ORIGINAL PAPER

ON WEAK INTERPOLATIVE CONTRACTIONS WITH APPLICATION TO HOMOTOPY THEORY

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Abstract. *In this paper, we firstly introduce definition of weak interpolative Hardy-Rogers type contractions and we give an example for a class of this mappings. We also prove a fixed point theorem under some appropriate conditions on complete $b-$metric spaces. Finally, we discuss an application to homotopy theory of our main result.*

Keywords: *Fixed point; weak interpolative contraction mapping; b-metric spaces.*

1. INTRODUCTION AND PRELIMINARIES

Classes of contraction mappings are very important in fixed point theory and there are its applications such as differential equations, matrix equations, boundary value problems, integral equations, homotopy theory, etc. These mapping classes also have many generalizations as in the literature [1-4].

Recently, Berinde [5] introduced the class of weak contraction defined as follows:

Definition 1. [5] A mapping $F: X \to X$ is said to be weak contraction if there exists $\mu \in [0,1)$ and $K \geq 0$ such that

$$d(Fp, Fq) \leq \mu d(p, q) + K d(q, Fp) \tag{1.1}$$

for all $p, q \in X$. Taking $\mu = 0$ at the above inequality, the contraction mapping is obtained. That is, weak contraction mappings are the generalizations of several contractive mappings.

Berinde also gave the following theorem for the existence of fixed points of weak contraction mappings on complete metric spaces.

Theorem 1. [5] Let (X, d) be a complete metric space and $F: X \to X$ be a weak contraction. Then F has a fixed point in X.

Recently, Roy and Saha [6] introduced a new type of weak contraction mapping as follows:

Definition 2. [6] Let (X, d) be a metric space. A mapping $F: X \to X$ is said to be special-type weak contraction if for some $\mu \in [0,1)$ and for some $K \geq 0$,

$$d(Fp, Fq) \leq \mu d(p, q) + K d(p, Fq) d(q, Fp) \tag{1.2}$$

for all $p, q \in X$.

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Theorem 2. [6] A special-type weak contraction mapping F over a complete metric space X possesses at least one fixed point in X.

In 2018, Karapinar [7] introduced a new concept of contractions with the notion of interpolation and he proved a fixed point theorem for such mappings on metric spaces.

Definition 3. [7] Let (X, d) be a metric space. The mapping $F: X \to X$ is called interpolative Kannan-type contractive mapping, if there are constants $\mu \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Fp, Fq) \leq \mu[d(p, Fp)]^{\alpha}[d(q, Fq)]^{1-\alpha} \tag{1.3}$$

for all $p, q \in X \setminus Fix(F)$ where $Fix(F)$ is the set of the fixed points of the mapping F.

Theorem 3. [7] Let (X, d) be a complete metric space and $F: X \to X$ be a interpolative Kannan-type contractive mapping. Then F has at least one fixed point in X.

After, Karapinar et al. [8] introduced the concept of interpolative Reich–Rus–Ciric type contractions as follows. They also proved that these contractions have a fixed point in partial metric spaces.

Definition 4. [8] Let (X, d) be a metric space. The mapping $F: X \to X$ is said to be an interpolative Reich–Rus–Ciric type contraction, if there are constants $\mu \in [0,1)$ and $\alpha, \beta \in (0,1)$ such that

$$d(Fp, Fq) \leq \mu d(p, q)]^{\alpha}[d(p, Fp)]^{\beta}[d(q, Fq)]^{1-\alpha-\beta} \tag{1.4}$$

for all $p, q \in X \setminus Fix(F)$.

From the definition above, Karapinar et al. [9] also gave the concept of interpolative Hardy-Rogers type contractions as follows, and, they showed that these contractions have a fixed point in complete metric spaces.

Definition 5. [9] Let (X, d) be a metric space. The mapping $F: X \to X$ is called interpolative Hardy-Rogers type contraction, if there are constants $\mu \in [0,1)$ and $\alpha, \beta, \eta \in (0,1)$ such that

$$\begin{aligned} d(Fp, Fq) \leq \mu d(p, q)]^{\alpha}[d(p, Fp)]^{\beta}[d(q, Fq)]^{\eta} \\ [\tfrac{1}{2}(d(p, Fq) + d(q, Fp)]^{1-\alpha-\beta-\eta} \end{aligned} \tag{1.5}$$

for all $p, q \in X \setminus Fix(F)$.

After then, some authors proved some fixed point results for the classes of these mappings in the setting of metric spaces (see, e.g., [10-13]).

The notion of b −metric space was introduced by Bakhtin [14] in 1989.

Definition 6. [14] Let X be a non-empty set and $s \geq 1$ be a real number. A mapping $d: X \times X \to \mathbb{R}$ is said to be b −metric if it satisfies the following conditions:

(B1) $d(p, q) \geq 0$ for all $p, q \in X$.
(B2) $d(p, q) = 0$ if and only if $p = q$.
(B3) $d(p, q) = d(q, p)$ for all $p, q \in X$.
(B4) $d(p, r) \leq s[d(p, q) + d(q, r)]$ for all $p, q, r \in X$.

We know that every metric space is a b −metric space but the reverse is not true.

Roy and Saha [15] introduced weak interpolative contraction mapping as follows and they proved some fixed point theorems for such contraction in complete b −metric space.

Definition 7. [15] Let (X,d) be a $b-$metric space with constant $s \geq 1$. The mapping $F: X \to X$ is called weak interpolative contraction mapping if there are constants $\mu \in \left(0, \frac{1}{s}\right)$; $\gamma, \delta > 0$ and $K \geq 0$ such that

$$d(Fp, Fq) \leq \mu d(p,q) + K d(q, Fp)^{\gamma} d(p, Fq)^{\delta} \tag{1.6}$$

for all $p, q \in X$.

Definition 8. [15] Let (X,d) be a $b-$metric space with constant $s \geq 1$. The mapping $F: X \to X$ is called weak interpolative Kannan-type contractive mapping if there exist constants $\mu \in \left(0, \frac{1}{s}\right)$, $\xi \in (0,1)$; $\gamma, \delta > 0$ and $K \geq 0$ such that

$$d(Fp, Fq) \leq \mu d(p, Fp)^{\xi} d(q, Fq)^{1-\xi} + K d(q, Fp)^{\gamma} d(p, Fq)^{\delta} \tag{1.7}$$

for all $p, q \in X \setminus Fix(F)$.

Definition 9. [15] Let (X,d) be a $b-$metric space with constant $s \geq 1$. The mapping $F: X \to X$ is called generalized weak interpolative Reich-type contractive mapping if there exist constants $\mu \in \left(0, \frac{1}{s}\right)$, $\xi \in (0,1)$, $\eta \in 0,1)$; $\gamma, \delta > 0$, and $K \geq 0$ such that

$$d(Fp, Fq) \leq \mu d(p,q)^{\xi} d(p, Fp)^{\eta} d(q, Fq)^{1-\xi-\eta} + K d(q, Fp)^{\gamma} d(p, Fq)^{\delta} \tag{1.8}$$

for all $p, q \in X \setminus Fix(F)$.

Inspired by the above works, we firstly introduced weak interpolative Hardy-Rogers type contraction mapping in complete $b-$metric space. We prove a fixed point result for such mappings. Further, we provide a application to homotopy theory to show the the impact of our main results.

2. MAIN RESULTS

In this section, firstly we give the definition of weak interpolative Hardy-Rogers type contraction mapping as follows.

Definition 10. Let (X,d) be a $b-$metric space with constant $s \geq 1$. The mapping $F: X \to X$ is called weak interpolative Hardy-Rogers type contractive mapping if there are constants $\mu \in (0, \frac{1}{s^{1+\frac{1}{1-\eta}}})$, $\alpha, \beta, \eta \in (0,1)$; $\gamma, \delta > 0$ and $K \geq 0$ such that

$$d(Fp, Fq) \leq \mu d(p,q)^{\alpha} d(p, Fp)^{\beta} d(q, Fq)^{\eta} [\frac{1}{2}(d(p, Fq) + d(q, Fp))]^{1-\alpha-\beta-\eta} + K d(q, Fp)^{\gamma} d(p, Fq)^{\delta} \tag{2.1}$$

for all $p, q \in X \setminus Fix(F)$.

Remark 1.

(i) Taking $\alpha + \beta + \eta = 1$ in the inequality (2.1), then it reduces to weak interpolative Reich-type contractive mapping (1.8).

(ii) In particular if we take $K=0$ and $\alpha,\beta,\eta\in(0,1)$, then the weak interpolative Hardy-Rogers contraction (2.1) reduces to the interpolative Hardy-Rogers contractive condition (1.5).

Theorem 4. Let (X,d) be a complete $b-$metric space and $F: X\to X$ be a weak interpolative Hardy-Rogers-type contraction map. Then F possesses a fixed point in X.

Proof: We suppose that $\{p_n\}$ is a Picard sequence given as $p_n=F^n(p_0)$ for starting point $p_0\in X$. If we take $p_n=p_{n+1}$ for some $n\in\mathbb{N}$, we get that p_n is a fixed point of F. So, we assume that $p_n\neq p_{n+1}$ for all $n\geq 0$. From (2.1) and Picard sequence, we get that

$$\begin{aligned} d(p_{n+1},p_n) = d(Fp_n,Fp_{n-1}) &\leq \mu d(p_n,p_{n-1})^{\alpha} d(p_n,Fp_n)^{\beta} d(p_{n-1},Fp_{n-1})^{\eta} \\ &[\tfrac{1}{2}(d(p_n,Fp_{n-1})+d(p_{n-1},Fp_n))]^{1-\alpha-\beta-\eta} \\ &+Kd(p_{n-1},Fp_n)^{\gamma} d(p_n,Fp_{n-1})^{\delta} \\ &= \mu d(p_n,p_{n-1})^{\alpha} d(p_n,Fp_n)^{\beta} d(p_{n-1},Fp_{n-1})^{\eta} \\ &[\tfrac{1}{2}(d(p_{n-1},Fp_n))]^{1-\alpha-\beta-\eta} \\ &\leq \mu d(p_n,p_{n-1})^{\alpha} d(p_n,Fp_n)^{\beta} d(p_{n-1},Fp_{n-1})^{\eta} \\ &s[\tfrac{1}{2}\left(d\left(p_{n-1,}p_n\right)+d(p_n,p_{n+1})\right)]^{1-\alpha-\beta-\eta}. \end{aligned} \tag{2.2}$$

Now, we suppose that $d(p_{n-1},p_n)<d(p_n,p_{n+1})$ for some $n\geq 1$. Thus,

$$\frac{1}{2}(d(p_{n-1,}p_n)+d(p_n,p_{n+1}))\leq d(p_n,p_{n+1}).$$

Using the inequality (2.2), we obtain

$$d(p_{n+1},p_n)^{\eta}\leq s\mu d(p_n,p_{n-1})^{\eta}$$

Since $s\mu<1$, it is clear that

$$d(p_{n+1},p_n)^{\eta}\leq d(p_n,p_{n-1})^{\eta}. \tag{2.3}$$

The inequality (2.3) contains a contradiction. That is, we have $d(p_n,p_{n+1})\leq d(p_n,p_{n-1})$ for all $n\geq 1$. Therefore, $\{d(p_{n-1},p_n)\}$ is a non-increasing sequence. We can write

$$\frac{1}{2}(d(p_{n-1,}p_n)+d(p_n,p_{n+1}))\leq d(p_{n-1},p_n) \tag{2.4}$$

for all $n\geq 1$. From (2.1), we have

$$\begin{aligned} d(p_1,p_2) = d(Fp_0,Fp_1) &\leq \mu d(p_0,p_1)^{\alpha} d(p_0,Fp_0)^{\beta} d(p_1,Fp_1)^{\eta} \\ &[\tfrac{1}{2}(d(p_0,Fp_1)+d(p_1,Fp_0))]^{1-\alpha-\beta-\eta} \\ &+Kd(p_1,Fp_0)^{\gamma} d(p_0,Fp_1)^{\delta} \\ &= \mu d(p_0,p_1)^{\alpha} d(p_0,p_1)^{\beta} d(p_1,p_2)^{\eta}[\tfrac{1}{2}(d(p_0,p_2))]^{1-\alpha-\beta-\eta} \\ &\leq \mu d(p_0,p_1)^{\alpha} d(p_0,p_1)^{\beta} d(p_1,p_2)^{\eta} \\ &s[\tfrac{1}{2}(d(p_{0,}p_1)+d(p_1,p_2))]^{1-\alpha-\beta-\eta}. \end{aligned}$$

Using (2.4) in the above inequality, we get

$$d(p_1,p_2) \le \mu d(p_0,p_1)^\alpha d(p_0,p_1)^\beta d(p_1,p_2)^\eta s[(d(p_{0,}p_1))]^{1-\alpha-\beta-\eta}$$

which implies that

$$d(p_1,p_2)^{1-\eta} \le (\mu s) d(p_0,p_1)^{1-\eta}.$$

Then

$$\begin{aligned} d(p_1,p_2) &\le (\mu s)^{\frac{1}{1-\eta}} d(p_0,p_1) \\ &\le \mu s^{\frac{1}{1-\eta}} d(p_0,p_1) \end{aligned}$$

as $s \ge 1$ and $\mu \in (0, \frac{1}{s^{1+\frac{1}{1-\eta}}})$. For all $r \ge 1$, we have

$$\begin{aligned} d(p_n,p_{n+r}) &\le s[d(p_n,p_{n+1}) + d(p_{n+1},p_{n+r})] \\ &\le s\mu^n s^{\frac{n}{1-\eta}} d(p_0,p_1) + s d(p_{n+1},p_{n+r}) \\ &\le s\mu^n s^{\frac{n}{1-\eta}} d(p_0,p_1) + s^2[d(p_{n+1},p_{n+2}) + d(p_{n+2},p_{n+r})] \\ &\le s\mu^n s^{\frac{n}{1-\eta}} d(p_0,p_1) + s^2\mu^{n+1} s^{\frac{n+1}{1-\eta}} d(p_0,p_1) + s^2 d(p_{n+2},p_{n+r}) \\ &\vdots \\ &\le [s\mu^n s^{\frac{n}{1-\eta}} + s^2\mu^{n+1} s^{\frac{n+1}{1-\eta}} + \ldots + s^{r-1}\mu^{n+r-2} s^{\frac{n+r-2}{1-\eta}}] d(p_0,p_1) + \\ & s^{r-1} d(p_{n+r-1},p_{n+r}) \\ &\le [(s\mu s^{\frac{1}{1-\eta}})^n + (s\mu s^{\frac{1}{1-\eta}})^{n+1} + \ldots + (s\mu s^{\frac{1}{1-\eta}})^{n+r-1}] d(p_0,p_1) \\ &= (s\mu s^{\frac{1}{1-\eta}})^n . \frac{1-(s\mu s^{\frac{1}{1-\eta}})^r}{1-s\mu s^{\frac{1}{1-\eta}}} d(p_0,p_1) \\ &\le \frac{(s\mu s^{\frac{1}{1-\eta}})^n}{1-s\mu s^{\frac{1}{1-\eta}}} d(p_0,p_1) \to 0, \text{as } n \to \infty. \end{aligned}$$

Then, $\{p_n\}$ is Cauchy sequence in X. Since X is complete, the sequence $\{p_n\}$ is convergent and let $\lim_{n\to\infty} p_n = p \in X$. Now,

$$d(p_{n+1},Fp) = d(Fp_n,Fp) \le \mu d(p_n,p)^\alpha d(p_n,Fp_n)^\beta d(p,Fp)^\eta \tag{2.5}$$

$$[\tfrac{1}{2}\big(d(p_n,Fp) + d(p,Fp_n)\big)]^{1-\alpha-\beta-\eta} + K d\big(p,Fp_n)^\gamma d\big(p_n,Fp)^\delta\big)].$$

Taking $n \to \infty$ in the inequality (2.5), we find $\lim_{n\to\infty} d(p_{n+1},Fp) \le 0$. This implies $p = Fp$, i.e. $p \in X$ is a fixed point of F.

Example 1. Let $X = [0,6] \subset \mathbb{R}$ and let the metric $d: X \times X \to \mathbb{R}$ be defined as $d(x,y) = |x-y|^2$ for all $x,y \in X$. We also know that (X,d) is a b −metric space with $s = 2$. We also define a mapping $F: X \to X$ as follows

$$F(p) = \begin{cases} 0 & \text{if } 0 \le p \le 3 \\ 6 & \text{if } 3 < p \le 6 \end{cases}.$$

We suppose that the constants like $\mu = \frac{1}{5}$, $\alpha = \beta = \eta = \frac{1}{9}$, $\gamma = \delta = 1$ and $K = \frac{1}{36}$. We claim that F satisfies the inequality (2.1), but F does not satisfy interpolative Hardy-Rogers type contraction (1.5) for $p = 3$ and $q = 4$. For this, we will consider the following three cases.

i) If $p, q \in [0,3]$, then it is easy to see that inequality (2.1) holds.

ii) If $p, q \in (3,6]$, then it is easy to see that inequality (2.1) holds.

iii) Assume that $p \in [0,3]$ and $q \in (3,6]$. Set

$$A = \mu d(p,q)]^{\alpha}[d(p,Fp)]^{\beta}[d(q,Fq)]^{\eta}[\frac{1}{2}(d(p,Fq)+d(q,Fp)]^{1-\alpha-\beta-\eta}.$$

From (2.1), we have

$$d(Fp,Fq) \leq A + Kd(q,Fp)^{\gamma}d(p,Fq)^{\delta}$$

If we write $p = 3$ and $q = 4$ in the above inequality,

$$\begin{aligned} 36 &\leq A + Kd(q,0)^{\gamma}d(p,6)^{\delta} \\ &\leq A + K.(36)^{\gamma}(36)^{\delta}. \end{aligned}$$

Since $\gamma = \delta = 1$, we have

$$\begin{aligned} 36 &\leq A + K.(36)^{\gamma}(36)^{\delta} \\ &= A + K.36^{2}, \end{aligned}$$

and, the above inequality holds for $K = \frac{1}{36}$. This implies that F satisfies weak interpolative Hardy-Rogers type contractive mapping (2.1). Now we will show that F does not satisfy interpolative Hardy-Rogers type contraction (1.5) for $p = 3$ and $q = 4$.

$$\begin{aligned} d(F3,F4) &\leq \frac{1}{5}d(3,4)^{\alpha}d(3,F3)^{\beta}d(4,F4)^{\eta}.[\frac{1}{2}(d(3,F4)+d(4,F3))]^{\frac{2}{3}} \\ 36 &= \frac{1}{5}(1)^{\alpha}(9)^{\beta}(4)^{\eta}.[\frac{9+36}{2}]^{\frac{2}{3}} \\ &= \frac{1}{5}(36)^{\frac{1}{9}}.[22,5]^{\frac{2}{3}} \\ &\approx \frac{1}{5}.1,489.7,969 \\ &\approx 2,373. \end{aligned}$$

which implies that a contradiction. Thus F is not interpolative Hardy-Rogers type contraction (1.5).

Now, we will give an application to homotopy theory of our main result Theorem 4. Next, we recall the following definition that will be useful in proving our result.

Definition 11. ([6, 16]) Let X, Y be two topological spaces, and let $R, S: X \to Y$ be two continuous mappings. Then, a homotopy from R to S is a continuous function $H: X \times [0,1] \to Y$ such that $H(p,0) = R(p)$ and $H(p,1) = S(p)$ for all $p \in X$ and $[0,1] \subset \mathbb{R}$. Here, R and S are called homotopic mappings.

Theorem 5. Let (X,d) be a complete metric space and U be an open subset of X. Let the operator $H:\overline{U}\times[0,1]\to X$ satisfy the following conditions:

(a) $p\neq H(p,t)$ for every $p\in\partial U$ (∂U denotes the boundary of U in X) and for any $t\in 0,1]$.

(b)

$$d(H(p,t),H(q,t))\leq \mu d(p,q)^{\alpha}d(p,H(p,t))^{\beta}d(q,H(q,t))^{\eta}\times$$

$$[\frac{d(p,H(q,t))+d(q,H(p,t))}{2}]^{1-\alpha-\beta-\eta}+Kd(q,H(p,t))^{\gamma}d(p,H(q,t))$$

for all $p,q\in\overline{U}$ and for any $t\in[0,1]$, where $\mu\in(0,1)$ and $K,\gamma>0$.

(c) There exists a continuous function $g:[0,1]\to\mathbb{R}$ such that

$$d(H(p,t),H(p,s))\leq|g(t)-g(s)|$$

for all $t,s\in[0,1]$ and for every $p\in\overline{U}$.

(d)

$$\sup_{p,q\in\overline{U},t\in 0,1]} d(q,H(p,t))<K^{-\frac{1}{\gamma}}.$$

Then $H(.,0)$ has a fixed point if and only if $H(.,1)$ has a fixed point.

Proof: We consider the following set

$$\Delta=\{t\in[0,1]:H(.,t)\text{ has a fixed point in }U\}.$$

Assume that $H(.,0)$ has a fixed point in $\overline{U}$. We know that there exists $p\in U$ such that $H(p,0)=p$ because of the condition (a). So $0\in\Delta$ and then Δ is nonempty subset of $[0,1]$. Next, we will show that Δ is both open and closed subset of $[0,1]$. We can say that $\Delta=[0,1]$ from connectedness of $[0,1]$. First we will show that Δ is open. Suppose that $t_0\in\Delta$. Then there exists $p_0\in U$ such that $H(p_0,t_0)=p_0$. Since the set U is open, there exists $r>0$ such that $B(p_0,r)\subset U$. Let $0<\epsilon\leq r[1-\mu-K(\sup_{p\in\overline{B(p_0,r)}} d(p_0,H(p,t_0))^{\gamma})]$. Since the mapping g is continuous, there exists $\delta(\epsilon)>0$ such that $|g(t)-g(t_0)|<\epsilon$ for all $t\in\left(t_0-\delta(\epsilon),t_0+\delta(\epsilon)\right)\subset[0,1]$. Now, assume that $p\in\overline{B(p_0,r)}$. From conditions (b), (c) and (d), we obtain that

$$d(H(p,t),p_0)=d(H(p,t),H(p_0,t_0))$$

$$\leq d(H(p,t),H(p,t_0))+d(H(p,t_0),H(p_0,t_0))$$

$$\leq|g(t)-g(t_0)|+\mu d(p,p_0)^{\alpha}d(p,H(p,t_0))^{\beta}d(p_0,H(p_0,t_0))^{\eta}\times$$

$$[\frac{d(p,H(p_0,t_0))+d(p_0,H(p,t_0))}{2}]^{1-\alpha-\beta-\eta}$$

$$+Kd(p_0,H(p,t_0))^{\gamma}d(p,H(p_0,t_0))$$

$$< \epsilon + \mu d(p,p_0)^{\alpha} d(p,H(p,t_0))^{\beta} d(p_0,p_0)^{\eta}$$

$$[\frac{d(p,p_0) + d(p_0,H(p,t_0))}{2}]^{1-\alpha-\beta-\eta}$$

$$+Kd(p_0,H(p,t_0))^{\gamma} d(p,H(p_0,t_0))$$

$$= \epsilon + Kd(p_0,H(p,t_0))^{\gamma} d(p,p_0)$$

$$< \epsilon + K(\sup_{p\in\overline{B(p_0,r)}} d(p_0,H(p,t_0))^{\gamma})r$$

$$\leq r$$

for all $t \in \left(t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)\right) \subset [0,1]$. Thus, $H(.,t)$ is self mapping on $\overline{B(p_0,r)}$ for every fixed $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon))$. Since $H(.,t)$ satisfies all the conditions of Theorem 4, we have $H(.,t)$ has a fixed point in $\overline{B(p_0,r)} \subset \overline{U}$. Since the condition (a) holds, it must be in U. Then $t \in \Delta$ for every $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon))$. Therefore $(t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset \Delta$, which implies that Δ is open in [0,1].

Now we will show that Δ is closed also. Let $\{t_n\} \subset \Delta$ be such that $t_n \to t^* \in [0,1]$ as $n \to \infty$. Then there exists $p_n \in U$ such that $p_n = H(p_n,t_n)$ for all $n \in \mathbb{N}$. Also, we have

$$d(p_n,p_m) = d(H(p_n,t_n),H(p_m,t_m))$$

$$\leq d(H(p_n,t_n),H(p_n,t_m)) + d(H(p_n,t_m),H(p_m,t_m))$$

$$\leq |g(t_n) - g(t_m)| + \mu d(p_n,p_m)^{\alpha} d(p_n,H(p_n,t_m))^{\beta} \times$$

$$d(p_m,H(p_m,t_m))^{\eta} [\frac{d(p_n,H(p_m,t_m) + d(p_m,H(p_n,t_m))}{2}]^{1-\alpha-\beta-\eta}$$

$$+Kd(p_m,H(p_n,t_m))^{\gamma} d(p_n,H(p_m,t_m))$$

$$= |g(t_n) - g(t_m)| + Kd(p_m,H(p_n,t_m))^{\gamma} d(p_n,p_m))$$

which implies that

$$[1 - Kd(p_m,H(p_n,t_m))^{\gamma}]d(p_n,p_m) \leq |g(t_n) - g(t_m)|$$

and

$$d(p_n,p_m) \leq [1 - K(\sup_{p,q\in\overline{U},t\in 0,1]} d(q,H(p,t))^{\gamma}]^{-1} |g(t_n) - g(t_m)|$$

for all $n,m \in \mathbb{N}$. If we take limit as $n,m \to \infty$ in above inequality, we get that $\{p_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, then the sequence $\{p_n\}$ is convergent. Let $p_n \to p^* \in \overline{U}$ as $n \to \infty$.

Finally, we will show that $H(p^*,t^*) = p^*$. From the conditions of Theorem 5, we write

$$d(p_n,H(p^*,t^*)) = d(H(p_n,t_n),H(p^*,t^*))$$

$$\leq d(H(p_n,t_n),H(p_n,t^*)) + d(H(p_n,t^*),H(p^*,t^*))$$

$$\leq |g(t_n) - g(t^*)| + \mu d(p_n,p^*)^\alpha d(p_n,H(p_n,t^*))^\beta \times$$

$$d(p^*,H(p^*,t^*))^\eta [\frac{d(p_n,H(p^*,t^*)) + d(p^*,H(p_n,t^*))}{2}]^{1-\alpha-\beta-\eta}$$

$$+Kd(p^*,H(p_n,t^*))^\gamma d(p_n,H(p^*,t^*))$$

$$\leq [1 - K(\sup_{p,q\in\overline{U}} d(q,H(p,t))^\gamma]^{-1} |g(t_n) - g(t^*)|$$

$$+\mu d(p_n,p^*)^\alpha d(p_n,H(p_n,t^*))^\beta d(p^*,H(p^*,t^*))^\eta$$

$$[\frac{d(p_n,H(p^*,t^*)) + d(p^*,H(p_n,t^*))}{2}]^{1-\alpha-\beta-\eta}.$$

Taking the limit of both sides as $n \to \infty$ in above inequality, we get that $p_n \to H(p^*,t^*)$. Then $H(p^*,t^*) = p^*$ which implies that $p^* \in \Delta$ and hence $t^* \in \Delta$. That is, Δ is closed and consequently $\Delta = [0,1]$. Therefore, from the definition of Δ, we can say that $H(.,1)$ has a fixed point in X.

3. CONCLUSION

In this study, the definition of weak interpolative Hardy-Rogers type contractions was defined. After, a fixed point theorem under some appropriate conditions on b −metric spaces was proved. Finally, an application to homotopy theory of our main result was showed.

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