*ORIGINAL PAPER*

# AN EXTENSION OF SOME SOLUTIONS OF THE FALKNER-SKAN EQUATION

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*Manuscript received: 10.01.2023; Accepted paper: 18.04.2023;*
*Published online: 30.06.2023.*

***Abstract.*** *The differential equation* $\varphi''' + \varphi\varphi'' + \alpha\left(\varphi'^2 - 1\right) = 0$ *where* $\alpha > 0$ *is appeared for studying the boundary layer flow past a semi infinitewedge. As a means to prove the existence of solutions verifying* $(0) = a \geq \sqrt{\frac{1}{1-\alpha}}$ *,* $\varphi'(0) = b \geq 0$ *and* $\varphi'(t) \to 1$ *or* $-1$ *as* $t \to +\infty$ *for* $0 < \alpha < 1$*. We utilize shooting technique and consider the initial conditions* $\varphi(0) = a$*,* $\varphi'(0) = b$ *and* $\varphi''(0) = c$*. We demonstrate that there exists an infinitely many solutions where* $\varphi'(+\infty) = 1$*.*

***Keywords:*** *third ordernonlinear differential equation; boundary layer; convex solution; shooting technique; concave solution; convex-concave solution.*

## 1. INTRODUCTION

The third order autonomous nonlinear differential equation

$$\varphi''' + \varphi\varphi'' + \alpha\left(\varphi'^2 - 1\right) = 0 \tag{1}$$

is introduced in 1931 by Falkner and Skan for studying the boundary layer flow past a semi infinite wedge and for this reason is called the Falkner-Skan equation. Many authors as in [1-6] have studied the solutions of this equation.

The general equation of (1) is

$$\varphi''' + \varphi\varphi'' + \rho(\varphi) = 0 \tag{2}$$

where $\rho: \mathbb{R} \to \mathbb{R}$ is some function. The Blasius equation is the most famous example, with $\rho(x) = 0$ and arises in the study of laminar boundary layer on a flat plate (see [7]).

Newly, the equation (2) with $g(x) = \beta x^2$ and $g(x) = \beta x(x-1)$ has been considered, for example, in the study of free convection and of mixed convection boundary layer flows over a vertical surface embedded in a porous medium (see [8, 9]).

Usually, to solve the boundary value problem $\left(P_{\alpha;a,b,\gamma}\right)$ where:

$$\left(P_{\alpha;a,b,\gamma}\right)\begin{cases} \varphi''' + \varphi\varphi'' + \alpha\left(\varphi'^2 - 1\right) = 0, \\ \varphi(0) = a, \\ \varphi'(0) = b, \\ \lim_{t\to+\infty} \varphi'(t) = \gamma. \end{cases}$$

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We will use the shooting technique. We denote $\varphi_c$ the solution of the initial value problem $P_i\ (a,b,c)$ consisting in the equation (1) with the initial conditions $\varphi(0)=a$, $\varphi'(0)=b$ and $\varphi''(0)=c$. We consider $[0,I_c)$ the right maximal interval of existence of $\varphi_c$. To get a solution of $\left(P_{\alpha;a,b,\gamma}\right)$ equivalent to find a value of c where $I_c=+\infty$ and $\varphi'_c(t)\to\gamma$ as $t\to+\infty$.

For condition of $\gamma$ one prove that if $\gamma$ is constant then $(\gamma^2-1)=0$. To have solutions, in our case of Falkner-Skan equation, the only relevant conditions are $\varphi'(t)\to-1$ or $\varphi'(t)\to 1$ as $t\to+\infty$.

In this work, we will study the existence of concave, convex, concave-convex, convex-concave, concave-convex-concave and convex-concave-convex solutions to the boundary value problem $\left(P_{\alpha;a,b,-1}\right)$ and $\left(P_{\alpha;a,b,1}\right)$ for $0<\alpha<1$, a $\geq\sqrt{\frac{1}{1-\alpha}}$ and b $\geq 0$.

## 2. PRELIMINARY RESULTS

We consider $\varphi$ as a solution to the equation (1) on some interval $J$, let $K_\varphi: J\to\mathbb{R}$ be the function defined by

$$K_\varphi=[\varphi''+\alpha\varphi(\varphi'-1)]exp^{(1-\alpha)F}. \tag{3}$$

where $F$ denote any primitive function of $\varphi$. This function is got by integrating the equation (1). In fact, if $\varphi$ is a solution of (1) then

$$K'_\varphi=\left[\alpha(\varphi'-1)\left((1-\alpha)\varphi^2-1\right)\right]exp^{(1-\alpha)F}.$$

In the following, we put lemmas that will be useful later.

**Lemma 2.1.** We consider $\varphi$ as a solution to (1) on some maximal interval $J$. If there exists $t_0\in J$ such that $\varphi'(t_0)\in\{-1,1\}$ and $\varphi''(t_0)=0$, then $J=\mathbb{R}$ and $\varphi''(t)=0$ for all $t\in\mathbb{R}$.

*Proof:* Cf [10]. Proposition 3.1 item 3.

**Lemma 2.2.** We consider $\alpha>0$ and $\varphi$ be a solution to equation (1) on some interval $J$, such that $\varphi'$is not constant.

1) If there exists $x<y\in J$ such that $\varphi''(x)\leq 0$ and $\left(\varphi'^2-1\right)>0$ on $]x,y[$, then $\varphi''(t)<0$ for all $t\in]x,y]$.
2) If there exists $x<y\in J$ such that $\varphi''(x)\geq 0$ and $(\varphi'^2-1)<0$ on $]x,y[$, then $\varphi''(t)>0$ for all $t\in]x,y]$.
3) If there exists $x<y\in J$ such that $\varphi''<0$ on $]x,y[$ and $\varphi''(y)=0$, then $(\varphi'^2(y)-1)<0$.
4) If there exists $x<y\in J$ such that $\varphi''>0$ on $]x,y[$ and $\varphi''(y)=0$, then $(\varphi'^2(y)-1)>0$.

*Proof:* We consider $F$ as a primitive function of $\varphi$. From (1) we deduce the relation

$$(\varphi''expF)'=-\alpha(\varphi'^2-1)expF.$$

The assertions 1-4 obtain easily from this relation and from preceding lemma. We verify the first and the third of these assertions. For the first one, as $\psi = \varphi'' expF$ is decreasing on $[x, y]$, we get

$$t \;\geq\; x \Rightarrow \psi(t) \leq \psi(x)$$

$$\Rightarrow\; \varphi''(t) expF(t) \leq \varphi''(s) expF(x)$$

$$\Rightarrow\; \varphi''(t) \leq \varphi''(x) exp(F(x) - F(t))$$

$$\Rightarrow\; \varphi''(t) \leq 0, \forall t \in ]x, y].$$

For the third one, as $\psi < 0$ on $]x, y[$ and $\psi(y) = 0$, then $\psi'(y) \geq 0$.

$$\psi'(y) = -\alpha({\varphi'}^2(y) - 1) expF \geq 0.$$

This and Lemma 2.1 imply that $({\varphi'}^2(y) - 1) < 0$.

**Lemma 2.3.** We consider $\varphi$ as a solution to (1) on some maximal interval $]I_-, I_+[$. If $I_+$ is finite, then $\varphi'$ and $\varphi''$ are unbounded in any neighborhood of $I_+$.

*Proof:* According [10], Proposition 3.1 item 6.

**Lemma 2.4.** We put $\alpha \neq 0$. If $\varphi$ is a solution of (1) on some interval $]\eta, +\infty[$ such that $\varphi'(t) \to \gamma$ as $t \to +\infty$, then $\gamma \in \{-1, 1\}$. In addition, if $\varphi$ is of constant sign at infinity, then $\varphi''(t) \to 0$ as $t \to +\infty$.

*Proof:* According [10], Proposition 3.1 item 4 and 5.

**Lemma 2.5.** We put $\alpha > 0$ and we consider $\varphi$ as a solution to (1) on some right maximal interval $J = [\eta, +\infty[$. If $\varphi \geq 0$ and $\varphi' \geq 0$ on $J$, then $I_+ = +\infty$ and $\varphi'$ is bounded on $J$.

*Proof:* We put $G = G_\varphi$ the function defined on $J$ by

$$G(t) = 3\varphi''(t)^2 + \alpha\varphi'(t)\big(2{\varphi'}^2(t) - 6\big) \tag{4}$$

Using (1), simply we get that

$$G'(t) = -6\varphi(t)\varphi''(t)^2 \qquad \forall t \in J$$

and, as $\varphi \geq 0$ on $J$ this implies that $G$ is decreasing. Therefore

$$\forall t \in J = [\eta, I_+[ : t > \eta \Rightarrow G(t) \leq G(\eta)$$
$$\alpha\varphi'(t)(2\varphi'(t)^2 - 6) \leq 3\varphi''(t)^2 + \alpha\varphi'(t)(2{\varphi'}^2(t) - 6) \leq G(\eta), \forall t \in J$$

It follows that $\varphi'$ is bounded on $J$ and thanks to lemma 2.3 that $I_+ = +\infty$.

**Lemma 2.6.** We put $\alpha > 0$ and we consider $\varphi$ as a solution to (1) on some right maximal interval $J = [\eta, I_+[$. If $\varphi(\eta) \geq 0$, $\varphi'(\eta) \geq 1$ and $\varphi''(\eta) > 0$, then there exists $t_0 \in ]\eta, I_+[$ where $\varphi'' > 0$ on $[\eta, t_0[$ and $\varphi''(t_0) = 0$.

*Proof:* Suppose for contradiction that $\varphi'' > 0$ on $J$. Then $\varphi(t) \geq 0,\ \varphi'(t) \geq 1$ for all $t \in J$. Then we get

$$\varphi''' = -\varphi\varphi'' - \alpha({\varphi'}^2 - 1) \leq 0 \tag{5}$$

It follows that $0 < \varphi''(t) \leq c$ for all $t \in J$ and therefore, by Lemma 2.4, we get $I_+ = +\infty$. After, we put $x > \eta$ and $\epsilon = \alpha(\varphi'(x)^2 - 1)$. One has $\epsilon > 0$ and, come again to (5), we get

$$\varphi''' \leq -\epsilon \text{ on } [x, +\infty[.$$

By integrating, we obtain

$$\forall t \geq x, \qquad \varphi''(t) - \varphi''(x) \leq -\epsilon(t - x)$$

and a contradiction with the fact that $\varphi''(t) > 0$.Therefore, there exists $t_0 \in ]\eta, I_+[$ where $\varphi'' > 0$ on $[\eta, t_0[$ and $\varphi''(t_0) = 0$.

**Lemma 2.7.** We put $\alpha \in [\frac{1}{2}, 1[$ and we consider $\varphi$ as a solution to (1) on some right maximal interval $J = ]I_-, I_+[$. If there exists $t_0 \in J$ where $\varphi(t_0) > \sqrt{\frac{1}{1-\alpha}}$, $\varphi'(t_0) > 1$ and

$$\alpha\varphi(t_0)(1 - \varphi'(t_0)) \leq \varphi''(t_0) \leq 0.$$

Then $I_+ = +\infty$ and $\varphi'(t) \to 1$ as $t \to +\infty$. In addition $\varphi'' \leq 0$ on $[t_0, +\infty[$ .

*Proof:* We put $\mu = \sup S(t_0)$, such that

$$S(t_0) = \{t \in ]t_0, I_+[ : 1 < \varphi' < \varphi'(t_0) \text{ and } \varphi'' < 0 \text{ on } ]t_0, t[\}$$

The set $S(t_0)$ is not empty. This is clear if $\varphi''(t_0) < 0$, and if $\varphi''(t_0) = 0$ it follows from the fact that

$$\varphi'''(t_0) = -\alpha({\varphi'}^2(t_0) - 1) < 0$$

We stay to prove that $\mu = I_+$, suppose for contradiction that $\mu < T_+$. From Lemma 2.2, item 1, we obtain that $\varphi''(\mu) < 0$, which implies, by definition of $\mu$, that $\varphi'(\mu) = 1$. Hence, as the function $K_f$ defined by (3) is increasing on $[t_0, \mu]$, we get

$$\mu \geq t_0 \Rightarrow K_\varphi(\mu) \geq K_\varphi(t_0)$$

$$\Rightarrow \varphi''(\mu) exp(1-\alpha)F(\mu) \geq [\varphi''(t_0) - \alpha\varphi(t_0)(1 - \varphi'(t_0))] exp(1-\alpha)F(\mu) \geq 0$$

$$\Rightarrow \varphi''(\mu) \geq 0.$$

a contradiction. In consequence, we have $\mu = I_+$. From Lemma 2.3, it follows that $I_+ = +\infty$. As $\varphi'' < 0$ on $[t_0, +\infty[$ , by vertue of Lemma 2.4, we obtain that $\varphi'(t) \to 1$ as $t \to +\infty$.

**Lemma 2.8.** We consider $\alpha \in [\frac{1}{2}, 1[$ and $\varphi$ a solution to equation (1) on some right maximal interval $J = ]I_-, I_+[$. If there exists $t_0 \in J$ such that $\varphi(t_0) > \sqrt{\frac{1}{1-\alpha}}$, $0 < \varphi'(t_0) < 1$ and

$$0 \leq \varphi''(t_0) \leq \alpha\varphi(t_0)(1 - \varphi'(t_0)).$$

Then $I_+ = +\infty$ and $\varphi'(t) \to 1$ as $t \to +\infty$. In addition $\varphi'' \geq 0$ on $[t_0, +\infty[$.

*Proof:* If we put $\lambda = supR(t_0)$, with

$$R(t_0) = \{t \in ]t_0, I_+[: \varphi'(t_0) < \varphi' < 1 \text{ and } \varphi'' > 0 \text{ on } ]t_0, t[\}.$$

The set $R(t_0)$ is not empty. This is clear if $\varphi''(t_0) > 0$, and if $\varphi''(t_0) = 0$ it follows from the fact that

$$\varphi'''(t_0) = -\alpha({\varphi'}^2(t_0) - 1) > 0.$$

We claim that $\lambda = I_+$, suppose for contradiction that $\lambda < I_+$. From Lemma 2.2, item 2, we obtain that $\varphi''(\lambda) > 0$, which implies, by definition of $\lambda$, that $\varphi'(\lambda) = 1$. Hence, as the function $K_\varphi$ defined by (3) is decreasing on $[t_0, \lambda]$, we get

$$\lambda \geq t_0 \Rightarrow K_\varphi(\lambda) \leq K_\varphi(t_0)$$

$$\Rightarrow \varphi''(\lambda)exp(1-\alpha)F(\lambda) \leq [\varphi''(t_0) - \alpha\varphi(t_0)(1 - \varphi'(t_0))]exp(1-\alpha)F(\lambda) \leq 0$$

$$\Rightarrow \varphi''(\lambda) \leq 0.$$

a contradiction. In consequence, we have $\lambda = I_+$. From Lemma 2.3, it follows that $I_+ = +\infty$. As $\varphi'' > 0$ on $[t_0, +\infty[$ , by vertue of Lemma 2.4, we obtain that $\varphi'(t) \to 1$ as $t \to +\infty$.

**Lemma 2.9.** We consider $\alpha \in [\frac{1}{2}, 1[$ and $\varphi$ a solution to equation (1) on some right maximal interval $J = ]I_-, I_+[$. If there exists $t_0 \in J$ such that $\varphi(t_0) > \sqrt{\frac{1}{1-\alpha}}$, $0 < \varphi'(t_0) < 1$ and

$$-\alpha\varphi(t_0)\varphi'(t_0) \leq \varphi''(t_0) \leq 0,$$

Then $I_+ = +\infty$ and $\varphi'(t) \to 1$ as $t \to +\infty$. In addition there exist $\eta \geq t_0$ such that $\varphi'' \leq 0$ on $[t_0, \eta[$ and $\varphi'' \geq 0$ on $[\eta, +\infty[$.

*Proof:* If $\varphi''(t_0) = 0$, the conclusion follows from Lemma 2.8.

If $\varphi''(t_0) < 0$, we put $\theta = sup\, T(t_0)$

$$T(t_0) = \{t \in ]t_0, I_+[: 0 < \varphi' < \varphi'(t_0) \text{ and } \varphi'' < 0 \text{ on } ]t_0, t[\,\}$$

The set $T(t_0)$ is not empty. We claim that $\theta < I_+$. Suppose for contradiction that $\theta = I_+$. From Lemma 2.3, we get that $\theta = I_+ = +\infty$, $0 < \varphi' < \varphi'(t_0)$ and $\varphi'' < 0$ on $]0, +\infty[$. Then $\varphi_c'$ is decreasing, and consequently $\varphi_c'$ has a finite limite $\gamma$ at infinity. By Lemma 2.4, we finally obtain that $\gamma \in \{-1,1\}$ a contraduction. In consequence we have $\theta < I_+$.

By definition of $\theta$, that $\varphi'(\theta) = 0$ or $\varphi''(\theta) = 0$. We claim that $\varphi'(\theta) \neq 0$, suppose for contraduction that $\varphi'(\theta) = 0$, we consider the function $L_\varphi: [t_0, \theta] \to \mathbb{R}$ defined by

$$L_\varphi = (\varphi'' + \alpha\varphi\varphi')exp(1-\alpha)F,$$

where$F$ denote any primitive function of $\varphi$, in fact, if $\varphi$ is a solution of (1) then

$$L_\varphi' = [((1-\alpha)\varphi^2\varphi' + 1)]exp(1-\alpha)F.$$

The function $L_\varphi$ is increasing on $[t_0, \theta]$, we get

$$\theta \ \geq \ t_0 \Rightarrow L_\varphi(\theta) \geq L_\varphi(t_0)$$

$$\Rightarrow \ \varphi''(\theta) exp(1-\alpha)F(\theta) \geq [\varphi''(t_0) + \alpha\varphi(t_0)\varphi'(t_0)] exp(1-\alpha)F(\theta) \geq 0$$

$$\Rightarrow \ \varphi''(\theta) \geq 0.$$

A contradiction. Consequently, we have $\varphi''(\theta) = 0$ and $0 < \varphi'(\theta) < 1$. From Lemma 2.8 it follows that $I_+ = +\infty$ and $\varphi'(t) \to 1$ as $t \to +\infty$.

## 3. DESCRIPTION OF OUR APPROACH WHEN $b \geq 1$

We consider $\alpha > 0$, $a \geq \sqrt{\frac{1}{1-\alpha}}$ and $b \geq 1$.The method we will use to obtain solutions of the boundary value problems $(P_{\alpha;a,b,-1})$ and $(P_{\alpha;a,b,1})$ is the shooting technique. Specifically, for $c \in \mathbb{R}$, we denote by $\varphi_c$ the solution of equation (1) verifying the initial conditions

$$\varphi_c(0) = a, \ \varphi_c'(0) = b \ \text{ and } \ \varphi_c''(0) = c \tag{6}$$

and we consider $[0, I_c[$ the right maximal interval of existence of $\varphi_c$. Therefore, finding a solution of one of the problems $(P_{\alpha;a,b,-1})$ and $(P_{\alpha;a,b,1})$ amounts to finding a value of $c$ where $I^+ = +\infty$ and $\varphi'(t) \to -1$ or 1 as $t \to +\infty$.

We divide $\mathbb{R}$ into the four sets $E_0$, $E_1$, $E_2$ and $E_3$ defined as follows. We put

$$E_0 = ]0, +\infty[ \, .$$

$$E_1 = \{c \leq 0;\ 1 \leq \varphi_c' \leq b \text{ and } \varphi_c'' \leq 0 \text{ on } [0, I_c[\} \ .$$

$$E_2 = \{c \leq 0;\ \exists\, z_c \in [0, I_c[, \exists\, \delta_c > 0 \text{ such that } \varphi_c' > 1 \text{ on } ]0, z_c[,$$

$$\varphi_c' < 1 \text{ on } ]z_c, z_c + \delta_c[ \text{ and } \varphi_c'' \leq 0 \text{ on } [0, z_c + \delta_c[\} \ .$$

and

$$E_3 = \{c \leq 0;\ \exists\, y_c \in [0, I_c[, \exists\, \sigma_c > 0 \text{ such that} \varphi_c'' < 0 \text{ on } ]0, y_c[$$

$$\varphi_c'' > 0 \text{ on } ]y_c, y_c + \delta_c[ \text{ and} \varphi_c' > 1 \text{ on } ]0, y_c + \sigma_c[\}.$$

This is evident that $E_0, E_1, E_2$ and $E_3$ are disjoint sets and that their union is the whole line of real numbers. Thanks to Lemma 2.3 and 2.4 if $c \in E_1$ then $I_+ = +\infty$ and $\varphi_c'(t) \to 1$ as $t \to +\infty$. In fact, $E_1$ is the set of values of c for which $\varphi_c$ is a concave solution of $(P_{\alpha;a,b,1})$.

As $\alpha > 0$, the study done in [10] (specialy in section 5.2) gives, on the one hand, that $E_3 = \emptyset$ (which can easily be conclude from Lemma 2.2, item 1) and, on the other hand, that either $E_1 = \emptyset$ and $E_2 = ]-\infty, 0]$, or there exist $c^* \leq 0$ such that $E_1 = [c^*, 0]$ and $C_2 = ]-\infty, c^*[$.

For the purpose of completing the stady, we divide the set $E_2$ into the following two subsets

$$E_{2,1} = \{c \in E_2 : \varphi_c' > 0 \text{ on } [0, I_c[\}$$

$$E_{2,2} = \{c \in E_2 : \exists\, x_c \in ]0, I_c[ \text{ such that } \varphi_c' > 0 \; on \; [0, x_c[ \text{ and } \varphi_c'(x_c) = 0\}$$

In the following, we give some characteristics of each of these subsets that hold for all $\alpha \in ]0,1]$.

**Lemma 3.1.** If $c \in \mathbb{R}$ where $\varphi_c' > 0$ on $[0, I_c[$, then $I_c = +\infty$ and $\varphi_c'$ is bounded.

In addition, if $c \leq 0$, then $\varphi_c' \leq max\{b, \sqrt{3}\}$ on $[0, +\infty[$.

*Proof:* We consider $c \in \mathbb{R}$ such that $\varphi_c' > 0$ on $[0, I_c[$, then $\varphi_c \geq a \geq 0$ on $[0, I_c[$ and thanks to Lemma 2.5, it follows that $I_c = +\infty$ and $\varphi_c'$ is bounded.

It stay to prove that $\varphi_c' \leq max\{b, \sqrt{3}\}$ in the case where $c \leq 0$. As in (4), we define the function $G_c$ on $[0, +\infty[$ by

$$G_c(t) = 3\varphi_c''(t)^2 + \alpha\, \varphi_c'(t)(2\, \varphi_c'(t)^2 - 6)$$

and, as $\varphi_c \geq 0$, it means that $G_c$ is nonincreasing.

If $\varphi_c'' \leq 0$ on $]0, +\infty[$, then $\varphi_c' \leq b$. Contrary, there exists $t_0$ such that $\varphi_c'' < 0$ on $]0, t_0[$ and $\varphi_c''(t_0) = 0$. By Lemma 2.2 item 3, it implies that $\varphi_c' < 1$, and consequently $G_c(t_0) < 0$. Then, $G_c < 0$ on $]t_0, +\infty[$ which implies that $\varphi_c' \leq \sqrt{3}$ on $]t_0, +\infty[$. As $\varphi_c' \leq b$ on $]0, t_0[$, the proof is complete.

**Proposition 3.2.** We put $c^* = inf\ (E_1 \cup E_{2,1})$. Then $c^*$ is finite.

*Proof:* We suppose $c \in E_1 \cup E_{2,1}$. By definition of $E_1$and $E_{2,1}$, and thanks to lemma 3.1, we get $I_c = +\infty$ and $0 < \varphi_c' < d$ on $[0, +\infty[$such that $d = max\{b, \sqrt{3}\}$. As

$$(\varphi_c'' + \varphi_c\varphi_c')' = \varphi_c''' + \varphi_c\varphi_c'' + \varphi_c'^2$$

$$= -\alpha(\varphi_c'^2 - 1) + \varphi_c'^2$$
$$= -\alpha\varphi_c'^2 + \alpha + \varphi_c'^2$$
$$\leq \alpha + d^2$$

By integrating, we get

$$\forall t \geq 0,\ f_c''(t) + f_c(t)f_c'(t) \leq c + ab + (\beta + d^2)t.$$

We integrate once again, for all $t \geq 0$, we obtain

$$0 < \varphi_c'(t) \leq \varphi_c'(t) + \frac{1}{2}\varphi_c^2(t) \leq b + \frac{1}{2}a^2 + (c + ab) + \frac{1}{2}(\alpha + d^2)t^2$$

Then we have

$$c \geq -ab - \sqrt{(2b + a^2)(\alpha + d^2)}.$$

**Remark 3.3.** If $C_1 \neq \emptyset$, then $E_1 = [c^*, 0]$ and moreover $E_{2,1} \subset [c_*, c^*[$.

## 4. THE CASE $\alpha \in [\frac{1}{2}, 1[$ AND $b \geq 1$

In this part we impose that $\alpha \in [\frac{1}{2}, 1[$, $a \geq \sqrt{\frac{1}{1-\alpha}}$ and $b \geq 1$.

**Proposition 4.1.** If $c > 0$, then $T_c = +\infty$. Moreover, $\varphi'(t) \to 1$ as $t \to +\infty$.

*Proof:* From Lemma 2.6, there exists $t_0 \in ]0, I_c[$ such that $\varphi_c'' > 0$ on $[0, t_0[$ and $\varphi_c''(t_0) = 0$. As $\varphi_c(t_0) > \sqrt{\frac{1}{1-\alpha}}$ and $\varphi_c'(t_0) > b > 1$. In consequence

$$\alpha\varphi_c(t_0)(1 - \varphi_c'(t_0)) \leq \varphi_c''(t_0) = 0.$$

The conclusion follows from Lemma 2.7.

**Remark 4.2.** Thanks to the preceding proposition, we note that $\varphi_c$ is a convex-concave solution of $(P_{\alpha;a,b,1})$for all $c > 0$.

**Proposition 4.3.** There exists $c^* \leq -\alpha a(b-1)$ such that $E_1 = [c^*, 0]$.

*Proof:* If $b = 1$, then $E_1 = \{0\}$.

If $b > 1$.on the one hand, from Lemma 2.7 with $t_0 = 0$ (or Lemma 5.12 of [10]), it follows that $[-\alpha a(b-1), 0] \subset E_1$. On the other hand, Lemma 5.12 of [10] implies that $E_2$ is an interval of the type $]-\infty, c^*[$. This complete the proof since $E_1 = ]-\infty, 0] \backslash E_2$.

**Remark 4.4.** From the preceding proposition, we have that $0 \notin E_2$.

**Proposition 4.5.** If $c \in E_{2,1}$, then $I_c = +\infty$ and $\varphi_c'(t) \to 1$ as $t \to +\infty$.

*Proof:* Let $c \in E_{2,1}$. By Proposition 4.3, we have $c < 0$.

We impose that $\varphi_c'' < 0$ on $]0, I_c[$. Then $\varphi_c'$ is decreasing and $0 < \varphi_c' \leq b$. From Lemma 2.3 and Lemma 2.4 we get that $I_c = +\infty$ and also $\varphi_c'$ has a limit $\gamma$ at infinity such that $\gamma \in \{-1, 1\}$. By definition of the set $E_{2,1}$ we obtain

$$\exists t_c \in [0, +\infty[ \text{ such that } \varphi_c'(t_c) = 1$$

Also we have $\varphi_c''$ vanishes on $]0, I_c[$, let $t_0$ be the first point where $\varphi_c''$ vanishes. Thanks to Lemma 2.2 item 3, we have $0 < \varphi_c'(t_0) \leq 1$, and the conclusion follows from Lemma 2.8.

**Remark 4.6.** If $c \in E_{2,1}$ then $\varphi_c$ is a concave-convex solution of $(P_{\alpha;a,b,1})$.

**Theorem 4.7.**We consider $\alpha \in [\frac{1}{2}, 1[$, $a \geq \sqrt{\frac{1}{1-\alpha}}$ and $b \geq 1$. There exists $c_* < 0$ where:

1) $\varphi_c$ is a solution of $(P_{\alpha;a,b,1})$ for all $c \in ]c_*, +\infty[$. Moreover, there exists $c^* \in [c_*, -\alpha\, a(b-1)]$;
2) $\varphi_c$ is a convex-concave solution of $(P_{\alpha;a,b,1})$ for all $c \in ]0, +\infty[$;
3) $\varphi_c$ is a concave solution of $(P_{\alpha;a,b,1})$ ; for all $c \in [c^*, 0]$;
4) $\varphi_c$ is a concave-convex solution of $(P_{\alpha;a,b,1})$ for all $c \in ]c_*, c^*[$;

5) $\varphi_c$ is a concave-convex-concave solution of $(P_{\alpha;a,b,1})$ for all $c\in]c_*,\ c^*[$.

*Proof:* All these results follow from the Lemmas 2.1, 2.7, 2.8 and 2.9, and the Propositions 4.1, 4.3 and 4.5.

## 5. THE CASE $\alpha\in]0,\frac{1}{2}]$ AND $-1<b<1$

We consider $\alpha\in]0,\frac{1}{2}]$, $a\geq 0$ and $-1<b<1$. In this case, we divide $\mathbb{R}$ into four sets $E'_{0,1}, E'_{0,2}, E_1{}'$ and $E_2{}'$ such that

$$E'_{0,1}=\{\ c<0:\ \varphi'_c>-1 \text{ on } [0,I_c[\ \}$$

$$E'_{0,2}=\{\ c<0: \exists\ x_c\in]0,I_c[ \text{ such that } \varphi'_c>-1 \text{ on } [0,x_c[ \text{ and } \varphi'_c(x_c)=-1\}$$

$$E'_1=\{\ c\geq 0;\ b\leq\varphi'_c\leq 1 \text{ and } \varphi''_c\geq 0 \text{ on } [0,I_c[\}$$

$$E'_2=\{c\geq 0;\ \exists\ z_c\in[0,I_c[,\exists\ \delta_c>0 \text{ such that } \varphi'_c>1 \text{ on } ]0,z_c[,$$

$$\varphi'_c>1 \text{ on } ]z_c,z_c+\delta_c[ \text{ and } \varphi''_c>0 \text{ on } [0,z_c+\delta_c[\ \}$$

The proofs employed in the preceding section, can be employed here. First, as $\rho(x)=\alpha(x^2-1)<0$ for $x\in]-1,b]$ such that $b\in]-1,0[$, the function $\rho$ is nonincreasing on $]-1,b]$, it follows from Theorem 5.5 of [10] that there exists a unique $c_*$where $\varphi_{c_*}$ is a concave solution of $(P_{\alpha;a,b,-1})$. In addition, we have $c_*<0$. As in the preceding section, this means that $E'_{0,2}=]-\infty,\ c_*[$. Moreover$E'_{0,1}=[c_*,0[$, and if $c\in]c_*,0[$, then $\varphi''_c$vanishes at a first point where $\varphi'_c<1$.

After, in the same manner as in the proof of Proposition 3.2, we prove that $c^*=inf E_1{}'$ is finite, and moreover that $E'_1=[0,c^*]$ and $E_2{}'=]c^*,+\infty[$. Hence, from Proposition 4.3, we get $c^*\leq-\alpha\ a(b-1)$. On the other hand, it follows from Lemma 2.6 that, if $c\in E_2{}'$, then $\varphi''_c$ vanishes at a first point where $\varphi'_c>1$.

All this, mixed with an appropriate employ of Lemmas 2.7, 2.8 and 2.9 permit to make the following theorem.

**Theorem 4.7.** We consider $\alpha\in\left]0,\frac{1}{2}\right]$, $a\geq 0$ and $-1<b<1$. There exists $c_*<0$ and $c^*\geq a(1-b)$ where:

1) $\varphi_c$ is a concave solution of $(P_{\alpha;a,b,-1})$ ; if $b\in]-1,0[$;
2) $\varphi_c$ is a concave-convex solution of $(P_{\alpha;a,b,1})$ for all $c\in]c_*,\ 0[$;
3) $\varphi_c$ is a concave-convex-concave solution of $(P_{\alpha;a,b,1})$ for all $c\in]c_*,\ 0[$;
4) $\varphi_c$ is a convex solution of $(P_{\alpha;a,b,1})$ for all $c\in[0,\ c^*]$;
5) $\varphi_c$ is a convex-concave solution of $(P_{\alpha;a,b,1})$ for all $c\in]c^*,+\infty[$.

## 6. CONCLUSIONS

In this work we have presented a set of new and important results for a problem arises when looking for similarity solutions to problem of boundary-layertheory. We studied the existence, uniqueness and the sign of concave, convex, convex-concave, concave- convex, concave-convex-concave and convex-concave-convex solutions to the autonomous third order nonlinear differential equation $\varphi''' + \varphi\varphi'' + \alpha\ (\varphi'^2 - 1) = 0$, where $0 < \alpha < 1$ and $\rho$ is a given continuous function. Associated with the above equation, we have the following boundary conditions $\varphi\ (0) = a \geq \sqrt{\frac{1}{1-\alpha}}$ , $\varphi'(0) = b \geq 0$ and $\varphi'(+\infty) = \gamma \in \{-1,1\}$, we use shooting technique and consider the initial conditions $\varphi\ (0) = a$, $\varphi'(0) = b$ and $\varphi''\ (0) = c$ where $a, b$ and $c \in \mathbb{R}$ we prove that there exists an infinitely many solutions such that $\varphi'(+\infty) = 1$.

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