ORIGINAL PAPER

INVESTIGATION OF CONFORMABLE DOUBLE SHEHU TRANSFORM FOR SOLVING SOME FRACTIONAL DIFFERENTIAL PARTIAL EQUATIONS

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Manuscript received: 19.11.2022; Accepted paper: 07.06.2023;
Published online: 30.06.2023.

Abstract: *In this paper, we generalize the concept of single conformable Sehu transformation (CSHT) to double conformable transformation (CDFSHT). Moreover, we are able to prove some theorems and properties related to this work. We apply the double conformable Sehu transform to solve the initial and boundary problems of linear and (non)homogenous conformable fractional partial differential equations (PDEs). The validity and the applicability of the proposed technique are shown by three numerical examples. Mathematica software is used for Euclidean division of polynomials and drawing graphs.*

Keywords: *conformable double Shehu transform; fractional partial differential equations; fractional Klein Gordon; partial derivatives.*

1. INTRODUCTION

Fractional derivatives provide an excellent gadget for describing the dynamic behavior of various complex materials and systems. It has been observed that many physical phenomena can be modeled successfully by means of fractional order differential equations, where integer order differential equations fail to model some problems. To better understand the fractional physical phenomena, there was a great need to identify the solutions of fractional differential models. In fact, finding an analytical solution to fractional nonlinear differential equations is a difficult task. In the literature, different computational schemes were developed to solve the fractional differential partial equation. Instances of such existing methods are the homotopy perturbation method, the homotopy analysis method, direct algebraic method, the wavelet method, the simplest equation method and functional variable method [1-4]. In our last work [5], we have been generalized the definition, some rules and important properties of the Single conformable fractional Shehu transform fractional order. The method of double integral transformations is a hot topic in recent research, and it is mainly based on applying one transformation twice to the functions of two variables. This new approach is a powerful tool for solving PDEs. Their properties and theorems are recent studies, attracting the interest of many mathematicians. Therefore, many researchers have studied new formulations, such as Double Laplace Transform, Double Sumudu Transform, others [6-7].

In this article, we introduce the definition of the Double Conformable Shehu Transform (CDSHT), in fractional order and derive a list of theorems and important properties of this transformation which have been demonstrated to solve $(1+1)$ linear

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fractional partial differential equations in the sense of conformable fractional derivatives in the homogeneous and nonhomogeneous case as Telegraph equations and (α, β) Klein Gordon equations.

This paper is structured as follows: In Section 2, some basic preliminaries of the conformable partial fractional calculus are presented. In Section 3, some properties of the Double Conformable Shehu Transform (CDSHT) are introduced. In section 4, the efficiency and validity of the proposed method are demonstrated by some numerical examples. Finally, conclusion is given in Section 5.

2. CONFORMABLE PARTIAL DERIVATIVES

Definition 1: [3] Let $0 < \alpha \leq 1$, and $f : \mathbb{R} \times \left(0, \infty\right) \to \mathbb{R}$. The conformable fractional partial derivative of $f(x,t)$ of order α with respect to x is given as:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha},t) - f(x,t)}{\varepsilon} \tag{1}$$

provided that the limit exists as a finite number.

Definition 2: [3] Let $0 < \beta \leq 1$, and $f : \mathbb{R} \times \left(0, \infty\right) \to \mathbb{R}$. The conformable space fractional partial derivative of $f(x,t)$ of order β is given as:

$$\frac{\partial^{\beta}}{\partial t^{\beta}} f(x,t) = \lim_{\delta \to 0} \frac{f(x, t + \delta t^{1-\beta}) - f(x,t)}{\delta}, \tag{2}$$

provided that the limit exists as a finite number.

Where $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ and $\frac{\partial^{\beta}}{\partial t^{\beta}}$ are called the fractional derivatives of order α and β, respectively.

Definition 3: [3] (The conformable fractional integral). Let $0 < \alpha \leq 1$, and $f : \left[0, \infty\right) \to \mathbb{R}$

The conformable fractional integral of f of order α from 0 to x is defined as:

$$I^{\alpha} f(x) = \int_{0}^{x} f(t) d_{\alpha} t = \int_{0}^{x} f(t) t^{1-\alpha} dt$$

where the above integral is the usual improper Riemann integral.Furthermore, if $f(x)$ is an α-differentiable function in some [0,∞] and 0<α≤1, then:

$$\begin{cases} D^{\alpha} f(x) = x^{1-\alpha} \dfrac{df(x)}{dx} \\ D^{\alpha} I^{\alpha} f(x) = f(x) \end{cases}$$

Theorem 1: Let $0<\alpha, \beta \le 1$ and $f(x,t)$ are an α,β-differentiable function at a point $x, t \in [0,\infty)$, then

$$\frac{\partial^{\alpha} f(x,t)}{\partial x^{\alpha}} = x^{1-\alpha} \frac{\partial f(x,t)}{\partial x} \tag{3}$$

$$\frac{\partial^{\beta} f(x,t)}{\partial t^{\beta}} = t^{1-\beta} \frac{\partial f(x,t)}{\partial t} \tag{4}$$

Proof: See[5].

Example 1: [6-7] Let $0<\alpha, \beta \le 1$ and a,b,m,n,λ and $\mu \in \mathbb{R}$.

1. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(au(x,t)+bv(x,t)\right) = a\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(x,t)+b\frac{\partial^{\alpha}}{\partial x^{\alpha}}v(x,t).$
2. $\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial t^{\beta}}\left(x^{\mu}t^{\lambda}\right) = \mu\lambda x^{\mu-\alpha}t^{\lambda-\beta}.$
3. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(e^{\mu\frac{x^{\alpha}}{\alpha}+\lambda\frac{t^{\beta}}{\beta}}\right) = \mu e^{\mu\frac{x^{\alpha}}{\alpha}+\lambda\frac{t^{\beta}}{\beta}}.$
4. $\frac{\partial^{\beta}}{\partial t^{\beta}}\left(e^{\mu\frac{x^{\alpha}}{\alpha}+\lambda\frac{t^{\beta}}{\beta}}\right) = \lambda e^{\mu\frac{x^{\alpha}}{\alpha}+\lambda\frac{t^{\beta}}{\beta}}.$
5. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\right)\left(\frac{t^{\beta}}{\beta}\right) = \frac{t^{\beta}}{\beta}.$
6. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\right)^{n}\left(\frac{t^{\beta}}{\beta}\right)^{m} = n\left(\frac{x^{\alpha}}{\alpha}\right)^{n-1}\left(\frac{t^{\beta}}{\beta}\right)^{m}.$
7. $\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{x^{\alpha}}{\alpha}\right)^{n}\left(\frac{t^{\beta}}{\beta}\right)^{m} = m\left(\frac{x^{\alpha}}{\alpha}\right)^{n}\left(\frac{t^{\beta}}{\beta}\right)^{m-1}.$
8. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\sin\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right) = \cos\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right).$
9. $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cos\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right) = -\sin\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right).$

3. CONFORMABLE DOUBLE FRACTIONAL SHEHU TRANSFORM (CDSHT)

Definition 4: [5] (The Single Conformable Fractional Shehu Transform (SCFSHT)):

Let $0<\alpha\le 1$, and $f:[0,\infty)\to\mathbb{R}$ be a real value function. Then, the conformable fractional Shehu transform of order α is defined by:

$$Sh_{\alpha}^{x}\left[f(x)\right]=V_{\alpha}^{x}(s,u)=\int_{0}^{\infty}e^{-\frac{s}{u}\frac{x^{\alpha}}{\alpha}}f(x)x^{1-\alpha}dx,$$

provided the integral exists.

Definition 5: We consider function in the set F:

$$F=\left\{v(x,t),\ \exists N,k_1,k_2>0,\left|v(x,t)\right|<N\exp\left(\frac{\frac{x^{\alpha}}{\alpha}+\frac{t^{\beta}}{\beta}}{k_j^{2}}\right),\ j=1,2,...,\frac{x^{\alpha}}{\alpha}+\frac{t^{\beta}}{\beta}\in\left[0,\infty\right[\right\},$$

The CDSHT is defined by:

$$Sh_{\alpha}^{x}Sh_{\beta}^{t}\left[f(x,t)\right]=V_{\alpha,\beta}\left((s,u),(r,v)\right)=\int_0^{\infty}\int_0^{\infty}e^{\left(-\frac{s}{u}\frac{x^{\alpha}}{\alpha}-\frac{r}{v}\frac{t^{\beta}}{\beta}\right)}f(x,t)x^{1-\alpha}t^{1-\beta}dxdt,$$

where $0<\alpha,\ \beta\le 1$ and $s,u,r,v>0$.

3.1. SOME PROPRIETIES OF THE CONFORMABLE DOUBLE SHEHU TRANSFORM

For certain functions the (CDSHT) is given by:

Theorem 2: Let a,b,c and $p,q\in\mathbb{R}$ and $0<\alpha,\ \beta\le 1$, then:

1. $V_{\alpha,\beta}\{c\}\left((s,u),(r,v)\right)=c\dfrac{uv}{sr}$.
2. $V_{\alpha,\beta}\left\{x^{p}t^{q}\right\}\left((s,u),(r,v)\right)=\alpha^{\frac{p}{\alpha}}\left(\dfrac{u}{s}\right)^{\frac{p}{\alpha}+1}\beta^{\frac{q}{\beta}}\left(\dfrac{v}{r}\right)^{\frac{q}{\beta}+1}\Gamma\left(\dfrac{p}{\alpha}+1\right)\Gamma\left(\dfrac{q}{\beta}+1\right)$.
3. $V_{\alpha,\beta}\left\{\exp\left(a\dfrac{x^{\alpha}}{\alpha}+b\dfrac{t^{\beta}}{\beta}\right)\right\}\left((s,u),(r,v)\right)=\dfrac{uv}{(s-au)(r-bv)},\ \dfrac{s}{u}>0,\ \dfrac{r}{v}>0.$
4. $V_{\alpha,\beta}\left\{\sin\left(a\dfrac{x^{\alpha}}{\alpha}\right)\sin\left(b\dfrac{t^{\beta}}{\beta}\right)\right\}\left((s,u),(r,v)\right)=\dfrac{abu^2v^2}{\left(s^2+a^2u^2\right)\left(r^2+b^2v^2\right)},\ \dfrac{s}{u}>0,\ \dfrac{r}{v}>0.$
5. $V_{\alpha,\beta}\left\{\cos\left(a\dfrac{x^{\alpha}}{\alpha}\right)\cos\left(b\dfrac{t^{\beta}}{\beta}\right)\right\}\left((s,u),(r,v)\right)=\dfrac{sruv}{\left(s^2+a^2u^2\right)\left(r^2+b^2v^2\right)},\ \dfrac{s}{u}>0,\ \dfrac{r}{v}>0.$

Proof:

1. Proof follows easily by the aid of definition.
2. We have:

$$V_{\alpha,\beta}\left\{x^p t^q\right\}\left((s,u),(r,v)\right)=V_\alpha^x\left\{x^p\right\}(s,u).V_\beta^t\left\{t^q\right\}(r,v)$$
$$=V_\alpha^x\left\{(\alpha x)^{\frac{p}{\alpha}}\right\}(s,u).V_\beta^t\left\{(\beta t)^{\frac{q}{\beta}}\right\}(r,v)$$
$$=\alpha^{\frac{p}{\alpha}}V\left\{x^{\frac{p}{\alpha}}\right\}\beta^{\frac{q}{\beta}}V\left\{t^{\frac{q}{\beta}}\right\}$$
$$=\alpha^{\frac{p}{\alpha}}\left(\frac{u}{s}\right)^{\frac{p}{\alpha}+1}\beta^{\frac{q}{\beta}}\left(\frac{v}{r}\right)^{\frac{q}{\beta}+1}\Gamma\left(\frac{p}{\alpha}+1\right)\Gamma\left(\frac{q}{\beta}+1\right).$$

If we put $p=n\alpha,\ q=m\beta$, we get:

$$V_{\alpha,\beta}\left\{x^{n\alpha}t^{m\beta}\right\}\left((s,u),(r,v)\right)=\alpha^n\left(\frac{u}{s}\right)^{n+1}\beta^m\left(\frac{v}{r}\right)^{m+1}\Gamma(n+1)\Gamma(m+1)$$
$$=\alpha^n\left(\frac{u}{s}\right)^{n+1}\beta^m\left(\frac{v}{r}\right)^{m+1}n!m!.$$

3. $V_{\alpha,\beta}\left\{\exp\left(a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}\right)\right\}\left((s,u),(r,v)\right)=V_\alpha^x V_\beta^t\left\{\exp(ax)\exp(bt)\right\}=\dfrac{uv}{(s-au)(r-bv)}.$

4. We have:

$$V_{\alpha,\beta}\left\{\sin\left(a\frac{x^\alpha}{\alpha}\right)\sin\left(b\frac{t^\beta}{\beta}\right)\right\}\left((s,u),(r,v)\right)=V_\alpha^x V_\beta^t\left\{\sin(ax)\sin(bt)\right\}$$
$$=\frac{abu^2v^2}{\left(s^2+a^2u^2\right)\left(r^2+b^2v^2\right)}.$$

5. We have:

$$V_{\alpha,\beta}\left\{\cos\left(a\frac{x^\alpha}{\alpha}\right)\cos\left(b\frac{t^\beta}{\beta}\right)\right\}\left((s,u),(r,v)\right)=V_\alpha^x V_\beta^t\left\{\cos(ax)\cos(bt)\right\}$$
$$=\frac{sruv}{\left(s^2+a^2u^2\right)\left(r^2+b^2v^2\right)}.$$

Theorem 3: The conformable double fractional Shehu transform is linear operator:

$$V_{\alpha,\beta}\left[\mu f(x,t)\pm\lambda g(x,t)\right]=\mu V_{\alpha,\beta}\left[f(x,t)\right]\pm\lambda V_{\alpha,\beta}\left[g(x,t)\right].$$

where μ and λ are constants.

Proof: Trivial by using the definition of (CDSHT).

Lemma 1: Let $0<\alpha,\ \beta\le 1$ and $\phi:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a real value function, the (CDSHT) of $\dfrac{\partial^\alpha\phi}{\partial x^\alpha},\ \dfrac{\partial^\beta\phi}{\partial t^\beta}$ are given respectively as follows*:*

$$V_{\alpha,\beta}\left(\frac{\partial^\alpha\phi}{\partial x^\alpha}\right)=\left(\frac{s}{u}\right)V_{\alpha,\beta}\left(\phi(x;t)\right)\left((s,u),(r,v)\right)-V_\beta^t\left(\phi(0;t)\right)\left((0,u),(r,v)\right). \tag{5}$$

$$V_{\alpha,\beta}\left(\frac{\partial^{\beta}\phi}{\partial t^{\beta}}\right)=\left(\frac{r}{v}\right)V_{\alpha,\beta}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-V_{\alpha}^{x}\left(\phi(x;0)\right)\left(\left(s,u\right),\left(0,v\right)\right). \quad (6)$$

Proof:

1. By using the definition of (CDSHT), we have:

$$\begin{aligned} V_{\alpha,\beta}\left(\frac{\partial^{\alpha}\phi}{\partial x^{\alpha}}\right)&=\int_0^{\infty}\int_0^{\infty}e^{\left(-\frac{s}{u}\frac{x^{\alpha}}{\alpha}-\frac{r}{v}\frac{t^{\beta}}{\beta}\right)}\frac{\partial^{\alpha}\phi}{\partial x^{\alpha}}x^{1-\alpha}t^{1-\beta}dxdt \\ &=\int_0^{\infty}e^{\left(-\frac{r}{v}\frac{t^{\beta}}{\beta}\right)}t^{1-\beta}\left(\int_0^{\infty}e^{\left(-\frac{s}{u}\frac{x^{\alpha}}{\alpha}\right)}\frac{\partial^{\alpha}\phi}{\partial x^{\alpha}}x^{1-\alpha}dx\right)dt. \end{aligned} \quad (7)$$

By substituting eq. (3) in eq. (7), we obtain:

$$\begin{aligned} V_{\alpha,\beta}\left(\frac{\partial^{\alpha}\phi}{\partial x^{\alpha}}\right)&=\int_0^{\infty}e^{\left(-\frac{r}{v}\frac{t^{\beta}}{\beta}\right)}t^{1-\beta}\left(\frac{s}{u}V_{\alpha}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-\left(\phi(0;t)\right)\right)dt \\ &=\frac{s}{u}V_{\alpha,\beta}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-\int_0^{\infty}e^{\left(-\frac{r}{v}\frac{t^{\beta}}{\beta}\right)}t^{1-\beta}\phi(0;t)dt \\ &=\frac{s}{u}V_{\alpha,\beta}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-V_{\beta}^{t}\left(\phi(0;t)\right)\left(\left(0,u\right),\left(r,v\right)\right). \end{aligned}$$

2. Similary, we can prove eq. (6).

Theorem 4: The (CDSHT) of the conformable partial derivativesof $\frac{\partial^{n\alpha}\phi}{\partial x^{n\alpha}}$, $\frac{\partial^{m\beta}\phi}{\partial t^{m\beta}}$ are given respectively as follows**:**

$$V_{\alpha,\beta}\left(\frac{\partial^{n\alpha}\phi}{\partial x^{n\alpha}}\right)=\left(\frac{s}{u}\right)^{n}V_{\alpha,\beta}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{k}V_{\beta}^{t}\left(\phi(0;t)\right)\left(\left(0,u\right),\left(r,v\right)\right). \quad (8)$$

$$V_{\alpha,\beta}\left(\frac{\partial^{m\beta}\phi}{\partial t^{m\beta}}\right)=\left(\frac{r}{v}\right)^{m}V_{\alpha,\beta}\left(\phi(x;t)\right)\left(\left(s,u\right),\left(r,v\right)\right)-\sum_{i=0}^{m-1}\left(\frac{r}{v}\right)^{i}V_{\alpha}^{x}\left(\phi(x;0)\right)\left(\left(s,u\right),\left(0,v\right)\right). \quad (9)$$

where $n, m\in\mathbb{N}^{*}$ and $0<\alpha, \beta\le 1$.

Proof: We can prove by using the Lemma 1.

4. ILLUSTRATIVE EXAMPLES

In this section, we study three examples using the (CDSHT) to solve conformable fractional partial differential equations.

Example 2: [6-9] Consider the conformable fractional partial Telegraph equation:

$$\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}}-\frac{\partial^{2\beta}\phi}{\partial t^{2\beta}}-\frac{\partial^{\beta}\phi}{\partial t^{\beta}}-\phi=1-\frac{x^{2\alpha}}{\alpha^2}-\frac{t^{\beta}}{\beta},\quad 0<\alpha,\ \beta\le 1. \tag{10}$$

subject to the initial conditions

$$\phi(x,0)=\frac{x^{2\alpha}}{\alpha^2},\ \ \phi(0,t)=\frac{t^{\beta}}{\beta},\ \ \frac{\partial^{\alpha}}{\partial x^{\alpha}}\phi(0,t)=0,\ \ \frac{\partial^{\beta}}{\partial t^{\beta}}\phi(x,0)=1. \tag{11}$$

We applying the(CDSHT) to both side eq (10) and conformable Fractional Shehu transform (CFHT) to initial conditions eq. (11), we get:

$$\left(\frac{s}{u}\right)^2 V_{\alpha,\beta}\left(\phi(x;t)\right)-V_{\beta}^{t}\left(\phi(0;t)\right)-\left(\frac{r}{v}\right)^2 V_{\alpha,\beta}+V_{\alpha}\left(\phi(x,0)\right)+\frac{r}{v}V_{\alpha}(1)-\frac{r}{v}V_{\alpha,\beta}\left(\phi(x;t)\right)$$
$$+V_{\alpha}\left(\phi(x,0)\right)-V_{\alpha,\beta}\left(\phi(x;t)\right)=\frac{uv}{sr}-2\left(\frac{u}{s}\right)^3\frac{v}{r}-\frac{u}{s}\left(\frac{v}{r}\right)^2.$$

By substituting the Conformable Single Shehu transform (SCFHT) to initial conditions, we obtain:

$$\left(\left(\frac{s}{u}\right)^2-\left(\frac{r}{v}\right)^2-\frac{r}{v}-1\right)V_{\alpha,\beta}=\frac{uv}{sr}-2\left(\frac{u}{s}\right)^3-\frac{u}{s}\left(\frac{v}{r}\right)^2+\frac{s}{u}\left(\frac{v}{r}\right)^2-2\left(\frac{u}{s}\right)^3\frac{v}{r}-2\left(\frac{u}{s}\right)^3\frac{r}{v}+\frac{uv}{sr}-\frac{u}{s}$$
$$=\frac{2uv}{sr}-2\left(\frac{u}{s}\right)^3-\frac{u}{s}\left(\frac{v}{r}\right)^2+\frac{s}{u}\left(\frac{v}{r}\right)^2-2\left(\frac{u}{s}\right)^3\frac{v}{r}-2\left(\frac{u}{s}\right)^3\frac{r}{v}-\frac{u}{s}. \tag{12}$$

After simplifying, eq. (12) become

$$V_{\alpha,\beta}(\phi(x,t))=2\left(\frac{u}{s}\right)^3\frac{v}{r}+\frac{u}{s}\left(\frac{v}{r}\right)^2. \tag{12}$$

By taking $V_{\alpha,\beta}^{-1}$ of both sides of eq. (13), we get:

$$\phi(x,t)=\frac{x^{2\alpha}}{\alpha^2}+\frac{t^{\beta}}{\beta}. \tag{13}$$

If we put $\alpha=1$and $\beta=1$, the exact solution for eq. (10) is:

$$\phi(x,t)=x^2+t.$$

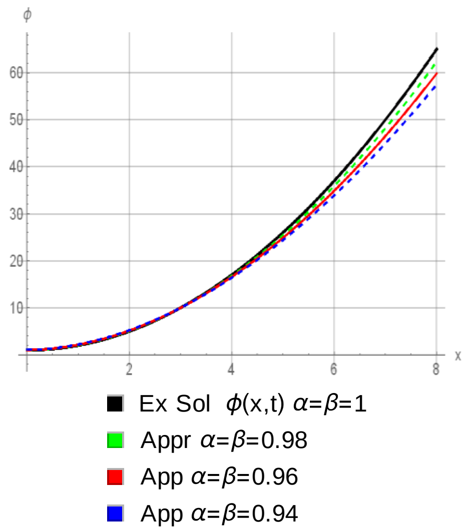

Figure 1. The exact and approximate solution $\phi(x,t)$ when $\alpha=\beta$.

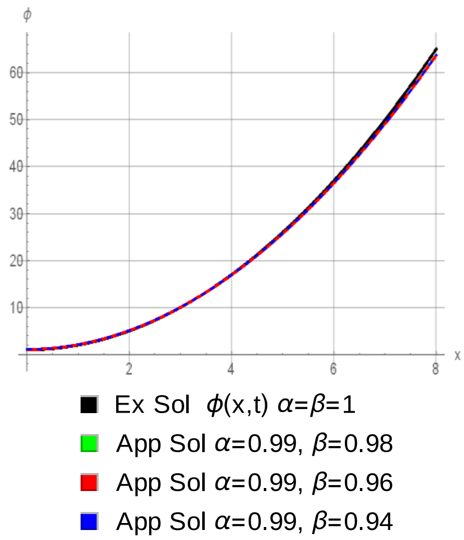

Figure 2. The exact and $\phi(x,t)$ different values when $\alpha=0.99$ and β takedifferent values offractional order.

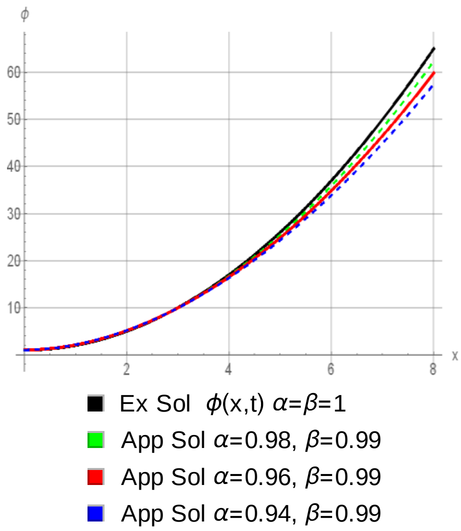

Figure 3. The exact and approximate solution $\phi(x,t)$ when $\beta=0.99$ and α take different values of fractional order.

Example 3: [5] Consider the following linear non-homogeneous (α,β) conformable fractional Klein Gordon equation:

$$\frac{\partial^{2\beta}\phi}{\partial t^{2\beta}}-\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}}-2\phi=-2\sin\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right),\quad 0<\alpha,\beta\leq 1. \tag{14}$$

with the initial condition

$$\phi(x,0)=\phi(0,t)=0,\ \ \frac{\partial^{\alpha}}{\partial x^{\alpha}}\phi(0,t)=\sin\left(\frac{t^{\beta}}{\beta}\right),\ \ \frac{\partial^{\beta}}{\partial t^{\beta}}\phi(x,0)=\sin\left(\frac{x^{\alpha}}{\alpha}\right). \tag{15}$$

We applying (CDSHT) of eq. (14) and (CSHT) of initial conditions eq. (15), we obtain:

$$V_{\alpha,\beta}\left(\frac{\partial^{2\beta}\phi}{\partial t^{2\beta}}\right)-V_{\alpha,\beta}\left(\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}}\right)-2V_{\alpha,\beta}(\phi)=-2V_{\alpha,\beta}\left[\sin\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right)\right]$$

$$\Leftrightarrow\left(\frac{r}{v}\right)^2V_{\alpha,\beta}-V_{\alpha}(\sin\left(\frac{x^{\alpha}}{\alpha}\right))-\left(\frac{s}{u}\right)^2V_{\alpha,\beta}+V_{\beta}(\sin\left(\frac{t^{\beta}}{\beta}\right))-2V_{\alpha,\beta}(\phi)=\frac{-2u^2v^2}{\left(s^2+u^2\right)\left(r^2+v^2\right)}$$

$$\Leftrightarrow\left(\left(\frac{r}{v}\right)^2-\left(\frac{s}{u}\right)^2-2\right)V_{\alpha,\beta}=\frac{ru^2}{v\left(s^2+u^2\right)}-\frac{sv^2}{u\left(r^2+v^2\right)}-\frac{2u^2v^2}{\left(s^2+u^2\right)\left(r^2+v^2\right)}.$$

Then, we have

$$V_{\alpha,\beta}\left(\phi(x,t)\right)=\frac{\dfrac{ru^2}{v\left(s^2+u^2\right)}-\dfrac{sv^2}{u\left(r^2+v^2\right)}-\dfrac{2u^2v^2}{\left(s^2+u^2\right)\left(r^2+v^2\right)}}{\left(\frac{r}{v}\right)^2-\left(\frac{s}{u}\right)^2-2}. \tag{16}$$

After simplifying, eq. (16) becomes:

$$V_{\alpha,\beta}(\phi(x,t))=\frac{u^2}{\left(s^2+u^2\right)}\frac{v^2}{\left(r^2+v^2\right)}. \tag{17}$$

By taking $V_{\alpha,\beta}^{-1}$ of both sides of eq. (17), we obtain:

$$\phi(x,t)=V_{\alpha,\beta}^{-1}\left[\frac{u^2}{\left(s^2+u^2\right)}\frac{v^2}{\left(r^2+v^2\right)}\right]=\sin\left(\frac{x^{\alpha}}{\alpha}\right)\sin\left(\frac{t^{\beta}}{\beta}\right).$$

If we put $\alpha=1$ and $\beta=1$, the exact solution for eq. (10) is:

$$\phi(x,t)=\sin(x)\sin(t).$$

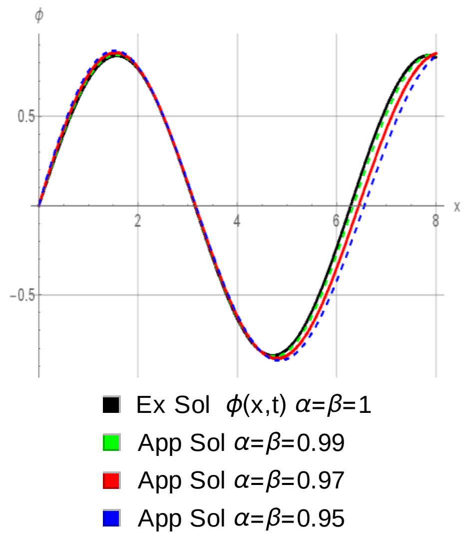

Figure 4. The exact and approximate solution $\phi(x,t)$ when $\alpha=\beta$.

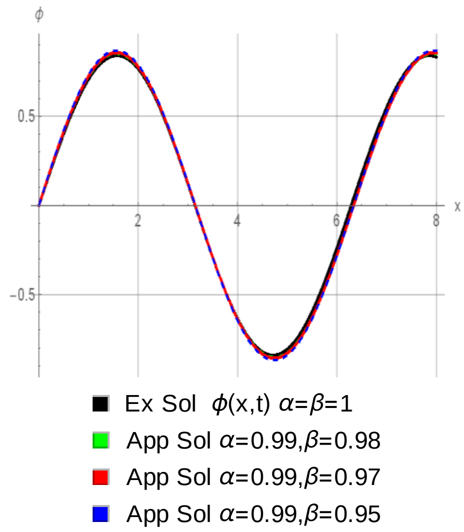

Figure 5. This The exact and approximate solution $\phi(x,t)$ when $\alpha=0.99$ and β take different values offractional order

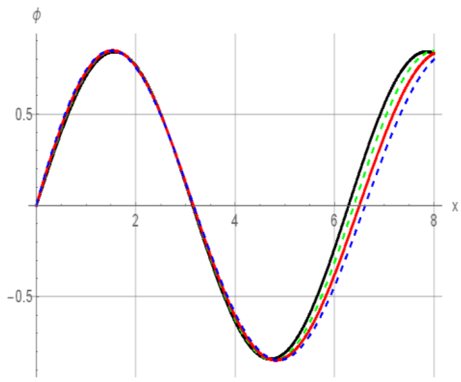

Figure 6. This The exact and approximate solution $\phi(x,t)$ when $\beta=0.99$ and α take different values of fractional order.

Example 3: [1] Consider the following linear non-homogeneous (α,β) conformable fractional Klein Gordon equation:

$$\frac{\partial^{2\beta}\phi}{\partial t^{2\beta}}-\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}}=\phi,\qquad 0<\alpha,\beta\le 1. \tag{18}$$

with the initial conditions:

$$\phi(x,0)=\cosh\left(\frac{x^{\alpha}}{\alpha}\right),\ \phi(0,t)=1+\sin\left(\frac{t^{\beta}}{\beta}\right),\ \frac{\partial^{\alpha}}{\partial x^{\alpha}}\phi(0,t)=0\ ,\frac{\partial^{\beta}}{\partial t^{\beta}}\phi(x,0)=0. \tag{19}$$

We applying (CDSHT) of eq(18) and (CSHT) of initial conditions eq. (19) , we obtain:

$$V_{\alpha,\beta}\left(\frac{\partial^{2\beta}\phi}{\partial t^{2\beta}}\right)-V_{\alpha,\beta}\left(\frac{\partial^{2\alpha}\phi}{\partial x^{2\alpha}}\right)-V_{\alpha,\beta}(\phi)=0,$$

by substitution, we get:

$$\left(\frac{r}{v}\right)^2 V_{\alpha,\beta}-V_{\alpha}\left(\cosh\left(\frac{x^{\alpha}}{\alpha}\right)\right)-\left(\frac{s}{u}\right)^2 V_{\alpha,\beta}+V_{\beta}\left(1+\sin\left(\frac{t^{\beta}}{\beta}\right)\right)-V_{\alpha,\beta}(\phi)=0$$

$$\Leftrightarrow\left(\left(\frac{r}{v}\right)^2-\left(\frac{s}{u}\right)^2-1\right)V_{\alpha,\beta}=\frac{su}{\left(s^2-u^2\right)}-\frac{u}{s}-\frac{u^2}{\left(s^2+u^2\right)}.$$

After simplifying, we obtain:

$$V_{\alpha,\beta}(\phi(x,t))=\frac{\dfrac{su}{\left(s^2-u^2\right)}-\dfrac{u}{s}-\dfrac{u^2}{\left(s^2+u^2\right)}}{\left(\dfrac{r}{v}\right)^2-\left(\dfrac{s}{u}\right)^2-1}=\frac{su}{\left(\left(s^2-u^2\right)\right)}+\frac{u^2}{\left(s^2+u^2\right)}. \tag{20}$$

By taking $V_{\alpha,\beta}^{-1}$ of both sides of eq. (20), we obtain:

$$\phi(x,t)=\cosh\left(\frac{x^{\alpha}}{\alpha}\right)+\sin\left(\frac{t^{\beta}}{\beta}\right).$$

In particular, if $\alpha,\ \beta\rightarrow 1$, we get the exact solution of eq. (18):

$$\phi(x,t)=\cosh x+\sin t.$$

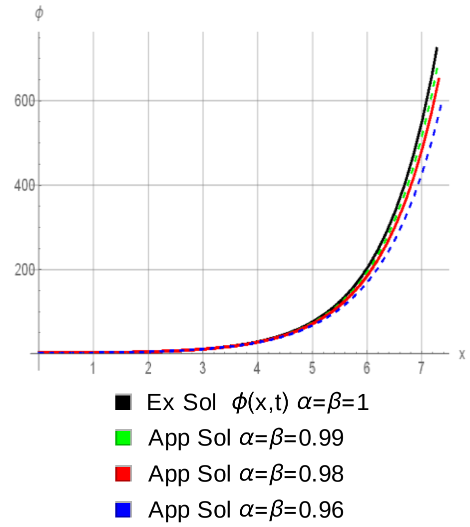

Figure 7. The exact and approximate $\phi(x,t)$ solution $\phi(x,t)$ when $\alpha=\beta$.

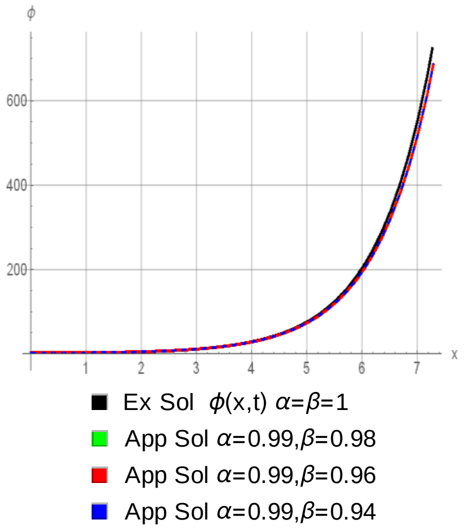

Figure 8. This The exact and approximatesolutionwhen $\alpha=0.99$ and β take different values of fractional order.

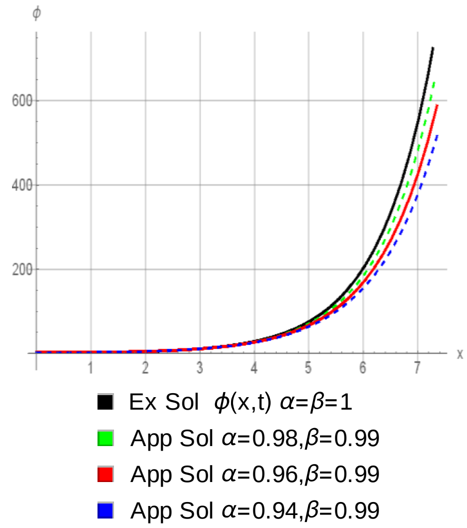

Figure 9. This The exact and approximate solution $\phi(x,t)$ when $\beta=0.99$ and α take different values of fractional order.

5. DISCUSSION

We discuss the precision and efficiency of the (CDSHT) by numerical results of $\phi(x,t)$ for the exact solution when $(\alpha,\beta \rightarrow 1)$ and approximate solutions at α and β taking different fractional values.

In Figs. 1, 4 and 7, the approximate solutions of equations (10),(14) and (18) at t = 1 taking different fractional values, are compared and we found that the numerical solution becomes close to the exact solution when the fractional value increases $(\alpha = \beta)$

In Figs. 2, 5 and 8, the approximate solution of equations (10),(14) and (18) decrease at the fractional derivative values $\alpha = 0.99$ and $\beta = 0.98, 0.96, 0.94$. Similarly, the exact solution and approximate solution are demonstrated.

In Figs. 3, 6 and 9, shows the approximate solution of equations (10), (14) and (18) with $0<\alpha\leq$, $\beta = 0.99$ and t = 1; in such a case, the function $\phi(x,t)$ gradually decreases.

6. CONCLUSION

In this paper, we introduce the definition of the double Shehu transform and its application to some functions, where we prove some of the theories and properties related to this transformation, this transformation is used to solve fractional partial differential equations in two-dimensional fractional space. Some numerical examples are given to show the ability and efficiency of the proposed method for solving homogeneous and inhomogeneous linear equations in two-dimensional fractional space. The obtained closed solutions correspond to the exact solutions found in the literature when $\alpha, \beta \rightarrow 1$ showing the feasibility and validity of the proposed method.

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