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TOPOLOGY BASED ON NEW GENERALIZED CLOSED SETS

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Abstract. *In this paper, we introduce a new classes of sets called $\tilde{E}$-closed sets and $\tilde{E}_\alpha$-closed sets in topological spaces and we study some of its basic properties. This class lies between the class of closed sets and the class of g-closed sets.*
Keywords: *Closed set; $\tilde{E}$-closed set; $\tilde{E}_\alpha$-closed set.*

1. INTRODUCTION

In 1963, Levine [1-2] was introduced by the notions of semi-open sets and g-closed sets are investigated its fundamental properties also. Since the advent of the notion of semi-open sets, many mathematicians worked on such sets and also introduced some other notions such as [3-15].

2. PRELIMINARIES

We recall the following definitions which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called
(i) semi-open set [8] if A $\subseteq$ cl(int(A));
(ii) preopen set [11] if A $\subseteq$ int(cl(A));
(iii) α -open set [12] if A $\subseteq$ int(cl(int(A)));
(iv) β-open set [1] (= semi-preopen set [2]) if A $\subseteq$ cl(int(cl(A)));
(v) regular open set [14] if A = int(cl(A)).

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.2. A subset A of a space (X, τ) is called
(i) a generalized closed (briefly g-closed) set [9] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of g-closed set is called g-open set;
(ii) a semi-generalized closed (briefly sg-closed) set [4] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ). The complement of sg-closed set is called sg-open set;

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(iii) a generalized semi-closed (briefly gs-closed) set [3] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gs-closed set is called gs-open set;
(iv) an α-generalized closed (briefly αg-closed) set [10] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of αg-closed set is called αg-open set;
(v) a generalized semi-preclosed (briefly gsp-closed) set [6] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set;
(vi) a $\hat{\hat{g}}$-closed set [7] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg-open in (X, τ). The complement of $\hat{\hat{g}}$--closed set is called $\hat{\hat{g}}$-open set.
(vii) a $\hat{g}$-closed set [15] (= ω-closed set [13]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open. The complement of $\hat{g}$-closed set is called $\hat{g}$-open.

Definition 2.3. [5] Let (X, τ) be a topological space and A $\subseteq$ X. We define the sg-closure of A (briefly, sg-cl(A)) to be the intersection of all sg-closed sets containing A.

3. ON Ẽ -CLOSED SETS

We introduce the following definitions.

Definition 3.1.
(i) A subset A of a space (X, τ) is called a A-closed set if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\hat{\hat{g}}$-open in (X, τ).
The complement of A-closed set is called A-open set.

(ii) A subset A of a space (X, τ) is called a B-closed set if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is A-open in (X, τ).
The complement of B-closed set is called B-open set.
The collection of all A-closed (resp. B-closed) sets in X is denoted by AC(X) (resp. BC(X)).

Definition 3.2. A subset A of a space (X, τ) is called a Ẽ-closed set if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is B-open in (X, τ).
The complement of Ẽ-closed set is called Ẽ-open set.
The collection of all Ẽ closed set in X is denoted by ẼC(X).

Remark 3.3.
(i) Each closed set is A-closed but not conversely.
(ii) Each closed set is B-closed but not conversely.
(iii) Each semi-open set is B-open but not conversely.

Example 3.4. Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then

AC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X};

BC(X) = {ϕ, {b}, {c}, {b, c}, X}

and

SO(X) = {ϕ, {a}, {a, b}, {a, c}, X}.

Then:

(i) {b} is A-closed but not closed.

(ii) {b} is B-closed but not closed.

Example 3.5. Let X = {a, b, c} with τ = {ϕ, {a, b}, X}. Then

$$BO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$

and

$$SO(X) = \{\phi, \{a, b\}, X\}.$$

We have A= {a} is B-open but not semi-open.

Proposition 3.6. Each closed set is Ẽ -closed in X.

Proof: If A is any closed set in X and G is any B-open set containing A, then G $\supseteq$ A = cl(A). Hence A is Ẽ -closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example 3.7. Let X and τ be as in Example 1.3.5. Then

$$\tilde{E}\,C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}.$$

We have A = {a, c} is Ẽ-closed set but not closed.

Definition 3.8. A subset A of a space (X, τ) is called a $\tilde{E}_\alpha$ -closed set if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is B-open in (X, τ).

The complement of $\tilde{E}_\alpha$ -closed set is called $\tilde{E}_\alpha$ -open set.

The collection of all $\tilde{E}_\alpha$ -closed (resp. $\tilde{E}_\alpha$ -open) sets in X is denoted by $\tilde{E}_\alpha$ C(X) (resp. $\tilde{E}_\alpha$ O(X)).

Proposition 3.9. Each Ẽ-closed set is $\tilde{E}_\alpha$ -closed in X.

Proof: If A is a Ẽ-closed set in X and G is any B-open set containing A, then G $\supseteq$ cl(A) $\supseteq$ α cl(A). Hence A is $\tilde{E}_\alpha$ -closed.

The converse of Proposition 3.9 need not be true as seen from the following example.

Example 3.10. Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then

$$\tilde{E}\,C(X) = \{\phi, \{a, c\}, X\}$$

and

$$\tilde{E}_\alpha C(X) = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}.$$

We have A = {a} is $\tilde{E}_\alpha$ -closed set but not Ẽ-closed.

Proposition 3.11. Each Ẽ-closed set is sg-closed in X.

Proof: If A is a Ẽ -closed set in X and G is any semi-open set containing A, since every semi-open set is B-open and A is Ẽ-closed, we have G $\supseteq$ cl(A) $\supseteq$ scl(A). Hence A is sg-closed.

The converse of Proposition 3.11 need not be true as seen from the following example.

Example 3.12. Let X and τ be as in Example 3.4. Then

$$\tilde{E}C(X) = \{\phi, \{b, c\}, X\}$$

and

$$SG\,C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}.$$

We have A = {b} is sg-closed set but not Ẽ-closed.

Proposition 3.13. Each Ẽ-closed set is g-closed in X.

Proof: If A is a Ẽ-closed set in (X, τ) and G is any open set containing A, since every open set is B-open, we have G $\supseteq$ cl(A). Hence A is g-closed.

The converse of Proposition 3.13 need not be true as seen from the following example.

Example 3.14. Let X and τ be as in Example 3.4. Then

$$G\,C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$$

and

$$\tilde{E}C(X) = \{\phi, \{b, c\}, X\}.$$

We have A = {a, b} is g-closed set but not Ẽ-closed.

Proposition 3.15. Each Ẽ-closed set is α g-closed in X.

Proof: If A is a Ẽ-closed set in X and G is any open set containing A, since every open set is B-open, we have G $\supseteq$ cl(A) $\supseteq$ α cl(A). Hence A is α g-closed.

The converse of Proposition 3.15 need not be true as seen from the following example.

Example 3.16. Let X and τ be as in Example 3.4. Then

$$\tilde{E}C(X) = \{\phi, \{b, c\}, X\}$$

and

$$\alpha G\,C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

We have A = {a, c} is αg -closed set but not Ẽ-closed.

Proposition 3.17. Each Ẽ-closed set is gs-closed in X.

Proof: If A is a Ẽ-closed set in X and G is any open set containing A, since every open set is B-open, we have G $\supseteq$ cl(A) $\supseteq$ scl(A). Hence A is gs-closed .

The converse of Proposition 3.17 need not be true as seen from the following example.

Example 3.18. Let X = {a, b, c} with τ = {ϕ, {b}, {a, b}, X}. Then

$$\tilde{E}C(X) = \{\phi, \{c\}, \{a, c\}, X\}$$

and

$$GS\,C(X) = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}.$$

We have A = {a} is gs-closed set but not Ẽ-closed.

Proposition 3.19. Each Ẽ-closed set is gsp-closed in X.

Proof: If A is a Ẽ-closed set in X and G is any open set containing A, every open set is B-open, we have G $\supseteq$ cl(A) $\supseteq$ spcl(A). Hence A is gsp-closed.

The converse of Proposition 3.19 need not be true as seen from the following example.

Example 3.20. Let X and τ be as in Example 3.10. Then

$$\tilde{E}C(X) = \{\phi, \{a, c\}, X\}$$

and

$$GSP\,C(X) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

We have A = {a} is gsp-closed set but not Ẽ-closed.

Remark 3.21. The following examples show that Ẽ-closed sets are independent of α-closed sets and semi-closed sets.

Example 3.22. Let X and τ be as in Example 3.5. Then

$$\tilde{E}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$$

and

$$\alpha\,C(X) = S\,C(X) = \{\phi, \{c\}, X\}.$$

We have A = {a, c} is Ẽ-closed set but it is neither α-closed nor semi-closed.

Example 3.23. Let X and τ be as in Example 1.3.4. Then

$$\tilde{E}C(X) = \{\phi, \{b, c\}, X\}$$

and

$$\alpha\,C(X) = S\,C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}.$$

We have A = {b} is both α-closed set and semi-closed set but not Ẽ-closed.

4. MORE PROPERTIES

In this section, we discuss some basic properties of Ẽ-closed sets.

Definition 4.1. The intersection of all B-open subsets in (X, τ) containing A is called the B-kernel of A and denoted by B-ker(A).

Lemma 4.2. A subset A of (X, τ) is Ẽ-closed if and only if cl(A) $\subseteq$ B-ker(A).

Proof: Suppose that A is Ẽ-closed. Then $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is B-open. Let $x \in cl(A)$. If $x \notin$ B-ker(A), then there is an B-open set U containing A such that $x \notin U$. Since U is an B-open set containing A, we have $x \notin cl(A)$ and this is a contradiction.

Conversely, let $cl(A) \subseteq$ B-ker(A). If U is any B-open set containing A, then $cl(A) \subseteq$ B-ker(A) $\subseteq U$. Therefore, A is Ẽ-closed.

Proposition 4.3. If A and B are Ẽ-closed sets in (X, τ) then $A \cup B$ is $\approx$g-closed in (X, τ).

Proof: If $A \cup B \subseteq G$ and G is B-open, then $A \subseteq G$ and $B \subseteq G$. Since A and B are Ẽ -closed, $G \supseteq cl(A)$ and $G \supseteq cl(B)$ and hence $G \supseteq cl(A) \cup cl(B) = cl(A \cup B)$. Thus $A \cup B$ is Ẽ-closed set in (X, τ).

Proposition 4.4. If a set A is Ẽ-closed in (X, τ) then $cl(A) - A$ contains no nonempty B-closed set in (X, τ).

Proof: Suppose that A is Ẽ-closed. Let F be a B-closed subset of $cl(A) - A$. Then $A \subseteq F^c$. Since A is Ẽ-closed, $cl(A) \subseteq F^c$. Consequently, $F \subseteq (cl(A))^c$. We already have $F \subseteq cl(A)$. Thus $F \subseteq cl(A) \cap (cl(A))^c$ and F is empty.

The converse of Proposition 4.4 need not be true as seen from the following example.

Example 4.5. Let $X = \{a, b, c, d\}$ with $\tau = \{\phi, \{a, d\}, X\}$. Then

$$\approx GC(X) = \{\phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$$

and

$$BC(X) = \{\phi, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

If $A = \{b\}$ then $cl(A) - A = \{c\}$ does not contain any nonempty B-closed set. But A is not Ẽ-closed.

Theorem 4.6. If a set A is Ẽ-closed, then $cl(A) - A$ contains no nonempty closed set.

Proof: Suppose that A is Ẽ-closed. Let S be a closed subset of $cl(A) - A$. Then $A \subseteq S^c$. Since A is Ẽ-closed, we have $cl(A) \subseteq S^c$. Consequently, $S \subseteq (cl(A))^c$. Hence, $S \subseteq cl(A) \cap (cl(A))^c = \phi$. Therefore S is empty.

Proposition 4.7. If A is Ẽ-closed set in (X, τ) and $A \subseteq B \subseteq cl(A)$, then B is Ẽ-closed set in (X, τ).

Proof: Let $B \subseteq U$ where U is B-open set in (X, τ). Since $A \subseteq B$, $A \subseteq U$. Since A is Ẽ-closed set, $cl(A) \subseteq U$. Since $B \subseteq cl(A)$, $cl(B) \subseteq cl(A) \subseteq U$. Therefore B is Ẽ-closed set in (X, τ).

Proposition 4.8. Let $A \subseteq Y \subseteq X$ and suppose that A is Ẽ-closed in (X, τ). Then A is Ẽ-closed relative to Y.

Proof: Let $A \subseteq Y \cap G$, where G is B-open in (X, τ). Then $A \subseteq G$ and hence $cl(A) \subseteq G$. This implies that $Y \cap cl(A) \subseteq Y \cap G$. Thus A is Ẽ-closed relative to Y.

Proposition 4.9. If A is an B-open and Ẽ-closed in (X, τ), then A is closed in (X, τ).

Proof: Since A is B-open and Ẽ-closed, cl(A) $\subseteq$ A and hence A is closed in (X, τ).

Theorem 4.10. Let A be a locally closed set of (X, τ). Then A is closed if and only if A is Ẽ-closed.

Proof: It is fact that every closed set is Ẽ-closed.

Conversely, by Proposition 3.3 of Bourbaki [9], A $\cup$ (X − cl(A)) is open in (X, τ), since A is locally closed. Now A $\cup$ (X − cl(A)) is B-open set of (X, τ) such that A $\subseteq$ A $\cup$ (X − cl(A)). Since A is Ẽ-closed, then cl(A) $\subseteq$ A $\cup$ (X − cl(A)). Thus, we have cl(A) $\subseteq$ A and hence A is a closed.

Proposition 4.11. For each x $\in$ X, either {x} is B-closed or $\{x\}^c$ is Ẽ-closed in (X, τ).

Proof: Suppose that {x} is not B-closed in (X, τ). Then $\{x\}^c$ is not B-open and the only B-open set containing $\{x\}^c$ is the space X itself. Therefore cl($\{x\}^c$) $\subseteq$ X and so $\{x\}^c$ is Ẽ-closed in (X, τ).

Theorem 4.12. Let A be a Ẽ-closed set of a topological space (X, τ). Then,
(i) If A is regular open, then pint(A) and scl(A) are also Ẽ-closed.
(ii) If A is regular closed, then pcl(A) is also Ẽ-closed.

Proof:
(i) Since A is regular open in X, A = int(cl(A)). Then scl(A) = A $\cup$ int(cl(A)) = A. Thus, scl(A) is Ẽ-closed in (X, τ). Since pint(A) = A $\cap$ int(cl(A)) = A, pint(A) is Ẽ -closed.

(ii) Since A is regular closed in X, A = cl(int(A)). Then pcl(A) = A $\cup$ cl(int(A)) = A. Thus, pcl(A) is Ẽ-closed in (X, τ). The converses of the statements in the Theorem 4.12 are not true as we can see in the following examples.

Example 4.13. Let X = {a, b, c} with τ = {ϕ, {a}, {a, b}, X}. Then

ẼC(X) = {ϕ, {c}, {b, c}, X}.

Then the set A = {c} is not regular open. However A is Ẽ-closed; scl(A) = {c} and pint(A) = ϕ are also Ẽ-closed.

Example 4.14. In Example 4.13, the set A = {c} is not regular closed. However A and pcl(A) = {c} are Ẽ-closed.

Lemma 4.15. Let F be a closed set of (X, τ). Then the following properties hold:

if A is B-closed in (X, τ), then A $\cap$ F is B-closed in (X, τ).

Corollary 4.16. If A is Ẽ-closed set and F is a closed set, then A $\cap$ F is Ẽ-closed set.

Proof: Let U be a B-open set of (X, τ) such that A $\cap$ F $\subseteq$ U. By Lemma 4.15, it shows that A $\subseteq$ U $\cup$ (X/F) and U $\cup$ (X/F) is B-open in (X, τ). Since A is Ẽ-closed in (X, τ), we have cl(A)

$\subseteq$ U $\cup$ (X\F) and so cl(A$\cap$F) $\subseteq$ cl(A) $\cap$ cl(F) = cl(A) $\cap$ F $\subseteq$ (U $\cup$ (X\F)) $\cap$ F = U $\cap$ F $\subseteq$ U. Therefore A $\cap$ F is Ẽ-closed in (X, τ).

5. CONCLUSION

In this article, we present two brand-new kinds of sets called Ẽ-closed sets and $\tilde{E}_{\alpha}$-closed closed sets in topological spaces, and we investigated some of their fundamental characterizations are arrived. This class is intermediate between the classes of closed sets and g-closed sets. And we can learn in various areas oftopological spaces with associated applications.

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