ORIGINAL PAPER

RESOLUTION NUMERICAL OF NON-LINEAR EQUATIONS

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Abstract. *In this work we have applied a very important the hyperbolic tangent (tanh) method in the analytical study of nonlinear coupled KdV systems of partial differential equations. Compared to existing sophisticated approaches, this proposed method gives more general exact traveling wave solutions without much extra effort. Two applications from the literature of non linear PDE systems have been solved by the method.*

Keywords: *Tanh method; non-linear system; exact solutions; nonlinear waves; Boussinesq equations*.

1. INTRODUCTION

The hyperbolic tangent (tanh) method is a powerful technique to symbolically compute traveling wave solutions of nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction diffusion phenomena play an important role [1]. Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especiallyin fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemicalphysics. The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeledby the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution [2-3]. In this study, we consider two coupled KdV equations. A variety of methods, such as the Adomian decomposition method [4], Backlund and Darboux transformation [5], inverse Scattering method [6], and Hirota's bilinear method [7] are used to obtain exact andnumerical solutions. In this study, the traveling wave solutions to the KdV equations will be considered withthe form $u(x,t)=u(\xi),\ \xi=k(x-\lambda t)$, where λ stands for the wave speed (see [8]). For completeness, we should mention that this techniqueis restricted to the search of traveling wave waves. Thus, we essentially deal with one-dimensional shock waves (kink type) and solitary-wave (pulse type) solutions in amoving frame of reference. Based on the tanh methodand its generalizations, several symbolic softwareprograms have been developed to find exact traveling wave solutions [9].

2. EXPLANATION OF THE TANH METHOD

The Tanh method will be introduced as presented by Malfliet [10] and by Wazwaz [11-13]. The Tanh method is based on a priori assumption that the travelingwave solutions can be expressed in terms of the tanh function to solve the coupled KdVequations.

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The tanh method is developed by Malfliet [10].The method is applied to find out an exact solution of a nonlinear ordinary differential equation

$$P(u,u_x,u_t,u_{xx},u_{xxx},..........)=0 \tag{2.1}$$

where P is a polynomial of the variable u and its derivatives. If we consider $u(x,t)=u(\xi),\ \xi=k(x-\lambda t)$, so that $u(x,t)=U(\xi)$, we can use the following changes:

$$\frac{\partial}{\partial t}=-k\lambda\frac{d}{d\xi},\ \frac{\partial}{\partial x}=k\frac{d}{d\xi},\ \frac{\partial^2}{\partial x^2}=k^2\frac{d^2}{d\xi^2},\ \frac{\partial^3}{\partial x^3}=k^3\frac{d^3}{d\xi^3},$$

and so on, then Eq. (2.1) becomes an ordinary differential equation

$$Q(U,U',U'',U''',..........)=0 \tag{2.2}$$

with Q being another polynomial form of its argument, which will be called the reduced ordinary differentialequation of Eq. (2.2). IntegratingEq. (2.2) as long as all terms contain derivatives, the integrationconstants are considered to be zeros in view of the localized solutions. However, the nonzero constantscan be used and handled as well [13]. Now finding the traveling wave solutions to Eq. (2.1) is equivalentto obtaining the solution to the reduced ordinary differential equation (2.2). For the tanh method, we introduce the new independent variable [14]:

$$Y(x,t)=\tanh(\xi) \tag{2.3}$$

that leads to the change of variables:

$$\begin{aligned}
\frac{d}{d\xi}&=(1-Y^2)\frac{d}{dY}\\
\frac{d^2}{d\xi^2}&=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2}\\
\frac{d^3}{d\xi^3}&=2(1-Y^2)(3Y^2-1)\frac{d}{dY}-6Y(1-Y^2)^2\frac{d^2}{dY^2}+(1-Y^2)^3\frac{d^3}{dY^3}
\end{aligned} \tag{2.4}$$

The next crucial step is that the solution we are looking for is expressed in the form

$$u(x,t)=U(\xi)=\sum_{i=1}^{m}a_iY^i \tag{2.5}$$

where the parameter m can be found by balancing the highest-order linear term with the nonlinear terms inEq. (2.2), and $k,\lambda,a_0,a_1,....,a_m$ are to be determined. Substituting (2.5) into (2.2) will yield a set of algebraicequations for $k,\lambda,a_0,a_1,....,a_m$ because all coefficients of Y^i have to vanish. From these relations, $k,\lambda,a_0,a_1,....,a_m$ can be obtained.

Having determined these parameters, knowing that m is a positive integerin most cases, and using (2.5) we obtain an analytic solution $u(x,t)$ in a closed form [13]. The tanh methodseems to be powerful tool in dealing with coupled nonlinear physical models. For a coupled system of nonlineardifferential equations with two unknowns:

$$\begin{aligned} P_1(u,v,u_x,v_x,u_t,v_t,u_{xx},v_{vv},..........) = 0 \\ P_2(u,v,u_x,v_x,u_t,v_t,u_{xx},v_{vv},..........) = 0 \end{aligned} \tag{2.6}$$

As for the traveling wave solutions to (2.6) concerned, we have to solve its corresponding reduced ordinarydifferential equations

$$\begin{aligned} Q_1(u,v,u',v',u'',v'',..........) = 0 \\ Q_2(u,v,u',v',u'',v'',..........) = 0 \end{aligned} \tag{2.7}$$

In most cases, the exact solvability of (2.7) depends on a delicate explicit assumption between the twounknowns or their derivatives, for more details see [13].

3. NUMERICAL APPLICATIONS

The Tanh method is generalized on two examples including systems of coupled KdV equations.These systems were studied from Sayed Tauseef [15] by applying the variational iteration method.

Example 1. Considerthe following (1+1) - dimensional nonlinear Boussinesq equations [14]:

$$\begin{aligned} u_t + v_x + u.u_x = 0 \\ v_t + (vu)_x + u_{xxx} = 0 \end{aligned} \tag{3.1}$$

Using the traveling wave transformations

$$u(x,t) = U(\xi) = \sum_{i=1}^{m} a_i Y^i \tag{3.2}$$

$$v(x,t) = V(\xi) = \sum_{i=1}^{n} b_i Y^i \tag{3.3}$$

where

$$\xi = k(x - \lambda t) \tag{3.4}$$

The nonlinear system of partial differential equations (3.1) is carried to a system of ordinary differentialequations

$$
\begin{aligned}
&-\lambda k \frac{dU}{d\xi} + k\frac{dV}{d\xi} + kU\frac{dU}{d\xi} = 0 \\
&-\lambda k \frac{dV}{d\xi} + kV\frac{dU}{d\xi} + kU\frac{dV}{d\xi} + k^3\frac{d^3U}{d\xi^3} = 0
\end{aligned} \tag{3.5}
$$

we postulate the following tanh series in Eq. (3.2), Eq. (3.3), Eq. (2.3) and the transformation given in (2.4), the first equation in (3.5) reduces to

$$
-\lambda k(1-Y^2)\frac{dU}{dY} + k(1-Y^2)\frac{dV}{dY} + kU(1-Y^2)\frac{dU}{dY} = 0 \tag{3.6}
$$

the second equation in (3.5) reduces to

$$
\begin{aligned}
&-\lambda k(1-Y^2)\frac{dV}{dY} + kV(1-Y^2)\frac{dU}{dY} + kU(1-Y^2)\frac{dV}{dY} + 2k^3(1-Y^2)(3Y^2-1)\frac{dU}{dY} \\
&-6k^3Y(1-Y^2)^2\frac{d^2U}{dY^2} + k^3(1-Y^2)^3\frac{d^3U}{dY^3} = 0
\end{aligned} \tag{3.7}
$$

Now, to determine the parameters m and n, we balance the linear term of highest-order with the highestordernonlinear terms. So, in Eq. (3.6) we balance $V^{/}$ with $UU^{/}$, to obtain

2 + n – 1 = 2 + m + m – 1, then n = 2m.

While in Eq. (3.7) we balance $U^{///}$ with $UV^{/}$, to obtain

6 + m – 3 = 2 + m + n – 1 then n = 2 , m = 1.

The tanh method admits the use of the finite expansion for both:

$$
u(x,t) = U(Y) = a_0 + a_1 Y, \quad a_1 \neq 0 \tag{3.8}
$$

and

$$
v(x,t) = V(Y) = b_0 + b_1 Y + b_2 Y^2, \quad b_2 \neq 0 \tag{3.9}
$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}$ from Eq. (3.8) and Eq. (3.9) respectively into Eq. (3.6), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$
\begin{aligned}
&Y^0 : -\lambda a_1 + a_1 a_0 +_1 = 0 \\
&Y^1 : 2b_2 + a_1^2 = 0
\end{aligned} \tag{3.10}
$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}$ from Eq. (3.8) and Eq. (3.9) respectively into Eq. (3.7), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$Y^0 : -\lambda b_1 + a_1 b_0 + a_0 b_1 - 2k^2 a_1 = 0$$
$$Y^1 : -\lambda b_2 + a_1 b_1 + a_0 b_2 = 0 \quad (3.11)$$
$$Y^2 : a_1 b_2 + 2c^2 a_1 = 0$$

Solving the nonlinear systems of equations (3.12) and (3.13) with help of Mathematica we can get:

$$a_0 = \lambda\ ,\ a_1 = 2k,\ b_0 = 2k^2\ ,\ b_1 = 0\ ,\ b_2 = -2k^2$$

Then:

$$u(x,t) = \lambda + 2k \tanh(k(x - \lambda t)) \quad (3.12)$$

and

$$v(x,t) = 2k^2 \operatorname{sech}^2(k(x - \lambda t)) \quad (3.13)$$

The solitary wave and behavior of the solutions $u(x,t)$ and $v(x,t)$ are shown in Figs. 3.1-3.2 respectively for some fixed values of the parameters ($\lambda = 0.5, k = 0.5$)

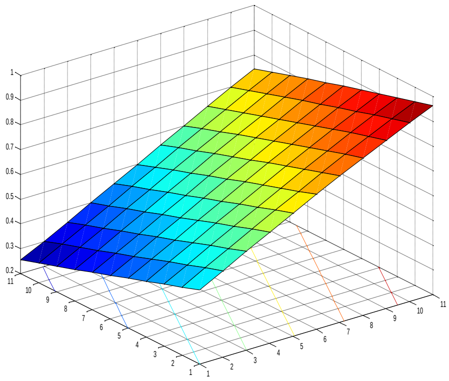

Figure 3.1. The solitary wave $u(x,t)$

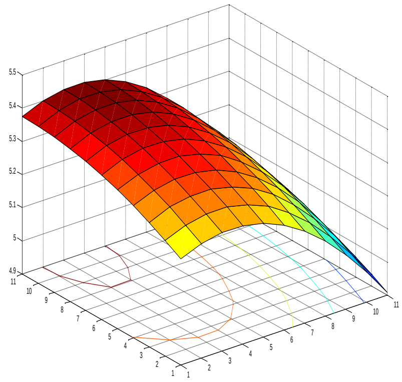

Figure 3.2. The solitary wave $v(x,t)$

Example 2. Considerthe following (1+1)- dimensional new coupled modified KdV nonlinear equations [14]:

$$u_t - \frac{1}{2}u_{xxx} + 3u^2 u_x - \frac{3}{2}v_{xx} - 3(uv)_x + 3\alpha . u_x = 0$$
$$v_t + v_{xxx} + 3vu_x - 3u_x v_x - 3u^2 v_x - 3\alpha . u_x = 0 \quad (3.14)$$

Using the traveling wave transformations

$$u(x,t) = U(\xi) = \sum_{i=1}^{m} a_i Y^i \quad (3.15)$$

$$v(x,t) = V(\xi) = \sum_{i=1}^{n} b_i Y^i \quad (3.16)$$

where

$$\xi = k(x - \lambda t) \quad (3.17)$$

The nonlinear system of partial differential equations (3.14) is carried to a system of ordinary differentialequations

$$
\begin{aligned}
&-\lambda k \frac{dU}{d\xi} - \frac{1}{2}k^3 \frac{d^3U}{d\xi^3} + 3kU^2 \frac{dU}{d\xi} - \frac{3}{2}k^2 \frac{d^2V}{d\xi^2} - 3kU \frac{dV}{d\xi} \\
&-3kV \frac{dU}{d\xi} + 3\alpha k \frac{dU}{d\xi} = 0 \\
&-\lambda k \frac{dV}{d\xi} + k^3 \frac{d^3V}{d\xi^3} + 3kV \frac{dU}{d\xi} - 3k^2 \frac{dU}{d\xi}\frac{dV}{d\xi} - 3kU^2 \frac{dV}{d\xi} \\
&-3\alpha k \frac{dU}{d\xi} = 0
\end{aligned}
\tag{3.18}
$$

We postulate the following tanh series in Eq. (3.2), Eq. (3.3), Eq. (2.3) and the transformation given in (2.4), the first equation in (3.18) reduces to

$$
\begin{aligned}
&-\lambda k(1-Y^2)\frac{dU}{dY} - \frac{1}{2}k^3[2(1-Y^2)(3Y^2-1)\frac{dU}{dY} - 6Y(1-Y^2)^2\frac{d^2U}{dY^2} + (1-Y^2)^3\frac{d^3U}{dY^3}] \\
&+3kU^2(1-Y^2)\frac{dU}{dY} - \frac{3}{2}k^2(1-Y^2)[(-2Y\frac{dV}{dY} + (1-Y^2)\frac{d^2V}{dY^2})] - 3kU(1-Y^2)\frac{dV}{dY} \\
&-3kV(1-Y^2)\frac{dU}{dY} + 3\alpha k(1-Y^2)\frac{dU}{dY} = 0
\end{aligned}
\tag{3.19}
$$

the second equation in (3.18) reduces to

$$
\begin{aligned}
&-\lambda k(1-Y^2)\frac{dV}{dY} + k^3[2(1-Y^2)(3Y^2-1)\frac{dV}{dY} - 6Y(1-Y^2)^2\frac{d^2V}{dY^2} + (1-Y^2)^3\frac{d^3V}{dY^3}] \\
&+3kV(1-Y^2)\frac{dU}{dY} - 3k^2(1-Y^2)^2\frac{dV}{dY}\frac{dU}{dY} - 3kU^2(1-Y^2)\frac{dV}{dY} - 3\alpha k(1-Y^2)\frac{dU}{dY} = 0
\end{aligned}
\tag{3.20}
$$

Now, to determine the parameters m and n, we balance the linear term of highest-order with the highestordernonlinear terms. So, in Eq. (3.19) we balance $U^{///}$ with $UV^{/}$, to obtain

6 + m – 3 =2 + m + n - 1, then n = 2.

while in Eq. (3.20) we balance $V^{///}$ with $UV^{/}$, to obtain

6 + n – 3 = 4 + n – 1 + m – 1, then m = 1.

The tanh method admits the use of the finite expansion for both

$$u(x,t) = U(Y) = a_0 + a_1 Y \tag{3.21}$$

and

$$v(x,t) = V(Y) = b_0 + b_1 Y + b_2 Y^2 \tag{3.22}$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}, V^{//}, V^{///}$ from Eq. (3.21) and Eq. (3.22) respectively into Eq. (3.19), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$Y^0 : -\lambda a_1 + k^2 a_1 + 3a_1 a_0^2 - 3kb_2 - 3a_0 b_1 - 3b_0 a_1 + 3\alpha a_1 = 0$$
$$Y^1 : 2a_0 a_1^2 + b_1 k - 2a_0 b_2 - 2a_1 b_1 = 0 \quad (3.23)$$

$$Y^2 : -k^2 a_1 + a_1^3 + 3kb_2 - 3a_1 b_2 = 0 \quad (3.23)$$

Substituting $U, U^{/}, U^{//}, U^{///}$ and $V, V^{/}, V^{//}, V^{///}$ from Eq. (3.21) and Eq. (3.22) respectively into Eq. (3.20), then equating the coefficient of Y^i ,i= 0, 1, 2, 3 leads to the following nonlinear system of algebraic equations

$$Y^0 : -\lambda b_1 - 2k^2 b_1 + 3a_1 b_0 - 3ka_1 b_1 - 3b_1 a_0^2 - 3\alpha a_1 = 0$$
$$Y^1 : -2\lambda b_2 - 4k^2 b_2 - 12k^2 b_2 + 3a_1 b_1 - 6ka_1 b_2 - 6b_1 a_0 a_1 - 6b_2 a_0^2 = 0$$
$$Y^2 : 2k^2 b_1 + a_1 b_2 + ka_1 b_1 - b_1 a_1^2 - 4b_2 a_0 a_1 = 0 \quad (3.24)$$
$$Y^3 : (4k^2 + ka_1 - a_1^2) b_2 = 0$$

Solving the nonlinear systems of equations (3.23) and (3.24) with help of Mathematica we can get:

$$a_0 = \frac{1}{4},\ a_1 = \frac{k}{2}(1 \pm \sqrt{17}),\ b_0 = \alpha,\ b_1 = 0,\ b_2 = \frac{k^2}{2}(9 \pm \sqrt{17}),$$
$$\lambda = \frac{-k^2}{2}(1 \pm 3\sqrt{17}) + \frac{3}{16}.$$

Then:

$$u(x,t) = \frac{1}{4} + \frac{k}{2}(1 \pm \sqrt{17}) \tanh(k(x - \lambda t)) \quad (3.25)$$

and

$$v(x,t) = \alpha + \frac{k^2}{2}(9 \pm \sqrt{17}) \tanh^2(k(x - \lambda t)) \quad (3.26)$$

The solitary wave and behavior of the solutions $u(x,t)$ and $v(x,t)$ are shown in Figs. 3.3-3.4 for some fixed values of the parameters $\alpha = 1.0$, $k = 1.0$.

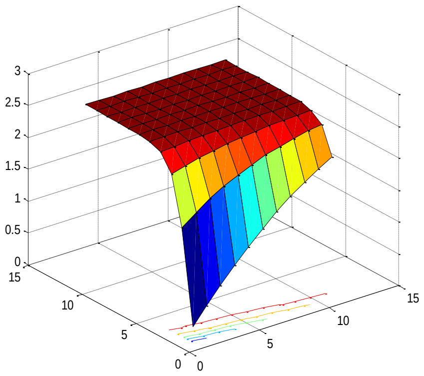

Figure 3.3. The solitary wave of $u(x,t)$

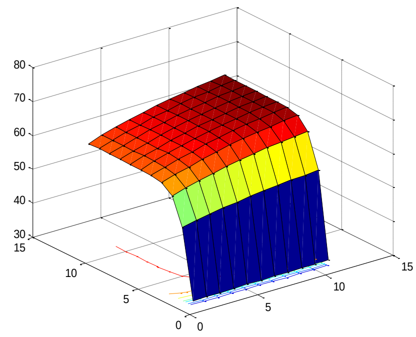

Figure 3.4. The solitary wave of $v(x,t)$

4. CONCLUSION

The powerful tanh methodwas employed for analytic treatment of nonlinear coupledpartial differential equations. The tanh method requires transformation formulas. Traveling wave solutions, kinks solutions were derived. Solution of coupled KdV systems of PDEs (3.1) and (3.14) are compatible and agreed with that solution of Sayed Tauseef [15] by applying the variational iteration method. The performance of the tanh method shows that it is reliable and effective.

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