ORIGINAL PAPER

ON THE (p, q) −NARAYANA n −DIMENSIONAL RECURRENCES

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Abstract. *In this study, a different perspective was brought to Narayana sequences and one-, two-, three- and n −dimensional recurrence relations of these sequences were created. Then, some identities ranging from one to n −dimensions of these recurrences were created.*

Keywords: *Narayana sequence; Hosoya's triangle; dimensional; complex number.*

1. INTRODUCTION

In this study, a new recurrence relation has been developed with a new approach to Narayana numbers in the complex plane. This study also has similar aspects with the Hosoya's triangle [1], which was also discussed by Hosoya in 1976. The Hosoya's triangle was later taken by the two-dimensional recurrence relation of the Fibonacci triangle also two- three-, …, n −dimensional recurrence relation of Jacobsthal numbers is handled [2]. Then these studies expanded to number sequences such as Padovan and Narayana [3, 4], which consist of three recurrences.

In this study, the focus is on the characteristic evaluation of the Narayana recurrence relation in the set of complex numbers in Gaussian integers in the horizontal triplet of any adjacent points. If we emphasize the recurrence relation formed in these numbers in more detail, it is said that the recurrence relation formed on the Narayana sequence has the symmetrical condition occurring in the horizontal and vertical triples of adjacent points.

Certain recurrence equations satisfied by the new numbers are outlined, and using them, some interesting new Narayana identities are readily obtained. Finally, it is shown that the numbers generalize in a natural manner to higher dimensions. The Narayana sequence N_n is defined by the recurrence relation

$$N_{n+3} = N_{n+2} + N_n,\ N_0 = 0, N_1 = 1, N_2 = 1 \text{ [5].}$$

There are many number sequences such as Padovan, Perrin, Leonardo and Tribonacci, which consist of three recurrences such as Narayana sequences. In addition to these number sequences, the Narayana number sequence also has many generalizations. One of these generalizations is the (p, q) −Narayana sequences.

For $n \in \mathbb{N}$ and since $-27q^2 - 4p^3q$ which is discriminant of third-degree equations, defined in every case. The (p, q) −Narayana sequence is defined as

$$N_{n+3}(p, q) = pN_{n+2}(p, q) + qN_n(p, q), N_0(p, q) = 0, N_1(p, q) = 1, N_2(p, q) = 1 \text{ [6].}$$

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We can write the first few (p,q) −Narayana sequences as follows:

$$N_3(p,q) = p, N_4(p,q) = p^2 + q, N_5(p,q) = p^3 + pq + q,$$
$$N_6(p,q) = p^4 + p^2q + 2pq, N_7(p,q) = p^5 + p^3q + 3p^2q + q^2, \dots$$

The number $G(n,m)$ is defined in set of Gauss integers as follow [7] Gauss integer

$$(n,m) = n + im;\ \ n,m \in \mathbb{Z}.$$

The $G(n,m)$ satisfies the following two-dimensional recurrence as follow:

$$G(n+2,m) = G(n+1,m) + G(n,m)$$
$$G(n,m+2) = G(n,m+1) + G(n,m)$$

where $G(0,0) = 0, G(1,0) = 1, G(0,1) = i, G(1,1) = 1 + i$ are conditions values. In addition to the dimensional studies in 2nd degree recurrences, some of the 3rd degree recurrence studies were handled on Perrin and Padovan sequences in [8, 9]. For more information, see all other properties and identities of the three and n-dimensional recurrences that similar techniques are used in [7, 10-12].

2. NEW RECURRENCES OBTAINED BY TWO-DIMENSIONAL APPROACH TO $(\boldsymbol{p},\boldsymbol{q})$ −NARAYANA SEQUENCE

In this section, the two-dimensional approach relation of Narayana sequences in the complex plane will be discussed and also important identities related to them will be given.

Definition 2.1. For $m,n \in \mathbb{N}$ and $-27q^2 - 4p^3q$ is defined, two-dimensional of Narayana sequence $N_{p,q}^{n,m}$ is defined as follow:

$$N_{p,q}^{n+3,m} = pN_{p,q}^{n+2,m} + qN_{p,q}^{n,m}$$

$$N_{p,q}^{n,m+3} = pN_{p,q}^{n,m+2} + qN_{p,q}^{n,m}$$

with initial conditions

$$N_{p,q}^{0,0} = 0, N_{p,q}^{1,0} = 1, N_{p,q}^{2,0} = 1$$

$$N_{p,q}^{0,1} = i, N_{p,q}^{1,1} = 1 + i, N_{p,q}^{2,1} = 1 + pi$$

$$N_{p,q}^{0,2} = i, N_{p,q}^{1,2} = p + i, N_{p,q}^{2,2} = p + pi$$

Proposition 2.2. The following equality holds:

- **a)** $N_{p,q}^{n,0} = N_n(p,q)$
- **b)** $N_{p,q}^{0,m} = iN_m(p,q)$
- **c)** $N_{p,q}^{1,m} = N_{m+1}(p,q) + iN_m(p,q)$
- **d)** $N_{p,q}^{n,1} = N_n(p,q) + iN_{n+1}(p,q)$
- **e)** $N_{p,q}^{n,m} = N_n(p,q)N_{m+1}(p,q) + iN_{n+1}(p,q)N_m(p,q)$

Proof:
a) By the recurrence

$$N_{p,q}^{n+3,m} = pN_{p,q}^{n+2,m} + qN_{p,q}^{n,m}, m = 0$$

and mathematical induction, we can prove this equality.

For $n = 0$, $N_{p,q}^{0,0} = N_0(p,q) = 0$. Suppose that for $k + 3 \leq n$ equality is true. So that

$$N_{p,q}^{k+3,0} = N_{k+3}(p,q)$$

Let's show that it is true for $k + 4$.

$$N_{p,q}^{k+4,0} = pN_{p,q}^{k+3,0} + qN_{p,q}^{k+1,0} = pN_{k+3}(p,q) + qN_{k+1}(p,q) = N_{k+4}(p,q).$$

From definition of (p,q) −Narayana sequences, this equation holds.

b) By the recurrence

$$N_{p,q}^{n,m+3} = pN_{p,q}^{n,m+2} + qN_{p,q}^{n,m}, n = 0$$

and mathematical induction, we can prove this equality. For $m = 0$, $N_{p,q}^{0,0} = iN_0(p,q) = 0$. Suppose that for $k + 3 \leq m$ equality is true. So that

$$N_{p,q}^{0,k+3} = N_{k+3}(p,q)$$

Let's show that it is true for $k + 4$.

$$N_{p,q}^{0,k+4} = pN_{p,q}^{0,k+3} + qN_{p,q}^{0,k+1} = ipN_{k+3}(p,q) + iqN_{k+1}(p,q) = iN_{k+4}(p,q).$$

From definition of (p,q) −Narayana sequences, this equation holds.

c) By the recurrence

$$N_{p,q}^{n,m+3} = pN_{p,q}^{n,m+2} + qN_{p,q}^{n,m}, n = 1$$

and mathematical induction, we can prove this equality.

For $m = 0$, $N_{p,q}^{1,0} = N_1(p,q) + iN_0(p,q) = 1$. Suppose that for $k + 2 \leq m$ equality is true. So that

$$N_{p,q}^{1,k+2} = N_{k+3}(p,q) + iN_{k+2}(p,q)$$

Let's show that it is true for $k + 3$

$$\begin{aligned} N_{p,q}^{1,k+3} &= pN_{p,q}^{1,k+2} + qN_{p,q}^{1,k} \\ &= p\big(N_{k+3}(p,q) + iN_{k+2}(p,q)\big) + q(N_{k+1}(p,q) + iN_k(p,q)) \\ &= pN_{k+3}(p,q) + qN_{k+1}(p,q) + i(pN_{k+2}(p,q) + qN_k(p,q)) \\ &= N_{k+4}(p,q) + iN_{k+3}(p,q) \end{aligned}$$

From definition of (p,q) −Narayana sequences, this equation holds.

d) By the recurrence

$$N_{p,q}^{n+3,m} = pN_{p,q}^{n+2,m} + qN_{p,q}^{n,m}, m = 1$$

and mathematical induction, we can prove this equality.

For $n = 0$,

$$N_{p,q}^{0,1} = N_0(p,q) + iN_1(p,q) = i.$$

Suppose that for $\mathrm{k} + 2 \leq \mathrm{n}$ equality is true. So that

$$N_{p,q}^{k+2,1} = N_{k+2}(p,q) + iN_{k+3}(p,q)$$

Let's show that it is true for $k+3$

$$\begin{gathered}N_{p,q}^{k+3,1} = pN_{p,q}^{k+2,1} + qN_{p,q}^{k,1}\\ = p\big(N_{k+2}(p,q) + iN_{k+3}(p,q)\big) + q(N_k(p,q) + iN_{k+1}(p,q))\\ = pN_{k+2}(p,q) + qN_k(p,q) + i(pN_{k+3}(p,q) + qN_{k+1}(p,q))\\ = N_{k+3}(p,q) + iN_{k+4}(p,q)\end{gathered}$$

From definition of (p,q) −Narayana sequences, this equation holds.

e) By the recurrence

$$N_{p,q}^{n+3,m} = pN_{p,q}^{n+2,m} + qN_{p,q}^{n,m}$$

and

$$N_{p,q}^{n,m+3} = pN_{p,q}^{n,m+2} + qN_{p,q}^{n,m}$$

For $m=0$,

$$N_{p,q}^{n,0} = N_n(p,q)N_1(p,q) + iN_{n+1}(p,q)N_0(p,q) = N_n(p,q)$$

and $n=0$

$$N_{p,q}^{0,m} = N_0(p,q)N_{m+1}(p,q) + iN_1(p,q)N_m(p,q) = iN_m(p,q) =$$

the equalities are true.

Suppose that for $k+2 \le n$ and $k+2 \le m$ equalities are true. So that

$$N_{p,q}^{k+2,m} = N_{k+2}(p,q)N_{m+1}(p,q) + iN_{k+3}(p,q)N_m(p,q)$$

and

$$N_{p,q}^{n,k+2} = N_n(p,q)N_{k+3}(p,q) + iN_{n+1}(p,q)N_{k+2}(p,q).$$

Let's show that it is true for $n = k+3$ and $m = k+3$ respectively.

$$\begin{gathered}N_{p,q}^{k+3,m} = pN_{p,q}^{k+2,m} + qN_{p,q}^{k,m}\\ = p\big(N_{k+2}(p,q)N_{m+1}(p,q) + iN_{k+3}(p,q)N_m(p,q)\big)\\ + q\big(N_k(p,q)N_{m+1}(p,q) + iN_{k+1}(p,q)N_m(p,q)\big)\\ = N_{m+1}(p,q)\big(pN_{k+2}(p,q) + qN_k(p,q)\big) + iN_m(p,q)(pN_{k+3}(p,q) + qN_{k+1}(p,q))\\ = N_{m+1}(p,q)N_{k+3}(p,q) + iN_m(p,q)N_{k+4}(p,q)\end{gathered}$$

and

$$\begin{gathered}N_{p,q}^{n,k+3} = pN_{p,q}^{n,k+2} + qN_{p,q}^{n,k}\\ = p\big(N_n(p,q)N_{k+3}(p,q) + iN_{n+1}(p,q)N_{k+2}(p,q)\big) + q(N_n(p,q)N_{k+1}(p,q)\\ + iN_{n+1}(p,q)N_k(p,q))\\ = N_n(p,q)\big(pN_{k+3}(p,q) + qN_{k+1}(p,q)\big) + iN_{n+1}(p,q)(pN_{k+2}(p,q) + qN_k(p,q))\\ = N_n(p,q)N_{k+4}(p,q) + iN_{n+1}(p,q)N_{k+3}(p,q)\end{gathered}$$

is desired.

3. NEW RECURRENCES OBTAINED BY THREE-DIMENSIONAL APPROACH TO $(\boldsymbol{p},\boldsymbol{q})$ −NARAYANA SEQUENCE

In this section, the three-dimensional approach relation of Narayana sequences in the complex plane will be discussed and also important identities related to them will be given.

Definition 3.1. For $m, n, k \in \mathbb{N}$ and $-27q^2 - 4p^3q$ is defined, three-dimensional Narayana sequence $N_{p,q}^{n,m,k}$ is defined as follow:

$$N_{p,q}^{n+3,m,k} = pN_{p,q}^{n+2,m,k} + qN_{p,q}^{n,m,k}$$

$$N_{p,q}^{n,m+3,k} = pN_{p,q}^{n,m+2,k} + qN_{p,q}^{n,m,k}$$

$$N_{p,q}^{n,m,k+3} = pN_{p,q}^{n,m,k+2} + qN_{p,q}^{n,m,k}$$

with initial conditions

$N_{p,q}^{0,0,0} = 0, N_{p,q}^{1,0,0} = 1, N_{p,q}^{0,1,0} = i, N_{p,q}^{0,0,1} = j$
$N_{p,q}^{1,1,0} = 1 + i, N_{p,q}^{1,0,1} = 1 + j, N_{p,q}^{0,1,1} = i + j, N_{p,q}^{1,1,1} = 1 + i + j$
$N_{p,q}^{2,0,0} = 1, N_{p,q}^{0,2,0} = i, N_{p,q}^{0,0,2} = j$
$N_{p,q}^{2,1,0} = 1 + pi, N_{p,q}^{0,2,1} = i + pj, N_{p,q}^{2,0,1} = 1 + pj$
$N_{p,q}^{1,2,0} = p + i, N_{p,q}^{0,1,2} = pi + j, N_{p,q}^{1,0,2} = p + j$
$N_{p,q}^{2,1,1} = 1 + pi + pj, N_{p,q}^{1,2,1} = p + i + pj, N_{p,q}^{1,1,2} = p + pi + j$
$N_{p,q}^{2,2,1} = p + pi + p^2j, N_{p,q}^{1,2,2} = p^2 + pi + pj, N_{p,q}^{2,1,2} = p + p^2i + pj$
$N_{p,q}^{2,2,0} = p + pi, N_{p,q}^{0,2,2} = pi + pj, N_{p,q}^{2,0,2} = p + pj, N_{p,q}^{2,2,2} = p^2 + p^2i + p^2j$

Proposition 3.2. The following equality holds.
a) $N_{p,q}^{n,0,0} = N_n(p,q)$,
b) $N_{p,q}^{0,m,0} = iN_m(p,q)$,
c) $N_{p,q}^{0,0,k} = jN_k(p,q)$,
d) $N_{p,q}^{n,1,0} = N_n(p,q) + iN_{n+1}(p,q)$,
e) $N_{p,q}^{n,0,1} = N_n(p,q) + jN_{n+1}(p,q)$,
f) $N_{p,q}^{n,1,1} = N_n(p,q) + iN_{n+1}(p,q) + jN_{n+1}(p,q)$,
g) $N_{p,q}^{1,m,0} = N_{m+1}(p,q) + iN_m(p,q)$,
h) $N_{p,q}^{0,m,1} = iN_m(p,q) + jN_{m+1}(p,q)$,
i) $N_{p,q}^{1,m,1} = N_{m+1}(p,q) + jN_m(p,q) + jN_{m+1}(p,q)$,
j) $N_{p,q}^{1,0,k} = N_{k+1}(p,q) + jN_k(p,q)$,
k) $N_{p,q}^{0,1,k} = iN_{k+1}(p,q) + jN_k(p,q)$,
l) $N_{p,q}^{1,1,k} = N_{k+1}(p,q) + iN_{k+1}(p,q) + jN_k(p,q)$,
m) $N_{p,q}^{n,m,0} = N_n(p,q)N_{m+1}(p,q) + iN_{n+1}(p,q)N_m(p,q)$,
n) $N_{p,q}^{n,0,k} = N_n(p,q)N_{k+1}(p,q) + jN_{n+1}(p,q)N_k(p,q)$,
o) $N_{p,q}^{0,m,k} = iN_m(p,q)N_{k+1}(p,q) + jN_{m+1}(p,q)N_k(p,q)$,
p) $N_{p,q}^{n,m,1} = N_n(p,q)N_{m+1}(p,q) + iN_{n+1}(p,q)N_m(p,q) + jN_{n+1}(p,q)N_{m+1}(p,q)$,
r) $N_{p,q}^{n,1,k} = N_n(p,q)N_{k+1}(p,q) + iN_{n+1}(p,q)N_{k+1}(p,q) + jN_{n+1}(p,q)N_k(p,q)$,
s) $N_{p,q}^{1,m,k} = N_{m+1}(p,q)N_{k+1}(p,q) + iN_m(p,q)N_{k+1}(p,q) + jN_{m+1}(p,q)N_k(p,q)$,
t) $N_{p,q}^{n,m,k} = N_n(p,q)N_{m+1}(p,q)N_{k+1}(p,q) + iN_{n+1}(p,q)N_m(p,q)N_{k+1}(p,q) + jN_{n+1}(p,q)N_{m+1}(p,q)N_k(p,q)$.

Proof: The proofs are easily shown in mathematical induction as in Proposition 2.2. The fourth-dimensional recurrences of the (p,q) −Narayana sequence are called quaternions. Similar studies can be found in [13-18].

4. NEW RECURRENCES OBTAINED BY *n*-DIMENSIONAL APPROACH TO (*p*,*q*)−NARAYANA SEQUENCE

In this section, the n-dimensional approach relation of Narayana sequences in the complex plane will be discussed and also important identities related to them will be given.

Definition 4.1. For $k_0,k_1,k_2,\dots,k_{n-1}\in\mathbb{N}$ and $-27q^2-4p^3q$, is defined, n −dimensional Narayana sequence $N_{p,q}^{k_0,k_1,k_2,\dots,k_{n-1}}$ is defined as follow:

$$N_{p,q}^{k_0+3,k_1,k_2,\dots,k_{n-1}}=pN_{p,q}^{k_0+2,k_1,k_2,\dots,k_{n-1}}+qN_{p,q}^{k_0,k_1,k_2,\dots,k_{n-1}},$$

$$N_{p,q}^{k_0,k_1+3,k_2,\dots,k_{n-1}}=pN_{p,q}^{k_0,k_1+2,k_2,\dots,k_{n-1}}+qN_{p,q}^{k_0,k_1,k_2,\dots,k_{n-1}},$$

$$\vdots$$

$$N_{p,q}^{k_0+3,k_1,k_2,\dots,k_{n-1}+3}=pN_{p,q}^{k_0+2,k_1,k_2,\dots,k_{n-1}+2}+qN_{p,q}^{k_0,k_1,k_2,\dots,k_{n-1}}$$

with initial condition

$N_{p,q}^{0,0,0,\dots,0}=0, N_{p,q}^{1,0,0,\dots,0}=1, N_{p,q}^{0,1,0,\dots,0}=c_1,\dots,N_{p,q}^{0,0,0,\dots,1}=c_{n-1},$
$N_{p,q}^{1,1,1,\dots,1}=1+c_1+c_2+c_3+\cdots+c_{n-1},$
$N_{p,q}^{2,0,0,\dots,0}=1, N_{p,q}^{0,2,0,\dots,0}=c_1,\dots,N_{p,q}^{0,0,0,\dots,2}=c_{n-1},$
$N_{p,q}^{2,2,2,\dots,2}=p^{n-1}+p^{n-1}c_1+p^{n-1}c_2+\cdots+p^{n-1}c_{n-1},$
$N_{p,q}^{2,2,2,\dots,2,0}=p^{n-2}+p^{n-2}c_1+p^{n-2}c_2+\cdots+p^{n-2}c_{n-2},$
$N_{p,q}^{2,2,2,\dots,2,0,0}=p^{n-3}+p^{n-3}c_1+p^{n-3}c_2+\cdots+p^{n-3}c_{n-3}$
$N_{p,q}^{2,1,1,1,\dots,1}=1+pc_1+pc_2+\cdots+pc_{n-1},$
$N_{p,q}^{1,2,1,1,\dots,1}=p+c_1+pc_2+\cdots+pc_{n-1},$
$N_{p,q}^{1,1,1,\dots,1,2}=p+pc_1+pc_2+\cdots+pc_{n-2}+c_{n-1},$
$N_{p,q}^{2,2,1,1,\dots,1}=p+pc_1+p^2c_2+p^2c_3+\cdots+p^2c_{n-1},$
$N_{p,q}^{1,1,1,\dots,1,2,2}=p^2+p^2c_1+p^2c_2+p^2c_3+\cdots+p^2c_{n-3}+pc_{n-2}+pc_{n-1},$
$N_{p,q}^{1,1,0,\dots,0,0}=1+c_1,$
$N_{p,q}^{0,0,0,\dots,0,1,1}=c_{n-2}+c_{n-1},$
$N_{p,q}^{1,1,1,0,\dots,0,0}=1+c_1+c_2,$
$N_{p,q}^{0,0,0,\dots,0,1,1,1}=c_{n-3}+c_{n-2}+c_{n-1},$
$N_{p,q}^{2,2,2,\dots,2,2,1}=p^{n-2}+p^{n-2}c_1+p^{n-2}c_2+\cdots+p^{n-2}c_{n-2}+p^{n-1}c_{n-1},$
$\vdots$
$N_{p,q}^{1,2,2,\dots,2,2,2}=p^{n-1}+p^{n-2}c_1+p^{n-2}c_2+\cdots+p^{n-2}c_{n-1},$
$N_{p,q}^{0,1,2,2,\dots,2,2}=p^2c_1+pc_2+pc_3+\cdots+pc_{n-1},$
$N_{p,q}^{2,1,0,0\dots,0}=1+pc_1, N_{p,q}^{2,2,1,1,0,0\dots,0}=p+pc_1+p^2c_2+p^2c_3,$
$N_{p,q}^{2,2,2,1,1,0,0,\dots,0}=p^2+p^2c_1+p^2c_2+p^3c_3+p^3c_4,$
$N_{p,q}^{2,2,1,0,0,\dots,0}=p+pc_1+p^2c_2,$

$N_{p,q}^{2,2,2,2,1,1,0,0,\dots,0} = p^3 + p^3c_1 + p^3c_2 + p^3c_3 + p^4c_4 + p^4c_5,$

$\vdots$

$N_{p,q}^{\overbrace{2,2,2,..2}^{k},1,1,\overbrace{0,0,\dots,0}^{n-k-2}} = p^{k-1} + p^{k-1}c_1 + \cdots + p^{k-1}c_{k-1} + p^kc_k + p^kc_{k+1} + 0c_{k+1} + \cdots + 0c_n.$

Note that here the unit vectors are $c_0 = 1, c_1, c_2, \dots, c_{n-1}$.

Proposition 4.2. The following equality holds.

a) $N_{p,q}^{k_0,0,0,..,0} = N_{k_0}(p,q),$

b) $N_{p,q}^{0,k_1,0,\dots,0} = c_1N_{k_1}(p,q),$

c) $N_{p,q}^{0,0,0,\dots,k_{n-1}} = c_{n-1}N_{k_{n-1}}(p,q),$

d) $N_{p,q}^{k_0,1,0,\dots,0} = N_{k_0}(p,q) + c_1N_{k_0+1}(p,q),$

e) $N_{p,q}^{k_0,0,0,\dots,1} = N_{k_0}(p,q) + c_{n-1}N_{k_0+1}(p,q),$

f) $N_{p,q}^{k_0,1,1,\dots,1} = N_{k_0}(p,q) + c_1N_{k_0+1}(p,q) + \cdots + c_{n-1}N_{k_0+1}(p,q),$

g) $N_{p,q}^{1,k_1,0,\dots,0} = N_{k_1+1}(p,q) + c_1N_{k_1}(p,q),$

h) $N_{p,q}^{0,k_1,0,\dots,1} = c_1N_{k_1}(p,q) + c_{n-1}N_{k_1+1}(p,q),$

i) $N_{p,q}^{1,k_1,1,\dots,1} = N_{k_1+1}(p,q) + c_1N_{k_1}(p,q) + \cdots + c_{n-1}N_{k_1+1}(p,q),$

j) $N_{p,q}^{1,0,0,\dots,k_{n-1}} = N_{k_n+1}(p,q) + c_{n-1}N_{k_{n-1}}(p,q),$

k) $N_{p,q}^{0,1,0,\dots,k_{n-1}} = c_1N_{k_n+1}(p,q) + c_{n-1}N_{k_{n-1}}(p,q),$

l) $N_{p,q}^{1,1,1,\dots,k_{n-1}} = N_{k_n+1}(p,q) + c_1N_{k_n+1}(p,q) + \cdots + c_{n-1}N_{k_{n-1}}(p,q),$

m)$N_{p,q}^{k_0,k_1,k_2,\dots,0} =$
$N_{k_0}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-2}+1}(p,q) + c_1N_{k_0+1}(p,q)N_{k_1}(p,q)\dots N_{k_{n-2}+1}(p,q) + \cdots +$
$c_{n-2}N_{k_0+1}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-2}}(p,q),$

n)$N_{p,q}^{k_0,0,k_2,\dots,k_{n-1}} =$
$N_{k_0}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) + c_2N_{k_0+1}(p,q)N_{k_2}(p,q)\dots N_{k_{n-1}+1}(p,q) + \cdots +$
$c_{n-1}N_{k_0+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}}(p,q),$

o)$N_{p,q}^{0,k_1,k_2,\dots,k_{n-1}} =$
$c_1N_{k_1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) + c_2N_{k_1+1}(p,q)N_{k_2}(p,q)\dots N_{k_{n-1}+1}(p,q) + \cdots +$
$c_{n-2}N_{k_1+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}}(p,q),$

p)$N_{p,q}^{k_0,k_1,k_2,\dots,1} =$
$N_{k_0}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-2}+1}(p,q) + c_1N_{k_0+1}(p,q)N_{k_1}(p,q)\dots N_{k_{n-2}+1}(p,q) + \cdots +$
$c_{n-2}N_{k_0+1}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-2}}(p,q) + c_{n-1}N_{k_0+1}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-1}+1}(p,q),$

r)$N_{p,q}^{k_0,1,k_2,\dots,k_{n-1}} =$
$N_{k_0}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) + c_1N_{k_0+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) +$
$c_2N_{k_0+1}(p,q)N_{k_2}(p,q)\dots N_{k_{n-1}+1}(p,q) + \cdots + c_{n-1}N_{k_0+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}}(p,q),$

s)$N_{p,q}^{0,k_1,k_2,\dots,k_{n-1}} =$
$N_{k_1+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) + c_1N_{k_1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}+1}(p,q) +$
$c_2N_{k_1+1}(p,q)N_{k_2}(p,q)\dots N_{k_{n-1}+1}(p,q) + \cdots + c_{n-2}N_{k_1+1}(p,q)N_{k_2+1}(p,q)\dots N_{k_{n-1}}(p,q),$

t)$N_{p,q}^{k_0,k_1,k_2,\dots,k_{n-1}} =$
$N_{k_0}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-1}+1}(p,q) + c_1N_{k_0+1}(p,q)N_{k_1}(p,q)\dots N_{k_{n-1}+1}(p,q) + \cdots +$
$c_{n-1}N_{k_0+1}(p,q)N_{k_1+1}(p,q)\dots N_{k_{n-1}}(p,q).$

Proof: The proofs are easily shown by mathematical induction as in Proposition 2.2.

5. CONCLUSION

In this study, significant extensions of Narayana sequences in the complex plane are mentioned. Firstly the two dimensional expansions $a+bi,\ a,b\in\mathbb{Z}$ and i imaginary units, then the three dimensional expansions $a+bi+cj,\ a,b,c\in\mathbb{Z}$ and i,j imaginary units, and finally $n-$dimensional expansions $a_1+a_2c_1+a_3c_3+\cdots+a_nc_n$ and $a_1,a_2,a_3,\ldots,a_n\in\mathbb{Z}$ $c_1,c_2,\ldots,c_n$imaginary units and important identities are discussed.

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