ORIGINAL PAPER

2-ABSORBING INTUITIONISTIC FUZZY IDEALS OF COMMUTATIVE SEMIRINGS

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Abstract. *The purpose of this study is to define the 2-absorbing intuitionistic fuzzy ideals and 2-absorbing intuitionistic fuzzy primary ideals of commutative semirings, as well as to describe their characteristics. Furthermore, we define the cartesian product of the 2-absorbing intuitionistic fuzzy primary ideals of commutative semirings.*

Keywords: *2-absorbing ideal; 2-absorbing primary ideal; 2-absorbing intuitionistic fuzzy ideal.*

1. INTRODUCTION

In commutative theory, prime ideals and primary ideals are essential elements. A number of generalizations of prime ideals have been looked various scientists. Badawi presents the idea of a 2-absorbing ideal (2AI) in [1] as a unique generalization of prime ideals, while 2-absorbing primary ideals (2API), which are a generalization of primary ideals, were proposed in [2]. Prime fuzzy ideals and primary fuzzy ideals are also offered in [3, 4]. In various ways, Badawi improved the research on these ideas in [5] and [2]. This notion was thoroughly studied by a number of additional researchers (see [6- 8]).

The use of mathematics in other fields of science is important, and during the past several decades, this has been one of the main goals of the work of Fuzzy Set Theory specialists all over the world. Numerous scientists began using fuzzy sets in a variety of scientific domains after Zadeh [9] introduced the traditional concept of fuzzy sets in 1965. In 1971, Rosenfeld [10] first the use of fuzzy sets in algebraic structures. The idea of a fuzzy ideal of a ring was then explained by Liu [11]. Fuzzy ideals were shown by Mukherjee and Sen by introducing the idea of prime fuzzy ideals [3].

The concept of an intuitionistic fuzzy set was first developed by Atanassov in 1986 as an extension of Zadeh's fuzzy sets [9]. Furthermore, Hur et al. concentrated on the idea of intuitionistic fuzzy subring in [12] and a number of authors have since tried to generalize the idea. Marashdeh et al. [13] provided an explanation of how the idea of intuitionistic fuzzy rings depends on the idea of fuzzy space. Sharma in [14] refined the translations of intuitionistic fuzzy subrings. Interval valued intuitionistic fuzzy sets were introduced to the broader environment by Chao

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et al. in [15]. Darani investigates and analyzes L-fuzzy 2-absorbing ideals in [16]. The idea of L-fuzzy 2-absorbing ideals in semiring was proved by Hashempoor et al. in [17].

The principal objective of this paper is to review the algebraic structure of 2AI by applying intuitionistic fuzzy set theory. The concept of 2-absorbing intuitionistic fuzzy ideals (2AIFIS) will be examined, along with some of its algebraic properties. Properties of the cartesian product of these ideals have been provided in accordance. Finally, the image and inverse image of these ideals has been defined under semiring homomorphism.

2. PRELIMINARIES

2.1. SEMIRING

A semiring is an algebraic structure $(\Im,+.,0,1)$ with the following axioms:

i) $(\Im,+,0)$ is a commutative monoid,

ii) $(\Im,.,1)$ is a monoid with $1\neq 0$,

iii) $r(s+t)=rs+rt$ and $(s+t)r=sr+tr$ for all $r,s,t\in\Im$,

iv) $r.0=0.r$ for all $r\in\Im$.

If a semiring $\Im$ is commutative satisfies $rs=sr$ for all $r,s\in\Im$. In this paper, all semirings are commutative.

Definition 1. [18] A subset $\Re$ of a semiring $\Im$ is a subsemiring of $\Im$ if it contains $0,1$ and is closed under the operations of addition and multiplication in $\Im$.

Example 1. [18] $\Im$ is a semiring then $P(\Im)=\{0\}\cup\{s+1:s\in\Im\}$ is a subsemiring of $\Im$.

Definition 2. [18] Let $\Im$ be a semiring. A nonempty subset Ω of $\Im$ is called a left ideal of $\Im$ if for all $r,s\in\Omega$ and for all $p\in\Im$, $r+s\in\Omega$, $pa\in\Omega$ $(\Omega\neq\Im)$.

Theorem 1. [18] The following conditions on an ideal Ω of a semiring $\Im$ are equivalent:

i) Ω is prime;

ii) $\{rps:p\in\Im\}\subseteq\Omega$ if and only if $r\in\Omega$ or $s\in\Omega$;

iii) If r and s are elements of $\Im$ satisfying $(r)(s)\subseteq\Omega$ then either $r\in\Omega$ or $s\in\Omega$.

Corollary 1. [18] An ideal Ω of a commutative semiring $\Im$ is prime if and only if $rs\in\Omega$ implies that $r\in\Omega$ or $s\in\Omega$ for all elements r and s of $\Im$.

Theorem 2. [18] If Ω is an ideal of a commutative semiring $\Im$ then $\sqrt{\Omega}$ is a prime ideal of $\Im$.

Definition 3. [1] Let $\Im$ be a commutative semiring with identity. A proper ideal Ω of $\Im$ is said to be 2- absorbing provided that whenever $r,s,t \in \Im$ with $rst \in \Omega$ then either $rs \in \Omega$ or $rt \in \Omega$ or $st \in \Omega$. $\Im$ is called a 2- absorbing semiring if and only if its zero ideal is 2-absorbing.

Definition 4. [2] A proper ideal Ω of $\Im$ is called 2- absorbing primary ideal of $\Im$ (2APIS($\Im$)) if whenever $r,s,t \in \Im$ with $rst \in \Omega$, then either $rs \in \Omega$ or $rt \in \sqrt{\Omega}$ or $st \in \sqrt{\Omega}$.

Theorem 3. [2] If Ω is a 2APIS($\Im$), then Ω is a 2-absorbing ideal of $\Im$ (2AIS($\Im$)).

Theorem 4. [1] Assume that Ω is a nonzero proper ideal of a semiring $\Im$ and the following conditions are equivalent

i) Ω is a 2AIS($\Im$).
ii) If $\Omega_1\Omega_2\Omega_3 \subseteq \Omega$ for some ideals $\Omega_1,\Omega_2,\Omega_3$ of $\Im$, then $\Omega_1\Omega_2 \subseteq \Omega$ or $\Omega_1\Omega_3 \subseteq \Omega$ or $\Omega_2\Omega_3 \subseteq \Omega$.

Example 2. Let the ideal $\Omega = (45)$ in $\mathbb{Z}$. On account of $3.3.5 \in \Omega$, but $3.3 \notin \Omega$ and $3.5 \notin \Omega$. So, Ω is not a 2AI. On the other hand, it is obvious that Ω is a 2API. It is obvious that 2-absorbing and 2-absorbing primary are various concepts. It seems that every primary ideal of commutative ring is a 2API, however, the opposite might not be true. Consider (15) is a 2API of $\mathbb{Z}$, though (15) is not a primary ideal of $\mathbb{Z}$.

2.2. INTUITIONISTIC FUZZY SETS.

Definition 5. [18] A fuzzy subset Φ in a set R is a function $\Phi : R \to [0,1]$.

Definition 6. [19] The intuitionistic fuzzy sets are defined on a non-empty set R as objects having the form

$$A = \{\langle r, \Phi(r), \Psi(r)\rangle \mid r \in R\}$$

where the functions $\Phi : R \to [0,1]$ and $\Psi : R \to [0,1]$ denote the degrees of membership and of non- membership of each element $r \in R$ to set A, respectively, $0 \le \Phi(r) + \Psi(r) \le 1$ for all $r \in R$.

Definition 7. [12] Let R be a nonempty set and let $A = \langle \Phi_A, \Psi_A \rangle$ and $B = \langle \Phi_B, \Psi_B \rangle$ be intuitionistic fuzzy sets in X. Then,

(1) $A \subset B$ iff $\Phi_A \le \Phi_B$ and $\Psi_A \ge \Psi_B$,
(2) $A = B$ iff $A \subset B$ and $B \subset A$,
(3) $A^c = \langle \Psi_A, \Phi_A \rangle$,
(4) $A \cap B = (\Phi_A \wedge \Phi_B, \Psi_A \vee \Psi_B)$,

(5) $A \cup B = (\Phi_A \vee \Phi_B, \Psi_A \wedge \Psi_B)$.

Definition 8. [20] Let $\gamma, \delta \in [0,1]$ with $\gamma + \delta \leq 1$. An intuitionistic fuzzy point, written as $r_{(\gamma,\delta)}$ is defined to be an intuitionistic fuzzy subset of R, given by

$$r_{(\gamma,\delta)}(s) = \begin{cases} (\gamma,\delta), & \text{if r=s,} \\ (0,1), & \text{if r} \neq \text{s.} \end{cases}$$

An intuitionistic fuzzy point $x_{(\gamma,\delta)}$ is said to belong in intuitionistic fuzzy set $\langle \Phi, \Psi \rangle$ denoted by $r_{(\gamma,\delta)} \in \langle \Phi, \Psi \rangle$ if $\Phi(r) \geq \gamma$ and $\Psi(r) \leq \delta$ and we have for $r, s \in R$

i) $r_{(p,q)} + y_{(\gamma,\delta)} = (r+s)_{(p \wedge \gamma, q \vee \delta)}$
ii) $r_{(p,q)} s_{(\gamma,\delta)} = (rs)_{(p \wedge \gamma, q \vee \delta)}$
iii) $\langle r_{(p,q)} \rangle \langle s_{(\gamma,\delta)} \rangle = \langle r_{(p,q)} s_{(\gamma,\delta)} \rangle$.

Definition 9. [21] Let Ω be an intuitionistic fuzzy set in a set R and $p, q \in [0,1]$ such that $p + q \leq 1$. Then, the set

$$\Omega^{(p,q)} = \{ r \in R \mid \Phi_{\Omega}(r) \geq p \text{ and } \Psi_{\Omega}(r) \leq q \}$$

is called (p,q) level subset of Ω.

2.3. INTUITIONISTIC FUZZY IDEALS OF SEMIRINGS.

Definition 10. [19] Let $\Im$ be a semiring. An intuitionistic fuzzy set $A = \{ \langle r, \Phi(r), \Psi(r) \rangle \mid r \in \Im \}$ is said to be an intuitionistic fuzzy ideal of $\Im$ if $\forall r, s \in \Im$

i) $\Phi(r+s) \geq \Phi(r) \wedge \Phi(s)$,
ii) $\Psi(r+s) \leq \Psi(r) \vee \Psi(s)$,
iii) $\Phi(rs) \geq \Phi(r) \vee \Phi(s)$,
iv) $\Psi(rs) \leq \Psi(r) \wedge \Psi(s)$.

Definition 11. [4] Let $\Omega = \langle \Phi_{\Omega}, \Psi_{\Omega} \rangle$ be an intuitionistic fuzzy ideal of $\Im$. Then, $\sqrt{\Omega}$ is called the radical of Ω and it is defined by $\sqrt{\Omega}(r) = \vee_{n \geq 1} \Omega(r^n)$.

Theorem 5. [24] If Ω and Θ are intuitionistic fuzzy ideals of $\Im$, then $\sqrt{\Omega \cap \Theta} = \sqrt{\Omega} \cap \sqrt{\Theta}$.

Theorem 6. [22] Let $f : \Im \to \Re$ be a semiring homomorphism and let Ω be an intuitionistic fuzzy ideal of $\Im$ such that Ω is an constant on $\ker f$ and Θ be an intuitionistic fuzzy ideal of $\Re$. Then,

i) $\sqrt{f(\Omega)} = f(\sqrt{\Omega})$,

ii) $\sqrt{f^{-1}(\Theta)} = f^{-1}(\sqrt{\Theta})$.

Also, we will give intuitionistic fuzzy prime and primary ideals.

Definition 12. [18] An intuitionistic fuzzy ideal $\Omega = \langle \Phi_\Omega, \Psi_\Omega \rangle$ of $\Im$ is called intuitionistic fuzzy prime ideal, if for any intuitionistic fuzzy ideals $A = \langle \Phi_A, \Psi_A \rangle$ and $B = \langle \Phi_B, \Psi_B \rangle$ of $\Im$
the condition $AB \subset \Omega$ implies that either $A \subset \Omega$ or $B \subset \Omega$.

Definition 13. [23] A fuzzy ideal Ω of a semiring $\Im$ is called intuitionistic fuzzy primary ideal of $\Im$

$$\Phi_\Omega(rs) = \Phi_\Omega(r) \text{ and } \Psi_\Omega(rs) = \Psi_\Omega(r)$$
$$\text{or}$$
$$\Phi_\Omega(rs) \le \Phi_\Omega(s^n) \text{ and } \Psi_\Omega(rs) \ge \Psi_\Omega(s^n)$$

for all $r, s \in \Im$, and $n \in \mathbb{Z}^+$.

3. 2-ABSORBING INTUITIONISTIC FUZZY IDEALS OF SEMIRING

In this part, we will focus on 2AIFIS.

Definition 14. Let Ω be an intuitionistic fuzzy set and $\Im$ be a semiring. An intuitionistic fuzzy ideal $\Omega =< \Phi, \Psi >$ of semiring $\Im$ is said to be 2AIFIS($\Im$)

$$\begin{aligned} &\Phi(rst) \ge \gamma \textit{ implies } \Phi(rs) \ge \gamma \textit{ or } \Phi(st) \ge \gamma \textit{ or } \Phi(rt) \ge \gamma, \\ &\Psi(rst) \le \delta \textit{ implies } \Psi(rs) \le \delta \textit{ or } \Psi(st) \le \delta \textit{ or } \Psi(rt) \le \delta, \end{aligned} \tag{1}$$

for all $r, s, t \in \Im$ and $\gamma, \delta \in [0,1]$.

Example 3: Let $\Im = \{0, d, f, h, g\}$ be a set under addition $+$ and the multiplication $\cdot$ given as shown

+	0	d	f	h	g
0	0	d	f	h	g
d	d	f	h	g	0
f	f	h	g	0	d
h	h	g	0	d	f
g	g	0	d	f	h

Figure 1. Addition

·	0	d	f	h	g
0	0	0	0	0	0
d	0	d	f	h	g
f	0	f	g	d	h
h	0	h	d	g	f
g	0	g	h	f	d

Figure 2. Multiplication

Then $(\Im, +, \cdot)$ forms a commutative semirings. Define intuitionistic fuzzy subset $\Omega =< \Phi, \Psi >$ of $\Im$ by $\Omega = (1,0)/0 + (0.4, 0.3)/d + (0.5, 0.7)/f + (0.2, 0.8)/h + (0.3, 0.5)/g$. Then $\Omega =< \Phi, \Psi >$ is a 2AIFIS($\Im$).

Example 4: Let $\Omega = \langle \lambda, \delta \rangle$ be an intuitionistic fuzzy ideal defined by

$$\lambda_\Omega(x) = \begin{Bmatrix} 1, & \text{if } x \in 18\mathbb{Z} \\ 0, & \text{otherwise} \end{Bmatrix} \text{ and } \delta_\Omega(\mathrm{x}) = \begin{Bmatrix} 1, & \text{if x} \in 18\mathbb{Z} \\ 0, & \text{otherwise} \end{Bmatrix}$$

It is obvious that Ω is a 2AIFPIS. However, Ω is not a 2AIFIS since $2_{(1,0)}3_{(1,0)}3_{(1,0)} \in \Omega$, but $2_{(1,0)}3_{(1,0)} \notin \Omega$ and $\%3_{(1,0)}3_{(1,0)} \notin \Omega$.

Definition 15. An intuitionistic fuzzy ideal Ω =< Φ, Ψ > of $\Im$ is said to be 2AIFPIS($\Im$)

$$\begin{aligned} &\Phi(rst) \geq \gamma \textit{ implies } \Phi(rs) \geq \gamma \textit{ or } \sqrt{\Phi}(st) \geq \gamma \textit{ or } \sqrt{\Phi}(rt) \geq \gamma, \\ &\Psi(rst) \leq \delta \textit{ implies } \Psi(rs) \leq \delta \textit{ or } \sqrt{\Psi}\%(st) \leq \delta \textit{ or } \sqrt{\Psi}(rt) \leq \delta, \end{aligned} \tag{2}$$

for all $r,s,t \in \Im$ and $\gamma, \delta \in [0,1]$.

Theorem 7. Let Ω be an intuitionistic fuzzy subset of $\Im$. Then, Ω is a 2AIFIS($\Im$) if and only if $\Omega_{(\gamma,\delta)}$ is a 2AIS($\Im$).

Proof: (⇒): Let Ω is a 2AIFIS($\Im$) and we want to show that $\Omega_{(\gamma,\delta)}$ is a 2AIS($\Im$). Take $rst \in \Omega_{(\gamma,\delta)}$ such that $r,s,t \in \Im$ and $\gamma, \delta \in [0,1]$. In view of the fact Ω is a 2AIFIS($\Im$), we get

$$\begin{aligned} rst \in \Omega_{(\gamma,\delta)} &\Rightarrow \Phi(rst) \geq \gamma \text{ and } \Psi(rst) \leq \delta \\ &\Rightarrow \Phi(rs) \geq \gamma \text{ or } \Phi(st) \geq \gamma \text{ or } \Phi(rt) \geq \gamma \text{ and} \\ &\quad \Psi(rs) \leq \delta \text{ or } \Psi(st) \leq \delta \text{ or } \Psi(rt) \leq \delta \\ &\Rightarrow rs \in \Omega_{(\gamma,\delta)} \text{ or } st \in \Omega_{(\gamma,\delta)} \text{ or } rt \in \Omega_{(\gamma,\delta)}. \end{aligned}$$

Therefore, $\Omega_{(\gamma,\delta)}$ is a a 2AIS($\Im$).

(⇐): On the contrary, assume Ω be intuitionistic fuzzy subset of $\Im$ with $\Omega_{(\gamma,\delta)}$ is a 2AIS ($\Im$) for every $\gamma, \delta \in [0,1]$, however Ω is not a 2AIFIS($\Im$). That being the case, $\Phi(rst) \geq \gamma$ doesn't indicate $\Phi(rs) \geq \gamma$ or $\Phi(st) \geq \gamma$ or $\Phi(rt) \geq \gamma$, and $\Psi(rst) \leq \delta$ doesn't show $\Psi(rs) \leq \delta$ or $\Psi(st) \leq \delta$ or $\Psi(rt) \leq \delta$ for $r,s,t, \in \Im$ and $\gamma, \delta \in [0,1]$. It means that $rst \in \Omega_{(\gamma,\delta)}$ doesn't give $rs \in \Omega_{(\gamma,\delta)}$ or $st \in \Omega_{(\gamma,\delta)}$ or $rt \in \Omega_{(\gamma,\delta)}$. This is a contradiction with $\Omega_{(\gamma,\delta)}$ is a 2AIS($\Im$). So, our basis assumption is not true. Hense, we reach Ω is a 2AIFIS($\Im$).

Theorem 8. Let Θ be a 2AIS($\Im$) and $\gamma, \delta \in [0,1]$. Then an intuitionistic fuzzy subset Ω of $\Im$ given as

$$\Theta(a) = \begin{cases} (1,0); & a \in \Omega \\ (A,w); & \text{otherwise} \end{cases} \tag{3}$$

Then Θ is a 2AIFIS($\Im$).

Proof: Take Θ be a 2AIFIS($\Im$) and $r,s,t \in \Im$. Presume that $\Phi(rst) \geq \gamma$ and $\Psi(rst) \leq \delta$.

Condition I: $\gamma = 1$ and $\delta = 0$ $rst \in \Theta_{(1,0)}$ implies that $rst \in \Theta$. Then, in case

$$\Phi(rs) < \gamma = 1 \text{ or } \Phi(st) < \gamma = 1 \text{ or } \Phi(rt) < \gamma = 1 \text{ and}$$
$$\Psi(rs) > \delta = 0 \text{ or } \Psi(st) > \delta = 0 \text{ or } \Psi(rt) > \delta = 0.$$

It follows that $rs \notin \Theta, st \notin \Theta$ and $rt \notin \Theta$. That is a contradiction in terms of Θ is a 2AIS($\Im$).

Condition II: $\gamma = A$ and $\delta = w$

In view of this $\Theta(rst) = (A, w)$ such that $\Phi(rst) = A$ and $\Psi(rst) = w$. Providing one of them $\Theta(rs)$ or $\Theta(st)$ or $\Theta(rt)$ is equal to (A, w). In all other respects, all of $\Theta(rs), \Theta(st)$ and $\Theta(rt)$ is equal to $(1,0)$. In that event, $rs, st, rt \in \Theta$, though $rst \notin \Theta$. This is a contradiction on Θ is a 2AIS($\Im$). Consequently, the proof is finished.

Theorem 9. Let Ω be a 2AIFPIS($\Im$). Then $\sqrt{\Omega}$ is a 2AIFIS($\Im$).

Proof: Let Ω be a 2AIFPI($\Im$). Assume that $\sqrt{\Omega}(rst) \geq (\gamma, \delta)$ for $r,s,t \in \Im$ and $\gamma, \delta \in [0,1]$. It follows that

$$\sqrt{\Phi}(rst) \geq \gamma \text{ and } \sqrt{\Psi}(rst) \leq \delta.$$

It means that

$$\sqrt{\Phi}(rst) \geq \gamma \Rightarrow \vee\{\Phi(rst)^k \mid k \in \mathbb{Z} \geq \gamma\}$$
$$\Rightarrow \vee\{\Phi(r^k s^k t^k) \mid k \in \mathbb{Z} \geq \gamma\}$$

and

$$\sqrt{\Psi}(rst) \leq \delta \Rightarrow \vee\{\Psi(rst)^k \mid k \in \mathbb{Z} \leq \delta\}$$
$$\Rightarrow \vee\{\Psi(r^k s^k t^k) \mid k \in \mathbb{Z} \leq \delta\}.$$

Presume taht $st, rt \notin \sqrt{\Omega}$ such that $st, rt \notin \sqrt{\Phi}$ and $st, rt \notin \sqrt{\Psi}$. Thus

$$\sqrt{\Phi}(s^k t^k) \ngeq \gamma \text{ and } \sqrt{\Phi}(r^k t^k) \ngeq \gamma$$
$$\sqrt{\Psi}(s^k t^k) \nleq \delta \text{ and } \sqrt{\Psi}(r^k t^k) \nleq \delta$$

for $\gamma, \delta \in [0,1]$ and $k \in \mathbb{Z}$.

In view of the fact that Ω is a 2AIFPIS($\Im$), we have $\Phi((r^k s^k)^l) \geq \gamma$ and $\Psi((r^k s^k)^l) \leq \delta$ for some $l \in \mathbb{Z}$ and hence

$$\vee\{\Phi(r^u s^u) \mid u \in \mathbb{Z} \geq \gamma\}$$
$$\wedge\{\Psi(r^u s^u) \mid u \in \mathbb{Z} \leq \delta\}.$$

It means that $rs \in \sqrt{\Phi}$ and $rs \in \sqrt{\Psi}$, so $rs \in \sqrt{\Omega}$. Consequently $\sqrt{\Omega}$ is a 2AIFIS($\Im$). Now, we will show intersection of non-empty family of 2AIFIS($\Im$) is also 2AIFIS($\Im$).

Theorem 10. Let Ω_i be a non-empty family of 2AIFIS($\Im$) for $i = 1, 2, \cdots$. Then $\bigcap \Omega_i$ is also
2AIFIS($\Im$).

Proof: It is trivial that intersection of a non-empty family intuitionistic fuzzy ideals of $\Im$ is also an intuitionistic fuzzy ideals of $\Im$. Then for 2AIS, let $\{\Omega_i : i \in J\}$ be a non-empty collection of 2AIFIS($\Im$) and $r, s, t \in \Im$. Then, we get

$$\Big(\bigcap_{i\in J}\Omega_i\Big)(rst) \geq (\gamma, \delta)$$
$$\Rightarrow \Big(\bigcap_{i\in J}\Phi_i\Big)(rst) \geq \gamma \text{ and } \Big(\bigcup_{i\in J}\Psi_i\Big)(rst) \leq \delta$$
$$\Rightarrow \wedge_{i\in J}\{\Phi_i(rst)\} \geq \gamma \text{ and } \vee_{i\in J}\{\Psi_i(rst)\} \leq \delta$$
$$\Rightarrow \wedge_{i\in J}\{\Phi_i(rs)\} \geq \gamma \text{ or } \wedge_{i\in J}\{\Phi_i(st)\} \geq \gamma \text{ or } \wedge_{i\in J}\{\Phi_i(rt)\} \geq \gamma$$

and

$$\bigvee_{i\in J}\{\Psi_i(rs)\} \leq \delta \text{ or } \vee_{i\in J}\{\Psi_i(st)\} \leq \delta \text{ or } \vee_{i\in J}\{\Psi_i(rt)\} \leq \delta$$
$$\Rightarrow \Big(\bigcap_{i\in J}\Phi_i\Big)(rs) \geq \gamma \text{ or } \Big(\bigcap_{i\in J}\Phi_i\Big)(st) \geq \gamma \text{ or } \Big(\bigcap_{i\in J}\Phi_i\Big)(rt) \geq \gamma$$

and

$$\Big(\bigcup_{i\in J}\Psi_i\Big)(rs) \leq \delta \text{ or } \Big(\bigcup_{i\in J}\Psi_i\Big)(st) \leq \delta \text{ or } \Big(\bigcup_{i\in J}\Psi_i\Big)(rt) \leq \delta$$
$$\Rightarrow \Big(\bigcap_{i\in J}\Omega_i\Big)(rs) \geq (\gamma, \delta) \text{ or } \Big(\bigcap_{i\in J}\Omega_i\Big)(st) \geq (\gamma, \delta) \text{ or } \Big(\bigcap_{i\in J}\Omega_i\Big)(rt) \geq (\gamma, \delta).$$

Thus, $\bigcap_{i\in J}\Omega_i$ is a 2AIFIS($\Im$).

Theorem 11. Let Ω_i be a non-empty family of 2AIFPIS($\Im$) for $i = 1, 2, \cdots \in J$. Then $\bigcap \Omega_i$ is also a 2AIFPIS($\Im$).

Proof: Straightforward.

Theorem 12. Let $\rho : \Im_1 \to \Im_2$ be a homomrphism of semirings and Ω be a 2AIFIS($\Im_2$). Then, $\rho^{-1}(\Omega)$ is also a 2AIFIS($\Im_1$).

Proof: Let $\rho : \Im_1 \to \Im_2$ be a homomrphism of semirings and Ω be a 2AIFIS($\Im_2$). Now for $r, s, t \in \Im$ and $\gamma, \delta \in [0,1]$.

$$\begin{aligned}
&\rho^{-1}(\Omega)(rst) \geq (\gamma,\delta) \\
&\quad\Rightarrow \rho^{-1}(\Phi)(rst) \geq \gamma \text{ and } \rho^{-1}(\Psi)(rst) \leq \delta \\
&\quad\Rightarrow \Phi(\rho(rst)) \geq \gamma \text{ and } \Psi(\rho(rst)) \leq \delta \\
&\Rightarrow \Phi(\rho(r)\rho(s)\rho(t)) \geq \gamma \text{ and } \Psi(\rho(r)\rho(s)\rho(t)) \leq \delta \\
&\Rightarrow \Phi(\rho(r)\rho(s)) \geq \gamma \text{ or } \Phi(\rho(s)\rho(t)) \geq \gamma \text{ or } \Phi(\rho(r)\rho(t)) \geq \gamma \textit{ and} \\
&\quad \Psi(\rho(r)\rho(s)) \leq \delta \text{ or } \Psi(\rho(s)\rho(t)) \leq \delta \text{ or } \Psi(\rho(r)\rho(t)) \leq \delta \\
&\Rightarrow \rho^{-1}(\Phi)(rs) \geq \gamma \text{ or } \rho^{-1}(\Phi)(st) \geq \gamma \text{ or } \rho^{-1}(\Phi)(rt) \geq \gamma \textit{ and} \\
&\quad \rho^{-1}(\Psi)(rs) \leq \gamma \text{ or } \rho^{-1}(\Psi)(st) \leq \gamma \text{ or } \rho^{-1}(\Psi)(rt) \leq \gamma \\
&\Rightarrow \rho^{-1}(\Omega)(rs) \geq (\gamma,\delta) \text{ or } \rho^{-1}(\Omega)(st) \geq (\gamma,\delta) \text{ or } \rho^{-1}(\Omega)(rt) \geq (\gamma,\delta).
\end{aligned}$$

Hence, $\rho^{-1}(\Omega)$ is also a 2AIFIS($\Im_1$).

Theorem 13. Let $\rho : \Im_1 \to \Im_2$ be a homomrphism of semirings and Θ be a 2AIFIS($\Im_1$). Then, $\rho(\Theta)$ is also a 2AIFIS($\Im_2$).

Proof: Straightforward.

Theorem 14. Let Ω_1, Ω_2 be two intuitionistic fuzzy subsets of $\Im$ such that Ω_1 is a 2AIFIS($\Im$)
and $\Omega_1 \subseteq \Omega_2$. If $\Omega_2(rst) \geq \Omega_1(rst) \geq (\gamma,\delta)$ for $r,s,t \in \Im$ and $\gamma,\delta \in [0,1]$, then Ω_2 is also a 2AIFIS($\Im$).

Proof: Assume that Ω_1, Ω_2 be two intuitionistic fuzzy subsets of $\Im$ such that $\Omega_1 \subseteq \Omega_2$ and Ω_1 is a 2AIFIS($\Im$). Let $\Omega_2(rst) \geq (\gamma,\delta)$ for $r,s,t \in \Im$ and $\gamma,\delta \in [0,1]$. On the grounds that $\Omega_2(rst) \geq \Omega_1(rst) \geq (\gamma,\delta)$ and Ω_1 is a 2AIFIS($\Im$), then we have

$$\Omega_1(rs) \geq (\gamma,\delta) \text{ or } \Omega_1(st) \geq (\gamma,\delta) \text{ or } \Omega_1(rt) \geq (\gamma,\delta)$$

it follows that

$$\Omega_2(rs) \geq \Omega_1(rs) \geq (\gamma,\delta) \text{ or } \Omega_2(st) \geq \Omega_1(st) \geq (\gamma,\delta) \text{ or } \Omega_2(rt) \geq \Omega_1(rt) \geq (\gamma,\delta).$$

As a result, Ω_2 is also a 2AIFIS($\Im$). Now, we will give cartesian product of intuitionistic fuzzy subsets of semirings.

Definition 16. Let Ω and Θ be two intuitionistic fuzzy subsets of $\Im$ and $\Re$, respectively. The cartesian product of Ω and Θ is given as the follows.

Let $\Omega = <\Phi_1, \Psi_1>$ and $\Theta = <\Phi_2, \Psi_2>$.

$$\begin{aligned}
&(\Omega \times \Theta)(r,s) = \Omega(r) \wedge \Theta(s), \text{ for all } (r,s) \in \Im \times \Re, \textit{ i.e} \\
&(\Omega \times \Theta)(r,s) = (\Phi_1(r) \wedge \Phi_2(s), \Psi_1(r) \vee \Psi_2(s)).
\end{aligned} \tag{4}$$

Definition 17. Let $\Re$ a semiring and Ω be a non-empty subset of $\Re$. The intuitionistic fuzzy characteristic function of Ω is denoted by $\mathcal{X}_{\Omega} =< \Phi_{\mathcal{X}_{\Omega}}, \Psi_{\mathcal{X}_{\Omega}} >$ and given as

$$\Phi_{\mathcal{X}_{\Omega}} : \Re \to [0,1] \qquad\qquad \Psi_{\mathcal{X}_{\Omega}} : \Re \to [0,1]$$

$$r \to \Phi_{\mathcal{X}_{\Omega}}(r) = \begin{cases} 1 \ ; & if \ \ r \in \Re \\ 0 \ ; & if \ \ r \notin \Re \end{cases} \text{ and } \qquad r \to \Psi_{\mathcal{X}_{\Omega}}(r) = \begin{cases} 0 \ ; & if \ \ r \in \Re \\ 1; & if \ \ r \notin \Re. \end{cases} \tag{5}$$

Theorem 15. Let $\Im$ and $\Re$ be two commutative semirings and Ω be a 2AIFIS($\Im$). Then $\Omega \times \mathcal{X}_{\Re}$ is a 2AIFIS of $\Im \times \Re$, where $\mathcal{X}_{\Re}$ is the characteristic function of $\Re$.

Proof: Admit that $\Im$ and $\Re$ be two commutative semirings and Ω, $\mathcal{X}_{\Re}$ be two 2AIFIS($\Im, \Re$) respectively.

Let $\mathcal{X}_{\Re} =< \Phi_{\mathcal{X}_{\Re}}, \Psi_{\mathcal{X}\%_{\Re}} >$.

$$(\Omega \times \mathcal{X}_{\Re})(r_1 s_1 t_1, r_2 s_2 t_2) \geq (\gamma, \delta)$$
$$\Rightarrow (\Phi \times \mathcal{X}_{\Phi_{\Re}})(r_1 s_1 t_1, r_2 s_2 t_2) \geq \gamma \ \text{ and } \ (\Psi \times \mathcal{X}_{\Psi_{\Re}})(r_1 s_1 t_1, r_2 s_2 t_2) \leq \delta$$
$$\Rightarrow \Phi(r_1 s_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(r_2 s_2 t_2) \geq \gamma \ \text{ and } \ \Psi(r_1 s_1 t_1) \vee \mathcal{X}_{\Psi_{\Re}}(r_2 s_2 t_2) \leq \delta$$
$$\Rightarrow \Phi(r_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(r_2 s_2 t_2) \geq \gamma \ \text{ or } \ \Phi(s_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(r_2 s_2 t_2) \geq \gamma \ \text{ or }$$
$$\Phi(r_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(r_2 s_2 t_2) \geq \gamma$$
$$\textit{and}$$
$$\Psi(r_1 s_1) \vee \mathcal{X}_{\Psi_{\Re}}(r_2 s_2 t_2) \leq \delta \ \text{ or } \ \Psi(s_1 t_1) \vee \mathcal{X}_{\Psi_{\Re}}(r_2 s_2 t_2) \leq \delta \ \text{ or }$$
$$\Psi(r_1 t_1) \wedge \mathcal{X}_{\Psi_{\Re}}(r_2 s_2 t_2) \leq \delta$$

As $r_2, s_2, t_2 \in \Re$, so $\mathcal{X}_{\Phi_{\Re}}(r_2 s_2 t_2) = 1$ and $\mathcal{X}_{\Psi_{\Re}}(r_2 s_2 t_2) = 0$. We infer that

$$\Phi(r_1 s_1) \wedge \mathcal{X}_{\Phi_{\Re}}(de) \geq \gamma \ \text{ or } \ \Phi(s_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(de) \geq \gamma \ \text{ or } \ \Phi(r_1 t_1) \wedge \mathcal{X}_{\Phi_{\Re}}(de) \geq \gamma$$
$$\text{and}$$
$$\Psi(r_1 s_1) \vee \mathcal{X}_{\Psi_{\Re}}(de) \leq \delta \ \text{ or } \ \Psi(s_1 t_1) \vee \mathcal{X}_{\Psi_{\Re}}(de) \leq \delta \ \text{ or } \ \Psi(r_1 t_1) \wedge \mathcal{X}_{\Psi_{\Re}}(de) \leq \delta$$

where $d, e \in \{r_2, s_2, t_2\}$. Therefore,

$$\Omega \times \mathcal{X}_{\Re}(r_1 s_1, de) \geq (\gamma, \delta) \ \text{ or } \ \Omega \times \mathcal{X}_{\Re}(r_1 t_1, de) \geq (\gamma, \delta) \ \text{ or } \ \Omega \times \mathcal{X}_{\Re}(r_1 t_1, de) \geq (\gamma, \delta).$$

For this reason, $\Omega \times \mathcal{X}_{\Re}$ is a 2AIFIS $\Im \times \Re$.

Corollary 2. We can give the given definitions as a diagram:

IFPIS($\Im$) $\Rightarrow$ 2AIFIS($\Im$) $\Rightarrow$ 2AIFPIS($\Im$)

4. CONCLUSION

In this article, we studied 2AIFIS from a theoretical point of view. we obtained novel structures in this article and the work was obtained on the several properties of cartesian product of intuitionistic fuzzy ideals of semirings. Next, the research focused on the properties of cartesian product of 2AIFIS. Finally, under a semiring homomorphism, these ideals were investigated. In order to extend this study, one could study other algebraic structures and do some further study on properties of them.

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