mid [(m,n)]_q$ and $d_1 \mid d_2$. Since $d_1 \mid d_2$ and $d_2 \mid d_1$, we conclude that $d_1 = d_2$.

4. CONCLUSION

In this paper, we have introduced two new identities for the q-integers and derived several divisibility properties based on these identities and on properties of the $q-$ integers for these integers.

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