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MATRIX THEORY OVER *DGC* NUMBERS

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Abstract. *Classical matrix theory for real, complex and hypercomplex numbers is a well-known concept. Is it possible to construct matrix theory over dual-generalized complex (DGC) matrices? The answer to this question is given in this paper. The paper is constructed as follows. Firstly, the fundamental concepts for DGC matrices are introduced and DGC special matrices are defined. Then, theoretical results related to eigenvalues/eigenvectors are obtained and universal similarity factorization equality (USFE) regarding to the dual fundamental matrix are presented. Also, spectral theorems for Hermitian and unitary matrices are introduced. Finally, due to the importance of unitary matrices, a method for finding a DGC unitary matrix is stated and examples for spectral theorem are given.*

Keywords: *dual-generalized complex number; matrices over special rings; eigenvalues and eigenvectors; fundamental matrix.*

1. INTRODUCTION

The set of complex numbers $\mathbb{C}$ with elements $z=a+b\mathrm{i}$ where $a,b\in\mathbb{R}$ with complex unit i $(\mathrm{i}^2=-1)$ is a field and also associative and commutative algebra with unity (for details [1]). Over the years, complex unit lead to construct other two dimensional number systems, such as hyperbolic numbers [2-4] (double, split complex, perplex) and dual numbers [4-6]. Additionally, the set of generalized complex numbers is given in [7-8]:

$$\mathbb{C}_{\mathfrak{p}} := \left\{ z = a + bJ : a,b \in \mathbb{R}, J^2 = \mathfrak{p}, \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R} \right\}.$$

The set $\mathbb{C}_{\mathfrak{p}}$ is a vector space over $\mathbb{R}$. It is analogous to complex numbers $\mathbb{C}$ for $\mathfrak{p}=-1$, to hyperbolic numbers $\mathbb{H}$ for $\mathfrak{p}=1$ and to dual numbers $\mathbb{D}$ for $\mathfrak{p}=0$. Several researchers give a different perspective to these number systems. By this way, complex-hyperbolic numbers (or hyperbolic-complex numbers) are examined in [8-11]. The n-dimensional hyperbolic-complex and bicomplex numbers are studied in [12-15]. Complex-dual numbers (or dual-complex numbers) are investigated in [11,16-17]. The notion of dual-complex numbers and their holomorphic functions are discussed in [18]. Hyperbolic-dual (or dual-hyperbolic) numbers are investigated in [11], and some properties of hyperbolic-dual numbers and hyperbolic-complex numbers are examined in [19]. Also, as an extension of dual numbers, hyper-dual numbers are presented in [20-22].

Motivated by papers above and using the Cayley-Dickson doubling procedure for construction, the dual-generalized complex (*DGC*), hyperbolic-generalized complex and

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complex-generalized complex numbers are investigated in [23]. The set of DGC numbers is defined as [23]:

$$\mathbb{DC}_{\mathfrak{p}} := \left\{ \tilde{a} = z_1 + z_2\varepsilon : z_1, z_2 \in \mathbb{C}_{\mathfrak{p}},\ \varepsilon^2 = 0,\ \varepsilon \neq 0,\ \varepsilon \notin \mathbb{R} \right\}$$

considering various properties and matrix representations.

It is known that matrix theory is used in various disciplines of science and engineering such as applied mathematics, data analysis, scientific computing, graphic software, optimization, electronics networks, airplane and spacecraft, robotics and automation and many more. For detailed information related matrix theory we refer to the studies [24-25]. In addition, a comprehensive study about matrix theory is conducted by Zhang [26] taking into account quaternions which are identified with $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and e_1, e_2, e_3 are quaternionic units [27]. Inspired by Zhang, many studies considering different types of quaternions are conducted over matrix theory in [28-30].

Non-commutativity is a common property of quaternions, which leads to many difficulties in applications of quaternions. In this paper, we consider *DGC* numbers which are commutative. Our purpose is to give *DGC* matrix theory with the perspective mentioned above. The paper is organized as follows: Section 2 contains a short knowledge about *DGC* numbers and vectors. In Section 3, the basic theory of *DGC* matrices which are treated as real matrices with *DGC* entries is developed and the special cases are classified. Section 4 establishes the fundamental properties, eigenvalues/eigenvectors of *DGC* matrices and universal similarity factorization equality (USFE) by using the dual fundamental matrix which gives several opportunities to analyze the concepts[3]. In the last section, a method to construct a *DGC* unitary matrix is introduced and examples for spectral theorem for *DGC* symmetric matrix are given. It can be concluded that our results generalize the classical matrix theory.

2. FUNDAMENTAL CONCEPTS

As for prerequisites, the reader is expected to be familiar with *DGC* numbers. Here and elsewhere, we consider $\mathfrak{p} \neq 0$. Let $\tilde{a}_1 = z_{11} + z_{12}\varepsilon, \tilde{a}_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_{\mathfrak{p}},\ \lambda \in \mathbb{R}$. Then, the algebraic operations on *DGC* numbers are given by:

$$\begin{aligned}
\tilde{a}_1 = \tilde{a}_2 &\Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}, \\
\tilde{a}_1 + \tilde{a}_2 &= \left(z_{11} + z_{21}\right) + \left(z_{12} + z_{22}\right)\varepsilon, \\
\lambda \tilde{a}_1 &= \lambda \left(z_{11} + z_{12}\varepsilon\right), \\
\tilde{a}_1 \tilde{a}_2 &= \left(z_{11} z_{21}\right) + \left(z_{11} z_{22} + z_{12} z_{21}\right)\varepsilon.
\end{aligned}$$

[3] One main motivation to investigate DGC matrices with respect to their dual fundamental matrices comes from dual matrices applicability to various areas of science and engineering like the kinematic analysis and synthesis of spatial mechanisms, and robot manipulators; see [31-35]. The linear algebra and matrix theory related to dual matrices have sparked increased and accompanied to science and engineering.

Additionally, the notation ^{†_k} $\left(k=1,2,3,4\right)$ represents the different conjugates and these conjugates are defined by:

$$\tilde{a}^{\dagger_1}=\bar{z}_1+\bar{z}_2\varepsilon,$$
$$\tilde{a}^{\dagger_2}=z_1-z_2\varepsilon,$$
$$\tilde{a}^{\dagger_3}=\bar{z}_1-\bar{z}_2\varepsilon,$$
$$\tilde{a}^{\dagger_4}=z_2-z_1\varepsilon.$$

Here $\bar{z}_1$ and $\bar{z}_2$ represent the usual conjugate of z_1, $z_2\in\mathbb{C}_\mathfrak{p}$. Different conjugates enable different norms. The norms^{†_k} are identified by $|\tilde{a}|^2_{\dagger_k}=\tilde{a}\tilde{a}^{\dagger_k}$ for $k=1,2,3$. The multiplications of the base elements are $J\varepsilon=\varepsilon J$ and $(J\varepsilon)^2=0$. Every *DGC* number $\tilde{a}$ can be written as $\tilde{a}=z_1+z_2\varepsilon=a_1+a_2J$ where a_1,a_2 are dual numbers. Based on this, it is clear that there is no difference between dual-generalized complex numbers and generalized complex-dual numbers, (see details in [23]).

Null (isotropic) *DGC* numbers are the numbers with zero norm and they identified by the following forms

- $0, c\varepsilon$, $dJ\varepsilon$, and $c\varepsilon+dJ\varepsilon$ with respect to ^{†_k},
- $\pm\sqrt{\mathfrak{p}}a+aJ$ with respect to ^{†_1} and ^{†_3} where $\mathfrak{p}>0$,
- $\pm\sqrt{\mathfrak{p}}a+aJ\pm\sqrt{\mathfrak{p}}c\varepsilon+cJ\varepsilon$ with respect to ^{†_1} and ^{†_3} where $\mathfrak{p}>0$ and $a\neq 0$,

where $a,b,c,d\in\mathbb{R}$.

Gathering all these facts, each non-null *DGC* number $\tilde{a}$ has an inverse^{†_k} $\left(k=1,2,3\right)$ where $\tilde{a}^{-1}_{\dagger_k}=\dfrac{\tilde{a}^{\dagger_k}}{|\tilde{a}|^2_{\dagger_k}}$. A chief motivation for our investigations over USFE for *DGC* numbers comes from the following basic USFE for complex number $a+b\mathrm{i}$:

$$\begin{bmatrix}1 & \mathrm{i}\\ -\mathrm{i} & -1\end{bmatrix}\begin{bmatrix}a+b\mathrm{i} & 0\\ 0 & a-b\mathrm{i}\end{bmatrix}\begin{bmatrix}1 & \mathrm{i}\\ -\mathrm{i} & -1\end{bmatrix}=2\begin{bmatrix}a & -b\\ b & a\end{bmatrix},$$

where $\mathrm{i}^2=-1$. Tian's result [36] is interesting as it provides a way to present a general method for establishing USFE. According to this, USFE for elliptic numbers [30] ($\mathfrak{p}<0$) is defined and it allows to expend USFE for *DGC* numbers.

Theorem 2.1. Let $\tilde{a}=a_1+a_2J\in\mathbb{DC}_\mathfrak{p}$ be given, where $a_1,a_2\in\mathbb{D}$. Then, $\tilde{a}$ and $\tilde{a}^{\dagger_1}$ satisfy USFE

$$\mathsf{P}^{-1}\begin{bmatrix}\tilde{a} & 0\\ 0 & \tilde{a}^{\dagger_1}\end{bmatrix}\mathsf{P}=\begin{bmatrix}a_1 & \mathfrak{p}a_2\\ a_2 & a_1\end{bmatrix}=\phi(\tilde{a}).$$

Here, $\mathsf{P}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -\frac{J}{\mathfrak{p}}\\ \frac{J}{\mathfrak{p}} & \frac{1}{\mathfrak{p}}\end{bmatrix}$ and $\mathsf{P}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & J\\ -J & \mathfrak{p}\end{bmatrix}$ have no relation with $\tilde{a}$.

Corollary 2.1. Let $\tilde{a}=a_1+a_2J\in\mathbb{DC}_{\mathfrak{p}}$ be given, where $a_1,a_2\in\mathbb{D}$. By USFE over *DGC* numbers, the facts below follow easily:

- The set of *DGC* numbers is algebraically isomorphic to

$$\mathsf{D}:=\left\{\begin{bmatrix}a_1 & \mathfrak{p}a_2\\ a_2 & a_1\end{bmatrix}: a_1,\ a_2\in\mathbb{D}\right\}\subset\mathbb{M}_2\left(\mathbb{D}\right)$$

  through the bijective map $\phi:\mathbb{DC}_{\mathfrak{p}}\to\ \mathsf{D},\quad \phi(\tilde{a})=\begin{bmatrix}a_1 & \mathfrak{p}a_2\\ a_2 & a_1\end{bmatrix}$.

- Every $\tilde{a}$ has a dual matix representation $\phi(\tilde{a})=\begin{bmatrix}a_1 & \mathfrak{p}a_2\\ a_2 & a_1\end{bmatrix}$ over the dual number ring.

For $\tilde{a}\cong\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ and $\tilde{b}=b_1+b_2J$, $\tilde{a}\tilde{b}=\tilde{b}\tilde{a}=\begin{bmatrix}a_1 & \mathfrak{p}a_2\\ a_2 & a_1\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}=\phi(\tilde{a})\tilde{b}$. Here $\phi(\tilde{a})$ is called fundamental matrix of $\tilde{a}$, [23].

Theorem 2.2. For $\tilde{a},\ \tilde{b}\in\mathbb{DC}_{\mathfrak{p}}$ and $\lambda\in\mathbb{R}$, the following universal identities are satisfied:

1) $\tilde{a}=\tilde{b}\Leftrightarrow\phi(\tilde{a})=\phi(\tilde{b})$,
2) $\phi(\tilde{a}+\tilde{b})=\phi(\tilde{a})+\phi(\tilde{b})$,
3) $\phi(\tilde{a}\tilde{b})=\phi(\tilde{a})\phi(\tilde{b})$,
4) $\phi(\phi(\tilde{a})\tilde{b})=\phi(\tilde{a})\phi(\tilde{b})$,
5) $\phi(\lambda\tilde{a})=\lambda\phi(\tilde{a})$,
6) $\phi(1)=I_2$,
7) $\det\left(\phi(\tilde{a})\right)=a_1^2-\mathfrak{p}a_2^2=\tilde{a}\tilde{a}^{\dagger_1}$,
8) $\tilde{a}_{\dagger_1}^{-1}$ exists $\Leftrightarrow$ $\phi^{-1}(\tilde{a})$ exists, in which case, $\phi(\tilde{a}_{\dagger_1}^{-1})=\phi^{-1}(\tilde{a})$
9) $\mathrm{tr}\left(\phi(\tilde{a})\right)=\tilde{a}+\tilde{a}^{\dagger_1}$.

Additionally, the set of *DGC* vectors is denoted by V^n and defined as: $V^n:=\left\{V=\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right):\tilde{a}_t\in\mathbb{DC}_{\mathfrak{p}},\ t=1,2...,n\right\}$. The set V^n is a module over $\mathbb{DC}_{\mathfrak{p}}$. The conjugate of $V\in V^n$ is the conjugate of its components. Let us extend the familiar definitions of scalar product, norm and cross product to *DGC* versions.

Definition 2.1. Let $V=\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)$, $U=\left(\tilde{b}_1,\tilde{b}_2,...,\tilde{b}_n\right)\in V^n$ be given. Then the standard scalar product and Hermitian^{†_k} scalar product over the module V^n are defined respectively by

$$V^n \times V^n \to \mathbb{DC}_{\mathfrak{p}}$$
$$(V,U) \mapsto \langle V,U\rangle = \sum_{r=1}^{n} \tilde{a}_r \tilde{b}_r = V^T U,$$
$$(V,U) \mapsto \langle V,U\rangle_{\dagger_k} = \sum_{r=1}^{n} \tilde{a}_r \tilde{b}_r^{\dagger_k} = V^T U^{\dagger_k},\ k = 1,2,3.$$

The standard norm and norm ^{†_k} of *DGC* vector V in V^n are defined as follows: $\|V\|^2 = \langle V,V\rangle$ and $\|V\|_{\dagger_k}^2 = \langle V,V\rangle_{\dagger_k}$, respectively, for $k = 1,2,3$. If the norm of a vector $V \in V^n$ equals 1, then it is called a unit vector.

One can see the details easily through the analogy between $V = (\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \in V^n$ and $V = \mathsf{V}_1 + \mathsf{V}_2 J + \mathsf{V}_3 \varepsilon + \mathsf{V}_4 J\varepsilon$ where $\mathsf{V}_i \in \mathbb{R}^n$, $i = 1,2,3,4$. A null (isotropic) *DGC* vector in V^n is a vector of zero norm. For example, $\left(J, \varepsilon, \sqrt{\mathfrak{p}}\right) \in V^3$ is null for $\mathfrak{p} > 0$ with respect to ^{†_1} and ^{†_3}. Moreover, if the components of a vector in V^n are multiple of the null *DGC* numbers, then the vector is also null. Hence $(\varepsilon - 2J\varepsilon, 0, 0)$ and $(\varepsilon, J\varepsilon, -\varepsilon)$ are null in V^3.

Definition 2.2. For $k = 1,2,3$, the standard cross and cross ^{†_k} (vector) products of two vectors V and U in V^3 are defined respectively as follows:

$$V \times U = \begin{vmatrix} i & j & k \\ \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \end{vmatrix}, \tag{1}$$

and

$$V \times_{\dagger_k} U = \begin{vmatrix} i & j & k \\ \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \end{vmatrix}^{\dagger_k} = \begin{vmatrix} i & j & k \\ \tilde{a}_1^{\dagger_k} & \tilde{a}_2^{\dagger_k} & \tilde{a}_3^{\dagger_k} \\ \tilde{b}_1^{\dagger_k} & \tilde{b}_2^{\dagger_k} & \tilde{b}_3^{\dagger_k} \end{vmatrix}.$$

Proposition 2.1. For any $V,U,W \in V^3$, $\lambda \in \mathbb{R}$, the cross ^{†_k} products ($k = 1,2,3$) satisfy the following properties:

1) $V \times_{\dagger_k} U = -U \times_{\dagger_k} V$,
2) $V \times_{\dagger_k} V = 0$,
3) $(\lambda V) \times_{\dagger_k} U = \lambda (V \times_{\dagger_k} U)$,
4) $0 \times_{\dagger_k} U = 0$,
5) $V \times_{\dagger_k} U = 0 \Leftrightarrow V = \lambda U$,
6) $V \times_{\dagger_k} (U + W) = (V \times_{\dagger_k} U) + (V \times_{\dagger_k} W)$.

Proof: The symbolic determinant of cross ^{†_k} product satisfies many of the familiar features of determinants, which can be checked easily for $k = 1,2,3$.

It is worthy of note that the properties in Proposition 2.1 can be given for the standard cross product given in equation (1).

3. *DGC* MATRICES

The matrix with DGC number entries is called DGC matrix. The DGC matrix $\tilde{A}$ of the order $m \times n$ is of the form: $\tilde{A} = \left[\tilde{a}_{ij}\right] = \left[a_{0ij} + a_{1ij}J + a_{2ij}\varepsilon + a_{3ij}J\varepsilon\right]$, where $\tilde{a}_{ij} \in \mathbb{DC}_{\mathfrak{p}}$, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. The set of all $m \times n$ matrices with DGC entries is denoted by:

$$\mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right) := \left\{\tilde{A} = \left[\tilde{a}_{ij}\right]_{m\times n} : \tilde{a}_{ij} \in \mathbb{DC}_{\mathfrak{p}},\ i = 1, 2, ..., m,\ j = 1, 2, ..., n\right\}.$$

A DGC matrix with all of the entries are zero, is called a zero-matrix and denoted by $\tilde{0}$. If $m = n$, then $\tilde{A}$ is called DGC square matrix.

Lemma 3.1. Every DGC matrix of the order $m \times n$ can be written as

$$\tilde{A} = A_0 + A_1 J + A_2 \varepsilon + A_3 J \varepsilon,$$

where A_0, A_1, A_2, A_3 are real matrices of the same order.

Lemma 3.2. Every DGC square matrix $\tilde{A} = A_0 + A_1 J + A_2 \varepsilon + A_3 J \varepsilon$ is also in the form:

$$\tilde{A} = A_0 + A_1 \tilde{J} + A_2 \tilde{\varepsilon} + A_3 \tilde{J} \tilde{\varepsilon},$$

where $\tilde{J} = JI_n$, $\tilde{\varepsilon} = \varepsilon I_n$.

Standard elementary matrix operations establish the following operations on DGC matrices. Let $\tilde{A} = \left[\tilde{a}_{ij}\right]$, $\tilde{B} = \left[\tilde{b}_{ij}\right] \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$, $\tilde{C} = \left[\tilde{c}_{js}\right] \in \mathbb{M}_{n\times r}\left(\mathbb{DC}_{\mathfrak{p}}\right)$ and $c \in \mathbb{R}$. $\tilde{A}$ and $\tilde{B}$ are equal if they have the same order and the corresponding entries are equal, in other words, $\tilde{a}_{ij} = \tilde{b}_{ij}$ for each $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. The addition (and hence subtraction) $\tilde{A} + \tilde{B}$ is defined as $\tilde{D} = \left[\tilde{d}_{ij}\right]$ with

$$\tilde{A} + \tilde{B} = \left[\tilde{a}_{ij}\right] + \left[\tilde{b}_{ij}\right] = \left[\tilde{a}_{ij} + \tilde{b}_{ij}\right] = \left[\tilde{d}_{ij}\right] = \tilde{D} \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right),$$

or $\tilde{A} + \tilde{B} = \left(A_0 + B_0\right) + \left(A_1 + B_1\right)J + \left(A_2 + B_2\right)\varepsilon + \left(A_3 + B_3\right)J\varepsilon$. The scalar multiplication of $\tilde{A}$ by c, $c\tilde{A}$, is defined as: $c\tilde{A} = \left[c\tilde{a}_{ij}\right] \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$, or $c\tilde{A} = cA_0 + cA_1 J + cA_2\varepsilon + cA_3 J\varepsilon$. The product $\tilde{A}\tilde{C}$ is defined as:

$$\tilde{A}\tilde{C} = \left[\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{c}_{js}\right] = \left[\tilde{e}_{is}\right] = \tilde{E} \in \mathbb{M}_{m\times r}\left(\mathbb{DC}_{\mathfrak{p}}\right),$$

or

$$\tilde{A}\tilde{C} = A_0C_0 + \mathfrak{p}A_1C_1 + \left(A_0C_1 + A_1C_0\right)J + \left(A_0C_2 + A_2C_0 + \mathfrak{p}\left(A_1C_3 + A_3C_1\right)\right)\varepsilon$$
$$+\left(A_0C_3 + A_1C_2 + A_2C_1 + A_3C_0\right)J\varepsilon.$$

As natural consequence of matrix product to find the power of $\tilde{A}$, $\tilde{A}$ is multiplied by itself as many times as the exponent (nonnegative integer) indicates. The *DGC* square matrix $\tilde{A}$ of the order n is said to be an invertible if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \tilde{I}_n$ for *DGC* square matrix $\tilde{B}$ of the same order.

With regard to the operations given above, $\mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$ is an abelian group, is a non-commutative ring and is a vector space over $\mathbb{R}$. It is also an algebra over $\mathbb{R}$ with matrix multiplication. Additionally, $\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ is a $\mathbb{M}_n\left(\mathbb{R}\right)$ module. The transpose of $\tilde{A}$ is denoted by $\tilde{A}^T$ and defined as: $\tilde{A}^T = \left[\tilde{a}_{ji}\right] \in \mathbb{M}_{n\times m}\left(\mathbb{DC}_{\mathfrak{p}}\right)$ or $\tilde{A}^T = A_0^T + A_1^T J + A_2^T\varepsilon + A_3^T J\varepsilon$. The trace of square $\tilde{A}$, denoted by $\mathrm{tr}(\tilde{A})$, is defined to be the sum of elements on the main diagonal of $\tilde{A}$: $\mathrm{tr}(\tilde{A}) = \sum_{i=1}^{n}\tilde{a}_{ii} = \mathrm{tr}(A_0) + \mathrm{tr}(A_1)J + \mathrm{tr}(A_2)\varepsilon + \mathrm{tr}(A_3)J\varepsilon.$

Proposition 3.1. For $\tilde{A},\tilde{B} \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$, $\tilde{C} \in \mathbb{M}_{n\times r}\left(\mathbb{DC}_{\mathfrak{p}}\right)$ and $c \in \mathbb{R}$, the matrix transpose satisfies the followings:

1) $\left(\tilde{A}^T\right)^T = \tilde{A}$,
2) $\left(\tilde{A}+\tilde{B}\right)^T = \tilde{A}^T + \tilde{B}^T$,
3) $\left(c\tilde{A}\right)^T = c\tilde{A}^T$,
4) $\left(\tilde{A}\tilde{C}\right)^T = \tilde{C}^T\tilde{A}^T$,
5) $\left(\tilde{A}^{-1}\right)^T = \left(\tilde{A}^T\right)^{-1}$.

Proposition 3.2. For $\tilde{A},\tilde{B} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ and $c \in \mathbb{R}$, the followings hold:

1) $\mathrm{tr}(\tilde{A}+\tilde{B}) = \mathrm{tr}(\tilde{A}) + \mathrm{tr}(\tilde{B})$,
2) $\mathrm{tr}(c\tilde{A}) = c\,\mathrm{tr}(\tilde{A})$,
3) $\mathrm{tr}(\tilde{A}\tilde{B}) = \mathrm{tr}(\tilde{B}\tilde{A})$,
4) $\mathrm{tr}(\tilde{A}^T) = \mathrm{tr}(\tilde{A})$.

Based on the following conjugates, we will define some special *DGC* matrices and spectral theorems in the next sections.

Definition 3.1. Let $\tilde{A} = \left[a_{ij}\right] = A_0 + A_1J + A_2\varepsilon + A_3J\varepsilon \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$. Then the conjugations of $\tilde{A}$ are defined as follow:

$$\begin{cases} \tilde{A}^{\dagger_1} = A_0 - A_1 J + A_2\varepsilon - A_3 J\varepsilon = (A_0 + A_2\varepsilon) + (-A_1 - A_3\varepsilon)J, \\ \tilde{A}^{\dagger_2} = A_0 + A_1 J - A_2\varepsilon - A_3 J\varepsilon = (A_0 - A_2\varepsilon) + (A_1 - A_3\varepsilon)J, \\ \tilde{A}^{\dagger_3} = A_0 - A_1 J - A_2\varepsilon + A_3 J\varepsilon = (A_0 - A_2\varepsilon) + (-A_1 + A_3\varepsilon)J, \\ \tilde{A}^{\dagger_4} = A_2 + A_3 J - A_0\varepsilon - A_1 J\varepsilon = (A_2 - A_0\varepsilon) + (A_3 - A_1\varepsilon)J. \end{cases}$$

Proposition 3.3. For $\tilde{A}, \tilde{B} \in \mathbb{M}_{m\times n}\left(\mathbb{DC}_{\mathfrak{p}}\right)$, $\tilde{C} \in \mathbb{M}_{n\times r}\left(\mathbb{DC}_{\mathfrak{p}}\right)$, the following assertions hold:

1) $\left(\tilde{A}^{\dagger_k}\right)^{\dagger_k} = \tilde{A}$, $k = 1,2,3$, $\left(\tilde{A}^{\dagger_4}\right)^{\dagger_4} = -\tilde{A}$,
2) $\left(\tilde{A}^{T}\right)^{\dagger_k} = \left(\tilde{A}^{\dagger_k}\right)^{T} = \tilde{A}^{\star_k}$, $k = 1,2,3,4$,
3) $\left(\tilde{A}+\tilde{B}\right)^{\dagger_k} = \tilde{A}^{\dagger_k} + \tilde{B}^{\dagger_k}$, $k = 1,2,3,4$,
4) $\left(\tilde{A}\tilde{C}\right)^{\dagger_k} = \tilde{A}^{\dagger_k}\tilde{C}^{\dagger_k}$, $k = 1,2,3$,
5) $\tilde{A}+\tilde{A}^{\dagger_1} = 2\left(A_0 + A_2\varepsilon\right)$, $\tilde{A}+\tilde{A}^{\dagger_2} = 2\left(A_0 + A_1 J\right)$, $\tilde{A}+\tilde{A}^{\dagger_3} = 2\left(A_0 + A_3 J\varepsilon\right)$.

Now, we also seek to identify special *DGC* matrices and their properties.

Definition 3.2. Let $\tilde{A} = \left[\tilde{a}_{ij}\right] \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$. If

- $\langle \tilde{A}V, U\rangle = \langle V, \tilde{A}U\rangle$, then $\tilde{A}$ is said to be *DGC* symmetric,
- $\langle \tilde{A}V, U\rangle = -\langle V, \tilde{A}U\rangle$, then $\tilde{A}$ is said to be *DGC* skew-symmetric,
- $\langle \tilde{A}V, U\rangle_{\dagger_k} = \langle V, \tilde{A}U\rangle_{\dagger_k}$, then $\tilde{A}$ is said to be *DGC* Hermitian^{†_k},
- $\langle \tilde{A}V, U\rangle_{\dagger_k} = -\langle V, \tilde{A}U\rangle_{\dagger_k}$, then $\tilde{A}$ is said to be *DGC* skew-Hermitian^{†_k},
- $\langle \tilde{A}V, \tilde{A}U\rangle = \langle V, U\rangle$, then $\tilde{A}$ is said to be *DGC* orthogonal,
- $\langle \tilde{A}V, \tilde{A}U\rangle_{\dagger_k} = \langle V, U\rangle_{\dagger_k}$, then $\tilde{A}$ is said to be *DGC* unitary^{†_k},

where V, $U \in V^n$ and $k = 1,2,3$.

The preceding definition allows us to construct the following general relations.

Theorem 3.1. For any *DGC* square matrix $\tilde{A}$ of the order n and $k = 1,2,3$, the followings are given:

1) $\tilde{A}$ is symmetric (skew-symmetric) if and only if $\tilde{A} = \tilde{A}^T$ ($\tilde{A} = -\tilde{A}^T$),
2) $\tilde{A}$ is Hermitian^{†_k} (skew-Hermitian^{†_k}) if and only if $\tilde{A}^T = \tilde{A}^{\dagger_k}$ ($\tilde{A}^T = -\tilde{A}^{\dagger_k}$),
3) $\tilde{A}$ is orthogonal if and only if $\tilde{A}\tilde{A}^T = \tilde{A}^T\tilde{A} = \tilde{I}_n$,
4) $\tilde{A}$ is unitary^{†_k} if and only if $\tilde{A}\tilde{A}^{\star_k} = \tilde{A}^{\star_k}\tilde{A} = \tilde{I}_n$.

Proof: Let $\tilde{A}=\left[\tilde{a}_{ij}\right]\in\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$, $V,\ U\in V^n$. The proof here is essentially an elaboration on Definition 2.1 and Definition 3.2, Proposition 3.1 and Proposition 3.3.
2) Considering equality $\langle\tilde{A}V,U\rangle_{\dagger_k}=\langle V,\tilde{A}U\rangle_{\dagger_k}$, we obtain:

$$\left(\tilde{A}V\right)^T U^{\dagger_k}=V^T\left(\tilde{A}U\right)^{\dagger_k}\Leftrightarrow V^T\tilde{A}^T U^{\dagger_k}=V^T\tilde{A}^{\dagger_k}U^{\dagger_k}\Leftrightarrow \tilde{A}^T=\tilde{A}^{\dagger_k}.$$

4) Using equality $\langle\tilde{A}V,\tilde{A}U\rangle_{\dagger_k}=\langle V,U\rangle_{\dagger_k}$, we have:

$$\left(\tilde{A}V\right)^T\left(\tilde{A}U\right)^{\dagger_k}=V^T U^{\dagger_k}\Leftrightarrow V^T\tilde{A}^T\tilde{A}^{\dagger_k}U^{\dagger_k}=V^T U^{\dagger_k}\Leftrightarrow \tilde{A}\tilde{A}^{\star_k}=\tilde{I}_n.$$

Lemma 3.3. For any *DGC* vector V with dimension n, *DGC* orthogonal transformation $\langle\tilde{A}V,\tilde{A}V\rangle=\langle V,V\rangle$ with matrix $\tilde{A}$ preserves lengths of vectors and maps orthonormal *DGC* bases to orthonormal *DGC* bases.

The preceding lemma gives rise to a geometric construction for *DGC* vectors. The rows of a *DGC* orthogonal matrix are mutually *DGC* orthogonal vectors with unit norm, so that the rows constitute an orthonormal *DGC* basis of V^n. The columns of the matrix form another orthonormal *DGC* basis of V^n.

Example 3.1. The *DGC* matrix $\tilde{A}=\begin{bmatrix}\frac{J}{2} & \frac{\varepsilon}{2} & \frac{J\varepsilon}{2}\\ -\frac{\varepsilon}{2} & \frac{J}{2} & \frac{J\varepsilon}{2}\\ -\varepsilon & -\varepsilon & 1\end{bmatrix}$ is orthogonal for $\mathfrak{p}=4$. Its rows (columns) are *DGC* orthogonal vectors with unit norm. Consequently, the system of rows is an orthonormal *DGC* basis of V^3.

Lemma 3.3 can also be given for *DGC* unitary^{†_k} matrices for $k=1,2,3$.

Definition 3.3. The *DGC* matrix $\tilde{A}$ is said to be normal^{†_k} if $\tilde{A}^{\star_k}$ and $\tilde{A}$ commute, that is, if $\tilde{A}\tilde{A}^{\star_k}=\tilde{A}^{\star_k}\tilde{A}$ for $k=1,2,3$.

The identity, diagonal, scalar, upper/lower triangular and triangular *DGC* matrices are defined by in a familiar way. For instance, the identity *DGC* matrix is denoted by $\tilde{I}_n$ and all diagonal entries are equal to 1.

In order to avoid distractions, several results involving additional properties of special *DGC* matrices are omitted[4]. In what follows, we now give the following simple fact concerning *DGC* square matrices.

[4] $\tilde{A}=A_0+A_1J+A_2\varepsilon+A_3J\varepsilon$ is unitary^{†_2} if and only if $A_0A_0^T+\mathfrak{p}A_1A_1^T=I_n$, $A_0A_1^T$ is skew-symmetric, $A_0A_2^T+\mathfrak{p}A_1A_3^T$ is symmetric, and $A_0A_3^T+A_1A_2^T$ is symmetric. The other special type of *DGC* matrices can be examined easily in same way.

Theorem 3.2. Every *DGC* square matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric *DGC* matrices of the same order.

For given any *DGC* square matrix $\tilde{A}=[\tilde{a}_{ij}]$ of the order n, the determinant of $\tilde{A}$ is defined as: $\det(\tilde{A})=\sum_{\sigma_i\in S_n}\operatorname{Sign}(\sigma_i)\tilde{a}_{1\sigma_1}\tilde{a}_{2\sigma_2}...\tilde{a}_{n\sigma_n}$, where the sum is over all permutations of n elements. This determinant exhibits the features of standard determinant. Namely, it act same as real matrices.

Corollary 3.1. Let $\det(\tilde{A})$ be a non-null *DGC* number. In this case, $\tilde{A}$ is invertible and its inverse can be obtained by the formula:

$$\tilde{A}^{-1}=\frac{1}{\det(\tilde{A})}\operatorname{adj}(\tilde{A}),$$

where $\operatorname{adj}(\tilde{A})$ is the classical adjoint of a matrix. $\tilde{A}$ is not invertible when $\det(\tilde{A})$ is null *DGC* number. In other respects, it is also possible to calculate $\tilde{A}^{-1}$ considering conjugates denoting as $\tilde{A}^{-1}_{\dagger_k}$, where the notation $\dagger_k$ represents $\det(\tilde{A})$ is non-null for conjugate $\dagger_k$.

Remark 3.1. It clear that, if any row or column of $\tilde{A}$ is a multiple of null *DGC* number, then $\tilde{A}$ is not invertible (because $\det(\tilde{A})$ is null *DGC* number). For example $\begin{bmatrix} 3J\varepsilon & 1 \\ \varepsilon-J\varepsilon & 2 \end{bmatrix}$ is not invertible.

Remark 3.2. Even if any row or column of $\tilde{A}$ is a null *DGC* vector, determinant may not be null *DGC* number. For instance, considering $\mathfrak{p}>0$ and $\dagger_1$, the first column of $\tilde{A}=\begin{bmatrix} J & 1 \\ \sqrt{\mathfrak{p}} & 2\varepsilon \end{bmatrix}$ is null *DGC* vector, however, $\det(\tilde{A})=2J\varepsilon-\sqrt{\mathfrak{p}}$ is non-null *DGC* number. Hence, $\tilde{A}^{-1}=\frac{1}{2J\varepsilon-\sqrt{\mathfrak{p}}}\begin{bmatrix} 2\varepsilon & -1 \\ -\sqrt{\mathfrak{p}} & J \end{bmatrix}$.

Remark 3.3. Taking $\tilde{A}=\begin{bmatrix} J+\sqrt{2} & 2J\varepsilon \\ \varepsilon & 1 \end{bmatrix}$ and $\mathfrak{p}=2$, $\det(\tilde{A})=J+\sqrt{2}$ is null *DGC* number for $\dagger_1$ but non-null number for $\dagger_2$. Thus, $\tilde{A}^{-1}_{\dagger_2}=\frac{1}{J+\sqrt{2}}\begin{bmatrix} 1 & -2J\varepsilon \\ -\varepsilon & J+\sqrt{2} \end{bmatrix}$.

4. *DGC* MATRICES CONSIDERING THE DUAL FUNDAMENTAL MATRIX

We frequently require the dual fundamental matrix in some of the theorems and therefore it plays a crucial role in our paper. Theorem 2.1 can be extended to *DGC* matrices as follows.

Theorem 4.1. Let $\tilde{A} = A_1 + A_2 J \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ be given, where A_1 and A_2 are dual matrices. Then, $\tilde{A}$ and $\tilde{A}^{\dagger_1}$ satisfy the following USFE

$$P^{-1}\begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}^{\dagger_1} \end{bmatrix} P = \begin{bmatrix} A_1 & \mathfrak{p}\, A_2 \\ A_2 & A_1 \end{bmatrix} = \chi\left(\tilde{A}\right).$$

Here, $P^{-1} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} I_n & -\dfrac{1}{\mathfrak{p}}\tilde{J} \\ \dfrac{1}{\mathfrak{p}}\tilde{J} & \dfrac{1}{\mathfrak{p}} I_n \end{bmatrix}$ and $P = \dfrac{1}{\sqrt{2}}\begin{bmatrix} I_n & \tilde{J} \\ -\tilde{J} & \mathfrak{p} I_n \end{bmatrix}$ have no relation with $\tilde{A}$.

Proof: By direct computation, one establishes USFE over *DGC* square matrices.

Corollary 4.1. Let $\tilde{A} = A_1 + A_2 J \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ be given, where A_1 and A_2 are dual matrices. By USFE over *DGC* square matrices, the facts below follow easily:

- The set of *DGC* square matrices is isomorphic to the matrix set:

$$D^* := \left\{ \begin{bmatrix} A_1 & \mathfrak{p} A_2 \\ A_2 & A_1 \end{bmatrix} :\ A_1,\ A_2 \in \mathbb{M}_n\left(\mathbb{D}\right) \right\} \subset \mathbb{M}_{2n}\left(\mathbb{D}\right)$$

through the bijective map $\chi : \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right) \to D^*, \chi(\tilde{A}) = \begin{bmatrix} A_1 & \mathfrak{p} A_2 \\ A_2 & A_1 \end{bmatrix}$.

- Every $\tilde{A}$ has a dual matrix representation $\chi(\tilde{A}) = \begin{bmatrix} A_1 & \mathfrak{p}\, A_2 \\ A_2 & A_1 \end{bmatrix}$.

One rewrite any *DGC* square matrix $\tilde{A} \cong \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Based on this, for $\tilde{A}$ and $\tilde{B} = B_1 + B_2 J$, we have $\tilde{A}\tilde{B} = \begin{bmatrix} A_1 & \mathfrak{p}\, A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \chi(\tilde{A})\tilde{B}$, where $\chi(\tilde{A})$ is called dual fundamental matrix of $\tilde{A}$.

Theorem 4.2. For $\tilde{A}, \tilde{B} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$, the followings hold:

1) $\chi\left(\tilde{I}_n\right) = I_{2n}$,
2) $\tilde{A} = \tilde{B} \Leftrightarrow \chi\left(\tilde{A}\right) = \chi\left(\tilde{B}\right)$,
3) $\chi\left(\tilde{A} + \tilde{B}\right) = \chi\left(\tilde{A}\right) + \chi\left(\tilde{B}\right)$,
4) $\chi\left(c\tilde{A}\right) = c\chi\left(\tilde{A}\right), c \in \mathbb{R}$,
5) $\chi\left(\tilde{A}\tilde{B}\right) = \chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right)$,
6) $\chi\left(\chi\left(\tilde{A}\right)\tilde{B}\right) = \chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right)$

7) $\chi\left(\tilde{A}^{-1}\right)=\left(\chi\left(\tilde{A}\right)\right)^{-1}=\chi\left(\tilde{A}\right)^{-1}$, if $\tilde{A}^{-1}$ exists,

8) $\chi\left(\tilde{A}^{\dagger_2}\right)=\overline{\chi\left(\tilde{A}\right)}$. However $\chi\left(\tilde{A}^{\dagger_k}\right)\neq\overline{\chi\left(\tilde{A}\right)}$ in general $(k=1,3)$,

9) $\chi\left(\tilde{A}^{T}\right)\neq\chi\left(\tilde{A}\right)^{T}$ in general,

10) $\chi\left(\tilde{A}^{\star_k}\right)\neq\overline{\chi\left(\tilde{A}\right)}^{T}$ in general $(k=1,2,3)$,

11) $\mathrm{tr}\left(\chi\left(\tilde{A}\right)\right)=\tilde{A}+\tilde{A}^{\dagger_1}$,

where the overline in part 8) indicates standard conjugate of a dual matrix.

Proof: The parts from 1) to 4), 6) and 11) can be easily shown. For parts 8), 9) and 10), an example can be found[5].
5) Take $\tilde{A}=A_1+A_2J,\tilde{B}=B_1+B_2J\in\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$. Then, the dual fundamental matrices of $\tilde{A}$ and $\tilde{B}$ are $\chi\left(\tilde{A}\right)=\begin{bmatrix}A_1 & \mathfrak{p}A_2\\ A_2 & A_1\end{bmatrix}$, $\chi\left(\tilde{B}\right)=\begin{bmatrix}B_1 & \mathfrak{p}B_2\\ B_2 & B_1\end{bmatrix}$. Calculating $\tilde{A}\tilde{B}$ gives $\tilde{A}\tilde{B}=\left(A_1B_1+\mathfrak{p}A_2B_2\right)+\left(A_1B_2+A_2B_1\right)J$ and we can assert that

$$\chi\left(\tilde{A}\tilde{B}\right)=\begin{bmatrix}A_1B_1+\mathfrak{p}A_2B_2 & \mathfrak{p}\left(A_1B_2+A_2B_1\right)\\ A_1B_2+A_2B_1 & A_1B_1+\mathfrak{p}A_2B_2\end{bmatrix}=\chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right).$$

7) If $\tilde{A}$ is invertible *DGC* matrix, $\chi\left(\tilde{A}^{-1}\right)=\left(\chi\left(\tilde{A}\right)\right)^{-1}$ is clear from substituting $\tilde{B}=\tilde{A}^{-1}$ into part 5) and using part 1).

Now we turn to important applications of the dual fundamental matrix.

Theorem 4.3. For any *DGC* matrices of the same order $\tilde{A}$ and $\tilde{B}$, if $\tilde{A}\tilde{B}=\tilde{I}_n$ then $\tilde{B}\tilde{A}=\tilde{I}_n$.

Proof: Let us write $\tilde{A}=A_1+A_2J$ and $\tilde{B}=B_1+B_2J$. Suppose $\tilde{A}\tilde{B}=\tilde{I}_n$. Then using part 1) and 5) of Theorem 4.2, we have: $\chi\left(\tilde{A}\tilde{B}\right)=\chi\left(I_n\right)$ and it follows $\chi\left(\tilde{A}\right)\chi\left(\tilde{B}\right)=I_{2n}$. $\tilde{A}$ and $\tilde{B}$ are dual matrices with order $2n$, so we have $\chi\left(\tilde{B}\right)\chi\left(\tilde{A}\right)=I_{2n}$. Hence we obtain $\tilde{B}\tilde{A}=I_n$.

Theorem 4.4. Let $\tilde{A}\in\mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$. Then, $\chi\left(\tilde{A}\right)$ is

1) symmetric if and only if $\tilde{A}$ is symmetric for $\mathfrak{p}=1$.

[5] $\tilde{A}=\begin{bmatrix}1 & J\\ \varepsilon & J\varepsilon\end{bmatrix}$ is example for the parts 8), 9) and 10) of Theorem 4.2.

2) Hermitian if and only if $\tilde{A}$ is Hermitian^{†_2} for $\mathfrak{p}=1$ and Hermitian^{†_3} for $\mathfrak{p}=-1$.
3) skew-Hermitian if and only if $\tilde{A}$ is skew- Hermitian^{†_2} for $\mathfrak{p}=1$ and skew-Hermitian^{†_3} for $\mathfrak{p}=-1$.
4) orthogonal if and only if $\tilde{A}$ is orthogonal for $\mathfrak{p}=1$.
5) unitary if and only if $\tilde{A}$ is unitary^{†_2} for $\mathfrak{p}=1$ and unitary^{†_3} for $\mathfrak{p}=-1$,
6) normal if and only if $\tilde{A}$ is normal^{†_2} for $\mathfrak{p}=1$ and normal^{†_3} for $\mathfrak{p}=-1$.

Proof: 2) If $\chi\left(\tilde{A}\right)$ is Hermitian then $\begin{bmatrix} A_1 & \mathfrak{p}\,A_2 \\ A_2 & A_1 \end{bmatrix}=\begin{bmatrix} \overline{A_1}^T & \overline{A_2}^T \\ \mathfrak{p}\overline{A_2}^T & \overline{A_1}^T \end{bmatrix}$. For $\mathfrak{p}=1$ we have $A_1=\overline{A_1}^T$ and $A_2=\overline{A_2}^T$. Then $\tilde{A}^{\star_2}=\overline{A_1}^T+J\overline{A_2}^T=A_1+JA_2=\tilde{A}$ gives $\tilde{A}$ is Hermitian^{†_2}. It is clear that for $\mathfrak{p}=-1$, $\tilde{A}^{\star_3}=\overline{A_1}^T-J\overline{A_2}^T=A_1+JA_2=\tilde{A}$ gives $\tilde{A}$ is Hermitian^{†_3}.

5) If $\chi\left(\tilde{A}\right)$ is unitary then $I_{2n}=\chi\left(\tilde{A}\right)\overline{\chi\left(\tilde{A}\right)}^T=\begin{bmatrix} A_1 & \mathfrak{p}A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} \overline{A_1}^T & \overline{A_2}^T \\ \mathfrak{p}\overline{A_2}^T & \overline{A_1}^T \end{bmatrix}$. By taking into account $\mathfrak{p}=1$, we have $A_1\overline{A_1}^T+A_2\overline{A_2}^T=I_n$ and $A_1\overline{A_2}^T+A_2\overline{A_1}^T=0_n$. Thus $\tilde{A}\tilde{A}^{\star_2}=I_n$ and $\tilde{A}$ is unitary^{†_2}. Additionally for $\mathfrak{p}=-1$, we have $\tilde{A}\tilde{A}^{\star_3}=I_n$ and $\tilde{A}$ is unitary^{†_3}.

Definition 4.1. Let $\tilde{A}\in\mathbb{M}_n\left(\mathbb{DC}_\mathfrak{p}\right)$ and $V\in V^n$ be a non-null vector. If $\tilde{A}V=\lambda V$ for some $\lambda\in\mathbb{DC}_\mathfrak{p}$, then λ is called an eigenvalue of $\tilde{A}$ and V is called an eigenvector of $\tilde{A}$ associated with λ. The set of all eigenvalues of $\tilde{A}$ is denoted by $\sigma(\tilde{A})$ and referred to as the spectrum of $\tilde{A}$. Those eigenvalues are said to be the standard eigenvalues of $\tilde{A}$.

Lemma 4.1. If λ is an eigenvalue of the *DGC* matrix $\tilde{A}$ corresponding to the eigenvector V, then $c\lambda$ is an eigenvalue of $c\tilde{A}$ corresponding to the same eigenvector V, where c is a non-zero real scalar.

Theorem 4.5. Every *DGC* square matrix $\tilde{A}$ of the order n has at most $2n$ distinct dual eigenvalues.

Proof: Let $\lambda\in\mathbb{DC}_\mathfrak{p}$ be an eigenvalue of $\tilde{A}=A_1+A_2J\in\mathbb{M}_n\left(\mathbb{DC}_\mathfrak{p}\right)$. Then, for non-pure dual column vectors V_1 and V_2, there exists a non-null *DGC* vector $V=V_1+V_2J\in V^n$ such that $(A_1+A_2J)(V_1+V_2J)=\lambda\left(V_1+V_2J\right)$. Hence,

$$A_1V_1+\mathfrak{p}A_2V_2=\lambda V_1 \quad \text{and } A_1V_2+A_2V_1=\lambda V_2.$$

So we have: $\begin{bmatrix} A_1 & \mathfrak{p}A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=\lambda\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$. Due to the fact that a dual matrix has at most $2n$ distinct dual eigenvalues, $\tilde{A}$ has at most $2n$ distinct dual eigenvalues.

Corollary 4.2. $\lambda = \lambda_1 + \lambda_2 J \in \mathbb{DC}_{\mathfrak{p}}$ is an eigenvalue of the *DGC* square matrix $\tilde{A} = A_1 + A_2 J$ of the order n if and only if

$$\begin{bmatrix} A_1 - \lambda_1 I_n & \mathfrak{p}\left(A_2 - \lambda_2 I_n\right) \\ A_2 - \lambda_2 I_n & A_1 - \lambda_1 I_n \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Corollary 4.3. For every $\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$,

$$\sigma(\tilde{A}) \cap \mathbb{D} = \sigma\left(\chi(\tilde{A})\right),$$

where $\sigma\left(\chi(\tilde{A})\right)$ is the spectrum of the dual fundamental matrix $\chi(\tilde{A})$.

Let us give an alternative version of our results in the setting of dual fundamental matrix and eigenvalues.

Theorem 4.6. For any *DGC* square matrix $\tilde{A}$, the followings are equivalent:

1) $\tilde{A}$ is invertible.
2) $\tilde{A}V = 0$ has a unique solution zero.
3) $\det\left(\chi\left(\tilde{A}\right)\right)$ is not pure dual number or zero, i.e., $\chi\left(\tilde{A}\right)$ is invertible.
4) $\tilde{A}$ has no zero eigenvalue. More precisely, if $\tilde{A}V = \lambda V$ for some λ and some non-null vector V, then λ is non-zero number in $\mathbb{DC}_{\mathfrak{p}}$.

Proof: 1) $\Rightarrow$ 2) This part is obvious.
2) $\Rightarrow$ 3) Take $\tilde{A} = A_1 + A_2 J \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$, $V = V_1 + V_2 J \in V^n$ where A_1, A_2 are dual matrices and V_1, V_2 are dual column vectors. Then,

$$\tilde{A}V = (A_1 + A_2 J)(V_1 + V_2 J) = (A_1 V_1 + \mathfrak{p} A_2 V_2) + (A_1 V_2 + A_2 V_1)J.$$

Writing $\tilde{A}V = 0$ gives: $\begin{cases} A_1 V_1 + \mathfrak{p} A_2 V_2 = 0 \\ A_1 V_2 + A_2 V_1 = 0 \end{cases}.$

So we have: $\tilde{A}V = 0$ if and only if $\begin{bmatrix} A_1 & \mathfrak{p} A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0.$ In other words, $\chi\left(\tilde{A}\right)(V_1, V_2)^T = 0$. Since $\tilde{A}V = 0$ has a unique solution, $\chi\left(\tilde{A}\right)(V_1, V_2)^T = 0$ has a unique solution and $\chi\left(\tilde{A}\right)$ is invertible.
3) $\Rightarrow$ 1) If $\chi\left(\tilde{A}\right)$ is invertible, then for $\tilde{A} = A_1 + A_2 J \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ there exists a unique dual matrix that satisfy: $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} A_1 & \mathfrak{p} A_2 \\ A_2 & A_1 \end{bmatrix} = I_{2n}$. Thus we get: $\begin{cases} B_1 A_1 + B_2 A_2 = I_n \\ \mathfrak{p} B_1 A_2 + B_2 A_1 = 0_n \end{cases}$. Using these,

we see that $(B_1A_1 + B_2A_2) + (\mathfrak{p}B_1A_2 + B_2A_1)J = \tilde{I}_n$ or $\tilde{B}\tilde{A} = \tilde{I}_n$, where $\tilde{B} = B_1 + B_2J$. This gives $\tilde{A}$ is invertible.
2) $\Leftrightarrow$ 4) Let $\tilde{A}V = 0$ has a unique solution zero for $\tilde{A}$, i.e., V is 0. However, assuming that $\tilde{A}$ has zero eigenvalue, $\tilde{A}V = \lambda V$ is satisfied for non-null vector V. This is a contradiction. It means that $\tilde{A}V = 0$ has a unique solution zero, then $\tilde{A}$ has no zero eigenvalue. Now assume that $\tilde{A}$ has no zero eigenvalue. In this case, $\tilde{A}V = 0 = \lambda V$ and by our assumption V is 0. Hence $\tilde{A}V = 0$ has a unique solution zero for $\tilde{A}$.

Remark 4.1. If any row or column of the dual fundamental matrix $\chi\left(\tilde{A}\right)$ is multiple of pure dual number, then the *DGC* square matrix $\tilde{A}$ is not invertible.

Now we turn to define determinant of the dual fundamental matrix of $\tilde{A}$.

Definition 4.2. Let $\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right)$ and $\chi\left(\tilde{A}\right)$ be the dual fundamental matrix of $\tilde{A}$. The $J-$ determinant is defined by $|\tilde{A}|_J = \det\left(\chi\left(\tilde{A}\right)\right)$. Here, $\det\left(\chi\left(\tilde{A}\right)\right)$ is the usual determinant of $\chi\left(\tilde{A}\right)$.

The general linear and special linear group of *DGC* matrices are:

$$GL_n\left(\mathbb{DC}_{\mathfrak{p}}\right) := \{\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right) : \ |\tilde{A}|_J \neq 0\},$$
$$SL_n\left(\mathbb{DC}_{\mathfrak{p}}\right) := \{\tilde{A} \in \mathbb{M}_n\left(\mathbb{DC}_{\mathfrak{p}}\right) : \ |\tilde{A}|_J = 1\},$$

respectively. According to above equalities, the $J-$determinant has an essential role in the concept of general linear and special linear groups to *DGC* matrices.

Theorem 4.7. For any *DGC* square matrices of the same order $\tilde{A}$ and $\tilde{B}$, the followings hold:

1) $\tilde{A}$ is invertible $\Leftrightarrow$ $|\tilde{A}|_J$ is not pure dual number or 0.
2) $|\tilde{A}\tilde{B}|_J = |\tilde{A}|_J |\tilde{B}|_J$, naturally $|\tilde{A}^{-1}|_J = |\tilde{A}|_J^{-1}$ if $\tilde{A}^{-1}$ exists.

Proposition 4.1. For any *DGC* square matrix $\tilde{A} = A_1 + A_2J$, if A_1 and A_2 are commutative, then $|\tilde{A}|_J = \det(\tilde{A})\det(\tilde{A}^{\dagger_1})$.

Proof: As A_1 and A_2 are commutative, we have:

$$\det(\tilde{A})\det(\tilde{A}^{\dagger_1}) = \det(\tilde{A}\tilde{A}^{\dagger_1}) = \det(A_1^2 - \mathfrak{p}A_2^2) = \det\left(\chi\left(\tilde{A}\right)\right) = |\tilde{A}|_J .$$

Example 4.1. Consider the *DGC* matrix $\tilde{A}=\begin{bmatrix}1+J\varepsilon & J\\ 0 & 1+J\end{bmatrix}$. According to $\chi\left(\tilde{A}\right)=\begin{bmatrix}1 & 0 & \mathfrak{p}\varepsilon & \mathfrak{p}\\ 0 & 1 & 0 & \mathfrak{p}\\ \varepsilon & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$ and $\det\left(\chi\left(\tilde{A}\right)\right)=|\tilde{A}|_J=1-\mathfrak{p}$, we can assert that $\tilde{A}$ is invertible *DGC* matrix if $\mathfrak{p}\neq 1$. As $\det(\tilde{A})=1+J+\mathfrak{p}\varepsilon+J\varepsilon$ and $\det(\tilde{A}^{\dagger_1})=1-J+\mathfrak{p}\varepsilon-J\varepsilon$, we verify that $|\tilde{A}|_J=\det(\tilde{A})\det(\tilde{A}^{\dagger_1})$.

Symmetric, Hermitian^{†_k}, orthogonal and unitary^{†_k} matrices have many special properties, the most important of which are expressed in the following theorems. One can establish Theorem 4.8 to characterize *DGC* symmetric, and Theorem 4.9 and Theorem 4.10 to characterize *DGC* orthogonal matrices.

Theorem 4.8. [Spectral theorem for *DGC* Hermitian^{†_k} matrices]
For any *DGC* Hermitian^{†_k} matrix $\tilde{A}$ ($k=1,2,3$):

1) the eigenvalues are of the form $a+c\varepsilon$ for ^{†_1}, $a+bJ$ for ^{†_2} and $a+dJ\varepsilon$ for ^{†_3} where $a,b,c,d\in\mathbb{R}$.
2) the eigenvectors corresponding to distinct eigenvalues are orthogonal^{†_k}.

Proof: 1) Let λ be an arbitrary eigenvalue of the Hermitian^{†_k} matrix $\tilde{A}$ and $V\in V^n$ be an eigenvector corresponding to the eigenvalue λ. So by using Definition 3.2 and $\tilde{A}V=\lambda V$, we have:

$$\begin{aligned}\langle\tilde{A}V,V\rangle_{\dagger_k}&=\langle V,\tilde{A}V\rangle_{\dagger_k}\\ \langle\lambda V,V\rangle_{\dagger_k}&=\langle V,\lambda V\rangle_{\dagger_k}\\ \lambda\langle V,V\rangle_{\dagger_k}&=\lambda^{\dagger_k}\langle V,V\rangle_{\dagger_k}.\end{aligned}$$

Hence, λ is of the form $a+c\varepsilon$ for ^{†_1}, $a+bJ$ for ^{†_2} and $a+dJ\varepsilon$ for ^{†_3} where $a,b,c,d\in\mathbb{R}$.

2) Consider V_1 and V_2 be two eigenvectors corresponding to the distinct eigenvalues λ_1 and λ_2. Since $\tilde{A}V_1=\lambda_1V_1$, $\tilde{A}V_2=\lambda_2V_2$, we have the following:

$$\lambda_1\langle V_1,V_2\rangle_{\dagger_k}=\langle\lambda_1V_1,V_2\rangle_{\dagger_k}=\langle\tilde{A}V_1,V_2\rangle_{\dagger_k}=\langle V_1,\tilde{A}V_2\rangle_{\dagger_k}=\lambda_2^{\dagger_k}\langle V_1,V_2\rangle_{\dagger_k}.$$

As $\lambda_1\neq\lambda_2$, we conclude that $\langle V_1,V_2\rangle_{\dagger_k}=0$.

Theorem 4.9 [Spectral theorem for *DGC* unitary^{†_k} matrices]
For any a *DGC* unitary^{†_k} matrix $\tilde{A}$ ($k=1,2,3$),

1) if λ is an eigenvalue then $|\lambda|^2_{\dagger_k}=1$.
2) the eigenvectors corresponding to distinct eigenvalues are orthogonal^{†_k}.

Proof: 1) Let λ be eigenvalue of $\tilde{A}$ corresponding to the eigenvector V. Then using Definition 3.2, we have: $\langle \tilde{A}V, \tilde{A}V \rangle_{\dagger_k} = \langle V, V \rangle_{\dagger_k} = \lambda\lambda^{\dagger_k} \langle V, V \rangle_{\dagger_k}$. Hence we state that $\lambda\lambda^{\dagger_k} = |\lambda|^2_{\dagger_k} = 1$.
2) Take V_1 and V_2 be two eigenvectors corresponding to the distinct eigenvalues λ_1 and λ_2. So, we have: $\langle \tilde{A}V_1, \tilde{A}V_2 \rangle_{\dagger_k} = \langle V_1, V_2 \rangle_{\dagger_k} = \lambda_1\lambda_2^{\dagger_k} \langle V_1, V_2 \rangle_{\dagger_k}$. From part 1), we get $\lambda_2^{-1} = \lambda_2^{\dagger_k}$. Since λ_1 and λ_2 are distinct eigenvalues, it follows that $\lambda_1\lambda_2^{-1} \neq 1$ and $\langle V_1, V_2 \rangle_{\dagger_k} = 0$.

Theorem 4.10. For any *DGC* unitary^{†_k} matrix $\tilde{A}$, $|\det(\tilde{A})|^2_{\dagger_k} = 1$, (for $k = 1, 2, 3$).

Proof: For *DGC* unitary^{†_k} matrix $\tilde{A}$, we have $\det(\tilde{A}\tilde{A}^{\star_k}) = \det(\tilde{I}_n)$. It follows $\det(\tilde{A})\det(\tilde{A}^{\star_k}) = 1$ and $\det(\tilde{A})(\det(\tilde{A}))^{\dagger_k} = 1$. Therefore $|\det(\tilde{A})|^2_{\dagger_k} = 1$.

Example 4.2. $\tilde{A} = \begin{bmatrix} 1+3J\varepsilon & -\varepsilon + J\varepsilon \\ \varepsilon + J\varepsilon & 1-2J\varepsilon \end{bmatrix}$ is a unitary^{†_1} matrix. Also, $\det\tilde{A} = 1 + J\varepsilon$ and $|\det(\tilde{A})|^2_{\dagger_k} = 1$.

5. COMPUTATIONAL RESULTS

We now want to give a method for finding unitary matrices, which has significant importance in quantum mechanics because they preserve norms, and thus, probability amplitudes.

The calculating steps to find *DGC* unitary^{†_k} matrix $(k = 1, 2, 3)$:

- Take an arbitrary unit vector $V_1 \in V^3$.
- Calculate a unit vector $V_2 \in V^3$ such that $\langle V_1, V_2 \rangle_{\dagger_k} = 0$.
- Compute another unit vector $V_3 \in V^3$ such that $V_3 = V_1 \times_{\dagger_k} V_2$.
- Write the *DGC* unitary^{†_k} matrix $[V_1, V_2, V_3]$.

Furthermore, these steps can be applied to find *DGC* orthogonal matrices. In the following, we will provide a computational example of finding *DGC* unitary^{†_k} matrix.

Example 5.1. For $k = 1$ and $\mathfrak{p} < 0$, take the unit *DGC* vector $V_1 = \frac{1}{\sqrt{-\mathfrak{p}}}(J, \varepsilon, J\varepsilon)$ in V^3. We can calculate the unknowns a, b, c, d for the *DGC* vector $U = (\varepsilon, J, a + bJ + c\varepsilon + dJ\varepsilon)$ where $\langle V_1, U \rangle_{\dagger_1} = aJ\varepsilon - \mathfrak{p}b\varepsilon = 0$. Hence, we have $a = 0,\ b = 0,\ c,\ d \in \mathbb{R}$ and we can choose $c = 0,\ d = -1$. Additionally, the norm of $\|U\|_{\dagger_1} = \sqrt{-\mathfrak{p}}$. So, we can write the desired unit vector as follows: $V_2 = \frac{1}{\sqrt{-\mathfrak{p}}}(\varepsilon, J, -J\varepsilon)$. Then, calculate the vector V_3 such that

$$V_3 = V_1 \times_{\dagger_1} V_2 = \frac{1}{-\mathfrak{p}} \begin{vmatrix} i & j & k \\ -J & \varepsilon & -J\varepsilon \\ \varepsilon & -J & J\varepsilon \end{vmatrix} = (\varepsilon, -\varepsilon, -1).$$

Finally, $\tilde{U} = \begin{bmatrix} \frac{J}{\sqrt{-\mathfrak{p}}} & \frac{\varepsilon}{\sqrt{-\mathfrak{p}}} & \frac{J\varepsilon}{\sqrt{-\mathfrak{p}}} \\ \frac{\varepsilon}{\sqrt{-\mathfrak{p}}} & \frac{J}{\sqrt{-\mathfrak{p}}} & -\frac{J\varepsilon}{\sqrt{-\mathfrak{p}}} \\ \varepsilon & -\varepsilon & -1 \end{bmatrix}$ is a *DGC* unitary^{†_1} matrix.

Let give an example for the spectral theorem for *DGC* symmetric matrix.

Example 5.2. Consider $\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & J+\varepsilon \end{bmatrix}$ for $\mathfrak{p} > 0$. The dual fundamental matrix of $\tilde{A}$: $\chi(\tilde{A}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \varepsilon \end{bmatrix}$. The eigenvalues of $\chi(\tilde{A})$ are calculated as $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -\sqrt{\mathfrak{p}} + \varepsilon$, $\lambda_4 = \sqrt{\mathfrak{p}} + \varepsilon$. One can easily obtain the following eigenvectors corresponding to these eigenvalues as: $\alpha_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$, $\alpha_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$, $\alpha_3 = \begin{bmatrix} 0 & -\sqrt{\mathfrak{p}} & 0 & 1 \end{bmatrix}^T$, $\alpha_4 = \begin{bmatrix} 0 & \sqrt{\mathfrak{p}} & 0 & 1 \end{bmatrix}^T$, respectively.

Corollary 4.2 and Corollary 4.3 allow us to move from the eigenvectors of $\chi(\tilde{A})$ to the eigenvectors of $\tilde{A}$, so we have: $V_1 = (J, 0)$, $V_2 = (1, 0)$, $V_3 = (0, J - \sqrt{\mathfrak{p}})$ and $V_4 = (0, J + \sqrt{\mathfrak{p}})$.

As an application of Spectral Theorem for *DGC* symmetric matrices (see Theorem 4.8), one can assert that the eigenvectors corresponding to distinct eigenvalues are orthogonal with respect to standard scalar product. Notice that two eigenvectors corresponding to the same eigenvector are not orthogonal.

Let P be the matrix whose columns are formed from the eigenvectors of $\chi(\tilde{A})$ and D be the diagonal matrix whose has eigenvalues of $\chi(\tilde{A})$ along its main diagonal. Thus, we have for $\chi(\tilde{A}) = PDP^{-1}$:

$$\chi(\tilde{A}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\mathfrak{p}} & \sqrt{\mathfrak{p}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\mathfrak{p}}+\varepsilon & 0 \\ 0 & 0 & 0 & \sqrt{\mathfrak{p}}+\varepsilon \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2\sqrt{\mathfrak{p}}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2\sqrt{\mathfrak{p}}} & 0 & \frac{1}{2} \end{bmatrix}.$$

Example 5.3. Consider $\tilde{A}=\begin{bmatrix} 1 & 0 \\ J+\varepsilon & J \end{bmatrix}$ for $\mathfrak{p}=9$. Then $\chi\left(\tilde{A}\right)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 0 & 9 & 9 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & \varepsilon & 0 \end{bmatrix}$. The eigenvalues of $\chi\left(\tilde{A}\right)$ are calculated as $-3,3$ and 1. Let P be a matrix whose columns are constructed by the eigenvectors of $\chi\left(\tilde{A}\right)$. Thus we have $\chi\left(\tilde{A}\right)=P\,\mathrm{diag}\{-3,3,1,1\}P^{-1}$, where

$$P=\begin{bmatrix} 0 & 0 & -\dfrac{8}{1+\varepsilon} & -\dfrac{9+\varepsilon}{1+\varepsilon} \\ -3 & 3 & \dfrac{9+\varepsilon}{1+\varepsilon} & \dfrac{9}{1+\varepsilon} \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

6. CONCLUSIONS

Consequently, we prove that classical results in matrix theory hold for *DGC* matrices as well. The main advantage in carrying out this construction is that *DGC* numbers are commutative. This theory is examined by proving several characteristic theorems. In this paper, the central focus is the dual fundamental matrix of any *DGC* square matrix. It is a known fact that matrices with dual number components make possible a concise representation of link proportions and joint parameters; together with the orthogonality properties of the matrices in kinematics. They are also used in closed-form solutions for the joint displacements of robot manipulators with special geometry in literature. For a more in-depth and application of dual-number matrices to the formulation of displacement equations of robot manipulators, see [32]. It is worth pointing out that this classical matrix theory can be established for dual-complex numbers for $\mathfrak{p}=-1$ and dual-hyperbolic numbers for $\mathfrak{p}=1$. We expect that the results obtained related to *DGC* matrices will become an important tool in many areas of science and provide a meaningful alternative to existing studies. Additionally, for future work, we intend to extend our work on similarity relations, exponential map, many different *DGC* matrix decompositions and investigate the solutions of linear equation systems.

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REFERENCES

[1] Yaglom, I. M., *Complex numbers in geometry,* Academic Press, New York, 1968.
[2] Fjelstad, P., *American Journal of Physics*, **54**, 416, 1986.
[3] Sobczyk, G., *The College Mathematics Journal*, **26**, 268, 1995.
[4] Yaglom, I. M., *A simple non-Euclidean geometry and its physical basis,* Springer-Verlag, New York, 1979.
[5] Pennestrì, E., Stefanelli, R., *Multibody System Dynamics,* **18**, 323, 2007.

[6] Study, E., *Geometrie der dynamen. Die zusammensetzung von kräften und verwandte gegenstände der geometrie bearb*, Leipzig, B.G. Teubner, 1903.
[7] Harkin, A. A., Harkin, J. B., *Mathematics Magazine*, **77**, 118, 2004.
[8] Kantor, I., Solodovnikov, A., *Hypercomplex Numbers,* Springer-Verlag, New York, 1989.
[9] Alfsmann, D., *European Signal Processing Conf. (EUSIPCO),* Florence, Italy, 2006.
[10] Cockle, J., *Philosophical magazine*, **3**, 37, 1849.
[11] Majernik, V., *Acta Physica Polonica*, **90,** 491, 1996.
[12] Fjelstad, P., Sorin, Gal G., *Advances in Applied Clifford Algebras*, **8**, 47, 1998.
[13] Price, G. B., *Monographs and textbooks in pure and applied mathematics*, New-York, 1991.
[14] Rochon, D., Shapiro, M., *Analele Universitatii din Oradea. Fascicola Matematica*, **11**, 71, 2004.
[15] Toyoshima, H., *IEICE Transactions on Information and Systems*, **81**, 236, 1989.
[16] Cheng, H. H., Thompson, S., *Proceeding of ASME 24th Biennial Mechanisms Conference*, Irvine, CA, 1996.
[17] Cheng, H. H., Thompson, S., *ASME Journal of Mechanical Design*, **121**, 200, 1999.
[18] Messelmi, F., *Dual-complex numbers and their holomorphic functions*, hal-01114178, 2015.
[19] Akar, M., Yüce, S., Şahin, S., *Journal of Computer Science and Computational Mathematics*, **8,** 1, 2018.
[20] Cohen, A., Shoham, M., *Mechanism and Machine Theory*, **125**, 101, 2018.
[21] Fike, J. A., Jongsma, S., Alonso, J. J., Weide, E.Van Der., *Proceeding of 29th AIAA Applied Aerodynamics Conference*, 2011.
[22] Fike, J .A., Alonso, J. J., *Lecture Notes in Computational Science and Engineering book series (LNCSE),* **87,** 163, 2011.
[23] Gürses, N., Şentürk, G. Y., Yüce, S., *Gazi University Journal of Science*, **34**, 180, 2021.
[24] Liitkepohl, H., *Handbook of matrices,* John Wiley and Sons Ltd., England, 1996.
[25] Horn, R. A., Johnson, C. R., *Matrix analysis,* Cambridge University Press, UK, 1985.
[26] Zhang, F., *Linear Algebra and Its Applications*, **251**, 21, 1997.
[27] Hamilton, W. R., *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science (3rd Series)*, **xxv–xxxvi**, 1844–1850.
[28] Alagöz, Y., Oral, K. H., Yüce, S., *Miskolc Mathematical Notes*, **13,** 223, 2012.
[29] Erdoğdu, M., Özdemir, M., *Advances in Applied Clifford Algebras*, **23**, 625, 2013.
[30] Kösal, H. H., *On commutative quaternion matrices*, Sakarya University Graduate School of Natural and Applied Sciences, Ph.D. Thesis, ID:434046, 2016.
[31] Pennock, G. R., Yang, A. T., *J. Journal of Mechanisms, Transmissions, and Automation in Design*, **107**, 201, 1985.
[32] Pennestrì, E., Valentini, P. P., *Multibody Dynamics Computational Methods and Applications*, **12**, 207, 2009.
[33] Angeles, J., *Latest Advances in Robot Kinematics*, Springer, Netherlands, 2012.
[34] Falco, D. De, Pennestrì, E., Udwadia, F. E., *Mechanism and Machine Theory*, **123,** 89, 2018.
[35] Udwadia, F. E., Pennestrì, E., Falco, D. De, *Mechanism and Machine Theory*, **151**, 103878, 2020.
[36] Tian, Y., *Advances in Applied Clifford Algebras*, **8**, 365, 1998.