ORIGINAL PAPER

EXPONENTIAL TYPE INEQUALITIES AND ALMOST COMPLETE CONVERGENCE OF THE OPERATOR ESTIMATOR OF FIRST-ORDER AUTOREGRESSIVE IN HILBERT SPACE GENERATED BY WOD ERROR

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Abstract. *In this paper, we establish a new concentration inequality and almost complete convergence of the value of the process of autoregressive Hilbertian of order one (ARH (1)), which directly stems from works of Serge Guillas, Denis Bosq, that is defined by $X_t = \rho(X_{t-1}) + \zeta_t;\ t \in \mathbb{Z}$ where the random variables are all Hilbertian, ρ is a linear operator on a space of separable Hilbert and ζ_t which constitute a widely orthant dependent error (WOD, in short) after recalling some results on the finite-dimensional model of this type, we introduce the mathematical and statistical tools which will be used afterwards. Then we build an estimator of the operator and we establish its asymptotic properties.*

Keywords: *autoregressive in Hilbert space; exponential inequalities, almost complete convergence; WOD random variables.*

1. INTRODUCTION AND PRELIMINARIES

Let $\{X_n, n \geq 1\}$ be a sequence of random variables that is defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As known that there are many results on probability limit theorems for independent random variables. In fact, the independence assumption is not always appropriate in applications.

So many authors introduced some dependent structure and mixing structure. Widely orthant dependent structure was one of the newest dependence structure that has attracted the interest of probabilists and statisticians, this structure contains most of negatively dependent random variables, some positively dependent random variables and some other random variables. The concept of widely orthant dependent random variables was introduced by Wang et al. [1] as follows.

Definition 1.1. For $\{X_n, n \geq 1\}$ a sequence of random variables:

(i) If there exists a sequence of real numbers $g_u(n), n \geq 1$ such that for each $n \geq 1$ and for all $x_i, i \in (-\infty, +\infty) 1 \leq i \leq n$;

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i > x_i\}\right) \leq g_u(n) \prod_{i=1}^{n} \mathbb{P}(X_i > x_i).$$

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Then we say that the random variables $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD) with dominating coefficients $g_u(n), n \geq 1$;

(ii) If there exists a sequence of real numbers $g_l(n), n \geq 1$ such that for each $n \geq 1$ and for all

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \leq x_i\}\right) \leq g_l(n)\prod_{i=1}^{n}\mathbb{P}(X_i \leq x_i).$$

Then we say that the random variables $\{X_n, n \geq 1\}$ are widely lowerorthant dependent (WLOD) with dominating coefficients $g_l(n), n \geq 1$;

(iii) If $\{X_n, n \geq 1\}$ are both WUOD and WLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are widely orthant dependent (WOD) with dominating coefficients $g_u(n), n \geq 1$ and $g_l(n), n \geq 1$.

(iv) An array $\{X_{ni}, i \geq 1, n \geq 1\}$ is said row-wise WOD random variables, if for each $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of WOD random variables.

Recall that when $g(n) = g_l(n) = M$ for some positive constant M, the random variables $\{X_n, n \geq 1\}$ are called extended negatively upper orthant dependent (ENUOD, in short) and extended negatively lower orthant dependent (ENLOD, in short), respectively. If they are both ENUOD and ENLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are extended negatively orthant dependent (END, in short). The concept of END random variables was proposed by Liu [2], and further promoted by Chen et al. [3], Shen [4], Wang and Wang [5], Wu and Guan [6], Qiu et al. [7], Wang et al. [8- 9], and so forth. When $g_u(n) = g_l(n) = 1$ for any $n \geq 1$, the random variables $\{X_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD, in short) and negatively lower orthant dependent (NLOD, in short), respectively. If they are both NUOD and NLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are negatively orthant dependent (NOD, in short). The concept of NOD random variables was introduced by Ebrahimi and Ghosh [10] and carefully studied by Joag and Proschan [11], Bozorgnia and al. [12], Taylor and al. [13], Wang and al. [14], Sung [15], Qiu et al. [16], Wu [17], Wu and Jiang [18], Shen [19-20], and so on. Joag and Proschan [11] pointed out that NA random variables are NOD. Hu [21] introduced the concept of negatively super-additive dependence (NSD, in short) and pointed out that NSD implies NOD. Christofides and Vaggelatou [22] indicated that NA implies NSD. By the statements above, we can see that the class of WOD random variables contains END random variables, NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Hence, studying the probability limiting behavior of WOD random variables is a great interest.

The concept of WOD random variables was introduced by Wang et al. [1] and many applications have been found subsequently. See, for example, Wang et al. [1] provided some examples which showed that the class of WOD random variables contains some common negatively dependent random variables, some positively dependent random variables and some others; in addition, they studied the uniform asymptotic for the finite-time ruin probability of a new dependent risk model with a constant interest rate. Wang and Cheng [23] presented some basic renewal theorems for a random walk with widely dependent increments and gave some applications. Wang and al. [24] studied the asymptotic of the finite-time ruin probability for a generalized renewal risk model with independent strong sub-exponential claim sizes and widely lower orthant dependent inter-occurrence times. Liu [25] gave the asymptotically equivalent formula for the finite-time ruin probability under a dependent risk model with constant interest rate. Chen et al. [26] considered uniform asymptotic for the finite-time ruin probabilities of two kinds of nonstandard bi-dimensional renewal risk models with constant interest forces and diffusion generated by Brownian motions. Shen [27] established the Bernstein type inequality for WOD random variables and gave some applications, Wang et al. [28] studied the complete convergence for WOD random variables and gave its applications in nonparametric regression models. Yang et al. [29] established the Bahadur representation of sample quantiles for WOD random variables under some mild conditions.

Definition 1.2. Let an array $\{X_{ni},\ i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C so that

$$\mathbb{P}(|X_{ni}| > x) \leq C\mathbb{P}(|X_n| > x)\ \forall\, x \geq 1, i \geq 1 \text{ and } n \geq 1.$$

Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with dominating coefficients $g_u(n),\ g_l(n),\ n \geq 1$ and $\{X_{ni},\ i \geq 1, n \geq 1\}$ be an array of row-wise WOD random variables with dominating coefficients $g_u(n),\ g_l(n),\ n \geq 1$ in each row. Denote $g(n) = \max\{g_u(n), g_l(n)\}$. $I(.)$denotes the indicator function, with C a positive constant, which value may be different places.

Lemma 1.1.

(i) Let random variables $\{X_n, n \geq 1\}$ be WOLD (WUOD) with dominating coefficients $g_l(n),\ n \geq 1(g_u(n),\ n \geq 1)$. If $\{f_n(.),\ n \geq 1\}$ arenondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_l(n),\ n \geq 1(g_u(n),\ n \geq 1)$; if $\{f_n(.),\ n \geq 1\}$are non-increasing, then $\{f_n(X_n),\ n \geq 1\}$ are WUOD (WLOD) with dominating coefficients$g_l(n),\ n \geq 1(g_u(n),\ n \geq 1)$.

(ii) If $\{X_n, n \geq 1\}$ are nonnegative WUOD with dominating coefficients $(g_u(n), n \geq 1$, then for each $n \geq 1$,

(iii)

$$\mathbb{E}\left[\exp\left\{t\sum_{i=1}^{n} X_i\right\}\right] \leq g_u(n)\prod_{i=1}^{n}\mathbb{E}(\exp\{tX_i\}).$$

Particularly, if $\{X_n, n \geq 1\}$ are WUOD with dominating coefficients $(g_u(n),\ n \geq 1)$, then for each $n \geq 1$ and any $t > 0$.

$$\mathbb{E}\left[\exp\left\{t\sum_{i=1}^{n} X_i\right\}\right] \leq g_u(n)\prod_{i=1}^{n}\mathbb{E}(\exp\{tX_i\}).$$

Wang et al.[25] obtained the following corollary by Lemma 1.1.

Corollary 1.1. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables.

(i) If $f_n(.)$ are all nondecreasing (or non-increasing), then $\{f_n(X_n),\ n \geq 1\}$ are still WOD.

(ii) For each $n \geq 1$ and any $t \in \mathbb{R}$,

(iii)

$$\mathbb{E}\left[\exp\left\{t\sum_{i=1}^{n} X_i\right\}\right] \leq g(n)\prod_{i=1}^{n}\mathbb{E}(\exp\{t\,X_i\}).$$

2. MAIN OF RESULT

Theorem 2.1. Let $(\zeta_t,\ t \in \mathbb{Z})$sequence of identically distributed widely orthant dependent (WOD, in short) random variables, of mean zero (i.e $\mathbb{E}\,\zeta_t = 0, \forall\, t \in \mathbb{Z}$) and hold $\sup_{j\in\mathbb{Z}} \|\zeta_t\| \le b < \infty$, indeed if, for any $\varepsilon < E|\zeta_n|\ \forall n \in \mathbb{Z}$.
Then

$$\mathbb{P}\left\{\left|\frac{\zeta_1 + \cdots + \zeta_n}{n}\right| > \varepsilon\right\} \le g(n)\, e^{-n\,(A_2 - A_1)}$$

where $A_1 = \frac{\mathbb{E}|\zeta|}{b}\left(\frac{\varepsilon}{\mathbb{E}|\zeta|} - 1\right)$ and $A_2 = \frac{\varepsilon}{b}\, ln\left(\frac{\varepsilon}{\mathbb{E}|\zeta|}\right)$.

Proof of Theorem 2.1: We notice that

$$\left\{\left|\frac{\zeta_1 + \cdots + \zeta_n}{n}\right|\right\} \subseteq \left\{\frac{|\zeta_1| + \cdots + |\zeta_n|}{n}\right\}$$

we obtain the following inequality valid for each $\lambda, \varepsilon > 0$:

$$\tilde{I} = \mathbb{P}\left\{\left|\frac{\zeta_1 + \cdots + \zeta_n}{n}\right| > \varepsilon\right\} \le \mathbb{E}\left[exp\left(\lambda\left(\sum_{j=1}^{n}|\zeta_j| - n\varepsilon\right)\right)\right]$$

$$= e^{-\lambda n\varepsilon}\mathbb{E}\left[exp\left(\lambda\sum_{j=1}^{n}|\zeta_j|\right)\right]$$

$$= e^{-\lambda n\varepsilon}\mathbb{E}\left[\prod_{j=1}^{n} e^{\lambda|\zeta_j|}\right]$$

We use Lemma 1.1 and the Corollary 1.1 we have

$$\tilde{I} = e^{-\lambda n\varepsilon}\, g(n)\prod_{j=1}^{n}\mathbb{E}\left[e^{\lambda|\zeta_j|}\right]$$

assumption $\sup_{j\in\mathbb{Z}} \|\zeta_t\| \le b < \infty$ of theorem and the elementary inequality $e^{xt} \le 1 + x(e^t - 1), 0 \le x < 1, t > 0$ then allow to bounded $\tilde{I}$ by:

$$e^{-\lambda n\varepsilon}\, g(n)\prod_{j=1}^{n}\mathbb{E}\left[1 + \frac{|\zeta_j|}{b}(\, e^{\lambda b} - 1)\right].$$

On another hand, for $\varepsilon, \lambda > 0$, we have:

$$\tilde{I} \le e^{-\lambda n\varepsilon}\, g(n)\prod_{j=1}^{n}\left[1 + \mathbb{E}|\zeta_j|\left(\frac{e^{\lambda b} - 1}{b}\right)\right]$$

$$\leq e^{-\lambda n\varepsilon}\, g(n) \prod_{j=1}^{n} exp\left[\mathbb{E}\left|\zeta_j\right|\left(\frac{e^{\lambda b}-1}{b}\right)\right] \ (using\ 1+x \leq e^{x}, \forall x \in \mathbb{R})$$

$$\leq e^{-\lambda n\varepsilon}\, g(n)\left(n\mathbb{E}\left|\zeta_j\right|\left(\frac{e^{\lambda b}-1}{b}\right) - \lambda n\varepsilon\right)$$

The equation $\frac{\partial \Lambda(\lambda)}{\partial \lambda} = \frac{\partial\left(n\mathbb{E}|\zeta_j|\left(\frac{e^{\lambda b}-1}{b}\right)-\lambda n\varepsilon\right)}{\partial \lambda} = 0$ a unique solution $\lambda = \frac{ln\left(\frac{\varepsilon}{\mathbb{E}|\zeta|}\right)}{b}$ that minimizes $\Lambda(\lambda)$. Then

$$\mathbb{P}\left\{\left|\frac{\zeta_1 + \cdots + \zeta_n}{n}\right| > \varepsilon\right\} \leq g(n)\, e^{-n\,(A_2 - A_1)}$$

where $A_1 = \frac{\mathbb{E}|\zeta|}{b}\left(\frac{\varepsilon}{\mathbb{E}|\zeta|} - 1\right)$ and $A_2 = \frac{\varepsilon}{b}\, ln\left(\frac{\varepsilon}{\mathbb{E}|\zeta|}\right)$. That concludes our Theorem.

Remark 2.1.

(a) $e^{-n\,(A_2 - A_1)} \leq \frac{4\,e^{-2}}{(A_2 - A_1)^2}$ (using the inequality: $e^{-x} \leq \frac{4\,e^{-2}}{x^2}, \forall x > 0$).

(b) $\mathbb{P}\left\{\left|\frac{\zeta_1 + \cdots + \zeta_n}{n}\right| > \varepsilon\right\} \leq A_3 \sum_{n=1}^{+\infty} \frac{n^{\gamma}}{n^2} < +\infty$

where

$$A_3 = \frac{4\,e^{-2}}{(A_2 - A_1)^2}$$

and

$$g(n) = O(n^{\gamma}), (0 \leq \gamma < 1).$$

Lemma 2.1.

$$(\tau_1) \sum_{j=1}^{r} j^{-1} \leq 1 + \ln r.$$

$$(\tau_2) \sum_{j=1}^{r} j^{\kappa-1} \leq \kappa^{-1} r^{\kappa}, 0 < \kappa < 1.$$

$$(\tau_3) \sum_{j=1}^{r} j^{-1-\omega} \leq \omega^{-1} r^{-\omega}.$$

Proof of Lemma 2.1:

- With reference to (τ_1): if $0 < j \leq k$, then

$$K^{-1} \leq j^{-1}$$

and

$$\int_{k-1}^{k} K^{-1}\, dj \leq \int_{k-1}^{k} j^{-1}\, dj = \ln k - \ln(k-1).$$

So

$$\sum_{j=1}^{r} K^{-1} \leq 1 + \ln r.$$

- As to (τ_2): if $0 < j \leq k$, then

$$K^{\kappa-1} \leq j^{\kappa-1}$$

and

$$\int_{k-1}^{k} K^{\kappa-1}\, dj \leq \int_{k-1}^{k} j^{\kappa-1}\, dj = \kappa^{-1}(k^{\kappa} - (k-1)^{\kappa})$$

So

$$\sum_{j=r+1}^{+\infty} K^{\kappa-1} \leq \kappa^{-1} r^{\kappa}, 0 < \kappa < 1.$$

- As to (τ_3): if $0 < j \leq k$, then

$$K^{-1-\omega} \leq j^{-1-\omega}$$

and

$$\int_{k-1}^{k} K^{-1-\omega}\, dj \leq \int_{k-1}^{k} j^{-1-\omega}\, dj = -\omega^{-1}(k^{-\omega} - (k-1)^{-\omega}).$$

Therefor

$$\sum_{j=r+1}^{+\infty} K^{-1-\omega} \leq \omega^{-1} r^{-\omega}, 0 < k < 1.$$

Theorem 2.2. Let $X = (X_t, t \in \mathbb{Z})$hilbertian autoregressive processes offirst order $(ARH(1))$. Assume that $\sup_{t\in\mathbb{Z}}\|\zeta_t\| \leq b < \infty$ where (ζ_t) are widely orthant dependent noise (WOD) and $(\lambda_j, j \geq 1)$ are eigenvalues of operator $C = C_{x_0}$. Then

$$\mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} \leq \eta(t+1)g(n)e^{-(A_2'-A_1')n} + \frac{g(n)}{\delta\varepsilon^2}\sum_{j>\eta}\lambda_j,$$

where

$$A_2' = \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\, b}\, ln\left[\frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\mathbb{E}|\langle\Phi_{\mathrm{t,k}}, v_j\rangle|}\right]$$

and

$$A_1' = \frac{\mathbb{E}|\Phi_{\mathrm{t,k}}, v_j\rangle|}{b}\left(\frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\mathbb{E}|\langle\Phi_{\mathrm{t,k}}, v_j\rangle|} - 1\right), 1 \leq j < \eta.$$

Proof of Theorem 2.2: Let $(v_j, j \geq 1)$ eigenvectors of operator Cthat form an orthonormal base of H. For each $\delta \in]0,1[$ we can write

$$\bar{I} \leq \mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} = \mathbb{P}\left\{\left\|\frac{S_n}{n}\right\|^2 > \varepsilon^2\right\}$$
$$= \mathbb{P}\left\{\sum_{j\geq 1}\left|< \frac{S_n}{n}, v_j >^2\right| \geq \varepsilon^2\right\}.$$

Using

$$\left\|\frac{S_n}{n}\right\|^2 = \sum_{j\geq 1}|x_j|^2, \qquad \frac{S_n}{n} = \sum_{j\geq 1} x_j v_j, \qquad x_j = <\frac{S_n}{n}, v_j>$$

we deduce

$$\bar{I} \leq \mathbb{P}\left\{\sum_{j=1}^{\eta} <\frac{S_n}{n}, v_j>^2 \geq (1-\delta)\varepsilon^2\right\} + \mathbb{P}\left\{\sum_{j>\eta} <\frac{S_n}{n}, v_j>^2 \geq \delta\varepsilon^2\right\},$$

we obtain the following bounded-up

$$I_1 \leq \mathbb{P}\left\{\sum_{j=1}^{\eta} <\frac{S_n}{n}, v_j>^2 \geq (1-\delta)\varepsilon^2\right\}$$
$$I_1 \leq \sum_{j=1}^{\eta} \mathbb{P}\left\{<\frac{S_n}{n}, v_j> \geq \varepsilon\,\frac{\sqrt{1-\delta}}{\sqrt{\eta}}\right\} \tag{1}$$

we have $\mathbb{P}\{T_1^2 + \cdots + T_\eta^2 \geq K\} \leq \mathbb{P}\left\{T_1^2 \geq \frac{K}{\eta}\right\} + \cdots + \mathbb{P}\left\{T_\eta^2 \geq \frac{K}{\eta}\right\}$.

We put $I_1' \leq \mathbb{P}\left\{<\frac{S_n}{n}, v_j> \geq \varepsilon\,\frac{\sqrt{1-\delta}}{\sqrt{\eta}}\right\}$ where $\frac{S_n}{n} = \frac{\sum_{t=1}^{n} X_t}{n}$, $X_t = \zeta_t + \rho(\zeta_{t-1}) + \cdots + \rho^{t-1}(\zeta_1)$, $\|\rho\| < 1$ and (ζ_t) are $H-$WOD.

So

$$I_1' \leq \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\langle X_t, v_j\rangle \geq \varepsilon\,\frac{\sqrt{1-\delta}}{\sqrt{\eta}}\right\}$$
$$= \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\langle\sum_{k=1}^{t+1}\rho^{k-1}\big(\xi_{t-(k-1))}\big), v_j\rangle \geq \varepsilon\frac{\sqrt{1-\delta}}{(\sqrt{\eta}}\right\}$$
$$= \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\sum_{k=1}^{t+1}\langle\rho^{k-1}\big(\xi_{t-(k-1))}\big), v_j\rangle \geq \varepsilon\frac{\sqrt{1-\delta}}{(\sqrt{\eta}}\right\}$$
$$\leq \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\langle\xi_t, v_j\rangle \geq \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}}\right\} + \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\langle\rho(\xi_{t-1}), v_j\rangle \geq \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}}\right\}$$
$$+ \cdots + \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\langle\rho^t(X_0), v_j\rangle \geq \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}}\right\}$$

we put

$$I_{k,1}' \leq \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n}\underbrace{\langle\rho^{k-1}\big(\xi_{t-(k-1))}\big)}_{\Phi_{t,k}}, v_j\rangle \geq \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}}\right\}.$$

By applying theorem 2.1 on $(\langle\Phi_{t,k}, v_j\rangle, t \in \mathbb{Z})$ we achieve

$$I_{k,1}' \leq g(n)e^{-(A_2'-A_1')n}, \forall 1 \leq k < t+1,$$

thus we conclude that

$$I_1' \leq (t+1)I_{k,1}' \leq (t+1)g(n)e^{-(A_2'-A_1')n}, \forall 1 \leq k < t+1, \tag{2}$$

From (1) and (2) we find:

$$I_1 \leq \sum_{j=1}^{\eta} (t+1) I'_{k,1} \leq \eta(t+1) g(n) e^{-(A'_2 - A'_1)n} \tag{3}$$

where

$$A'_2 = \frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\, b} ln\left[\frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\mathbb{E}\left|\langle \Phi_{\mathrm{t,k}}, v_j\rangle\right|}\right]$$

and

$$A'_1 = \frac{\mathbb{E}\left|\Phi_{\mathrm{t,k}}, v_j\rangle\right|}{b}\left(\frac{\varepsilon\sqrt{1-\delta}}{(t+1)\sqrt{\eta}\mathbb{E}\left|\langle \Phi_{\mathrm{t,k}}, v_j\rangle\right|} - 1\right), 1 \leq j < \eta.$$

On another hand, from Markov inequality

$$\mathbb{P}\left\{\sum_{j>\eta} \langle \frac{S_n}{n}, v_j\rangle^2 \geq \delta\, \varepsilon^2\right\} \leq \frac{g(n)}{\delta\varepsilon^2} \sum_{j>\eta}\left[\mathbb{E}\, \langle \frac{S_n}{n}, v_j\rangle^2\right].$$

Since

$$\left(\frac{1}{n}\sum_{t=1}^{\eta}\langle X_t, v_j\rangle\right)^2 \leq \frac{1}{n}\sum_{t=1}^{\eta}\langle X_t, v_j\rangle^2,$$

we obtain

$$\begin{aligned}\mathbb{P}\left\{\sum_{j>\eta} \langle \frac{S_n}{n}, v_j\rangle^2 \geq \delta\, \varepsilon^2\right\} &\leq \frac{g(n)}{\delta\varepsilon^2} \sum_{j>\eta}\left[\mathbb{E}\langle X_0, v_j\rangle^2\right] \\ &\leq \frac{g(n)}{\delta\varepsilon^2}\sum_{j>\eta}\lambda_j.\end{aligned} \tag{4}$$

Finally, from (3) and (4), we have:

$$\bar{I} = \mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} \leq \eta(t+1) g(n) e^{-(A'_2 - A'_1)n} + \frac{g(n)}{\delta\varepsilon^2}\sum_{j>\eta}\lambda_j.$$

Thus the result.

Corollary 2.1. Indeed, if $\exists a > 0, \beta > 1$ and $g(n) = O(n^\gamma), 0 \leq \gamma < 1$ such that

$$\sum_{j>\eta}\lambda_j \leq \sum_{j>n}\lambda_j \; (n > \eta) \tag{5}$$

and

$$\lambda_j \leq a\, j^{-1-\beta}\; \forall j \geq 1. \tag{6}$$

Then

$$\sum_{n=1}^{+\infty}\mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} \leq \eta(t+1)\sum_{n=1}^{+\infty} n^\gamma e^{-(A'_2 - A'_1)n} + \frac{a}{\beta\delta\varepsilon^2}\sum_{n=1}^{+\infty} n^{\gamma-\beta} < \infty \tag{7}$$

(i.e.$\frac{S_n}{n} \to 0$ completely almost when $n \to \infty$ with $\frac{S_n}{n}$ is an element of Hilbert space H).

Proof of Corollary 2.2: From theorem we have that

$$\mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} \leq \eta(t+1)g(n)e^{-(A'_2-A'_1)n} + \underbrace{\frac{g(n)}{\delta\varepsilon^2}\sum_{j>\eta}\lambda_j}_{\tilde{I}} < \infty$$

The expression $\tilde{I}$ be bounded up by $\frac{1}{\delta\varepsilon^2}\ a\ \beta^{-1}n^{\delta}\ n^{-\beta}$ using lemma **(2.1).** Finally, from (4) and (5) we deduce

$$\sum_{n=1}^{+\infty}\mathbb{P}\left\{\left\|\frac{S_n}{n}\right\| > \varepsilon\right\} \leq \eta(t+1)\sum_{n=1}^{+\infty}n^{\gamma}e^{-(A'_2-A'_1)n} + \frac{a}{\beta\delta\varepsilon^2}\sum_{n=1}^{+\infty}n^{\gamma-\beta} < \infty$$

That is to say that $\frac{S_n}{n}$ converges almost completely to 0 when n tends to ∞.

3. CONCLUSION

In this article we established a new concentration inequality and the almost complete convergence of the value of the process of autoregressive Hilbertian of order one ARH(1).

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