ORIGINAL PAPER

ON RULED SURFACES CONSTRUCTED BY THE EVOLUTION OF A POLYNOMIAL SPACE CURVE

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Abstract. *In this paper, we present the evolutions of the ruled surfaces constructed by the tangent, normal-like, and binormal-like vector fields of a polynomial space curve. These evolutions of the ruled surfaces depend on the evolutions of their directrices using the Flc (Frenet like curve) frame along a polynomial space curve. Therefore, the evolutions of a polynomial curve are expressed in the first step of this study. Then, some geometric properties of the special ruled surfaces are investigated and examples of these surfaces are given and their graphics are drawn using the Mathematica 9 program.*

Keywords: *polynomial curve; ruled surface; Flc frame; evolution of curve.*

1. INTRODUCTION

The most important research topics of differential geometry are curves and surfaces. The time variation of curves and surfaces in E^3 is produced by their corresponding flow to these curves and surfaces. Considering the time variation of the curve and surface in E^3, if the arc-length of the curve (in other words, if the arc-length variation of the curve is zero) is maintained in their initial and final positions, at the same time, if the main curvature is preserved at the initial and final positions of the surface, this curve and surface are called an inelastic curve and surface [1]. Physically, movements that occur in the absence of tension energy cause the flow of surfaces and inelastic curves. For example, the oscillating motion of a fixed-length cable or the oscillating motion of a piece of paper created by the wind is expressed by inelastic curves and surface flows [1, 2]. In other words, for all t, the evolution of the surface $X(s, u, t)$ is expressed as the isometric image of the original surface $X(s, u, t_0)$ defined at the initial time t_0 [1, 2]. Inelastic curves and surface flows occur in the context of many problems in computer imaging [3, 4], computer animation [5], and even mechanical motion science [6]. Although there is a lot of research on curved flows, there is not much research on surface flows. In this study, we obtained ruled surfaces with the help of polynomial evaluation using the Flc frame, which is defined along the moving polynomial curve. This Flc frame is an alternative frame to the Frenet frame. Because the Frenet frame cannot be defined at points where the second and higher order derivatives of any curve are zero. To solve this problem, Dede defined the Flc frame for moving polynomial curves [7, 8]. The Flc frame [9-13] and ruled surfaces on different frames [14-23] have been investigated by many researchers. Inspired by these studies, we conducted this research to create a new resource on the subject of surfaces and to form a basis for future studies.

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Our aim in this study is to present the evolutions of the ruled surfaces constructed by the tangent, normal-like and binormal-like vector fields of a polynomial space curve. These evolutions of the ruled surfaces depend on the evolutions of their directrices using the Flc frame along a polynomial space curve. Some geometric properties of the ruled surfaces are investigated and examples of these surfaces are given and their graphics are drawn using the Mathematica 9 program.

2. PRELIMINARIES

In Euclidean 3-space E^3, Euclidean scalar product is given by

$$<\alpha,\beta>=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3,$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3),\ \beta=(\beta_1,\beta_2,\beta_3)\in E^3$. The norm of $\alpha\in E^3$ is $\|\alpha\|=\sqrt{<\alpha,\alpha>}$. Let T, D_2 and D_1 be tangent, principal normal-like and binormal-like vectors at point $\alpha(s)$ of a polynomial space curve α, respectively, then the Frenet like curve frame is given by matrix form

$$\begin{bmatrix} T' \\ D_2' \\ D_1' \end{bmatrix} = v\begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix}\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix},$$

where $v^2=\left\langle \frac{\partial\alpha(s,t)}{\partial s},\frac{\partial\alpha(s,t)}{\partial s}\right\rangle$, d_1, d_2 and d_3 are the curvatures of the polynomial curve α with the arc-length s (see for more details [7-9]), respectively. Let X_s and X_u be tangent vectors of a surface $X(s,u)$, then the normal vector field of the surface $X(s,u)$ can be defined by

$$U=\frac{X_s\times X_u}{\|X_s\times X_u\|}, \tag{1}$$

where $X_s=\frac{\partial X}{\partial s}$ and $X_u=\frac{\partial X}{\partial u}$. The coefficients of first and second fundamental form of the surface $X(s,u)$ are, respectively, given by

$$E=\left\langle \frac{\partial X}{\partial s},\frac{\partial X}{\partial s}\right\rangle, F=\left\langle \frac{\partial X}{\partial s},\frac{\partial X}{\partial u}\right\rangle, G=\left\langle \frac{\partial X}{\partial u},\frac{\partial X}{\partial u}\right\rangle \tag{2}$$

and

$$e=\left\langle \frac{\partial^2 X}{\partial s^2},U\right\rangle,\ f=\left\langle \frac{\partial^2 X}{\partial s\partial u},U\right\rangle, g=\left\langle \frac{\partial^2 X}{\partial u^2},U\right\rangle. \tag{3}$$

Moreover, the Gaussian curvature and the mean curvature of the surface are defined by

$$K = \frac{eg - f^2}{EG - F^2} \text{ and } H = \frac{1}{2}\frac{Eg - 2Ef + Ge}{EG - F^2} \tag{4}$$

respectively. Also, the surfaces with vanishing Gaussian curvature are called developable and the surfaces with vanishing mean curvature are called minimal.

2.1. INTRINSIC EQUATIONS OF A MOVING POLYNOMIAL CURVE

Now, let's express the basic concepts about the evolution of a moving polynomial curve in Euclidean 3-space and we give the intrinsic equations expressing with curvatures of it corresponding with moving Flc frame $\{T, D_2, D_1\}$. Let α be a moving polynomial curve described in parametric form by a position vector $\alpha(s,t)$ where s is the arc-length parameter and t is the time parameter. Thus, the moving Flc frame $\{T, D_2, D_1\}$ of the curve with respect to s and t can be given in matrix form as follows:

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}_s = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} \text{ and } \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}_t = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ -\lambda_1 & 0 & \lambda_3 \\ -\lambda_2 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix},$$

where λ_1, λ_2 and λ_3 are smooth functions of s and t. By considering the compatible condition $T_{st} = T_{ts}$, $D_{2st} = D_{2ts}$ and $D_{1st} = D_{1ts}$, one can determine

$$\lambda_1 = \frac{\left(v(d_2 - d_3)\right)_t + (\lambda_3 - \lambda_2)_s + v d_1 (\lambda_2 + \lambda_3)}{v(d_2 + d_3)},$$
$$\lambda_2 = \frac{\left(v(d_3 - d_1)\right)_t + (\lambda_1 - \lambda_3)_s + v d_2 (\lambda_1 + \lambda_3)}{v(d_1 + d_2)},$$
$$\lambda_3 = \frac{\left(v(d_1 + d_2)\right)_t - (\lambda_1 + \lambda_2)_s - v d_3 (\lambda_1 - \lambda_2)}{v(d_2 - d_1)}.$$

Moreover, the curvatures of the evolving polynomial curve are found as

$$d_{1t} = \lambda_3 d_2 - \lambda_2 d_3 + v^{-1}\left(\lambda_{1s} - v_t d_1\right),$$
$$d_{2t} = \lambda_1 d_3 - \lambda_3 d_1 + v^{-1}\left(\lambda_{2s} - v_t d_2\right),$$
$$d_{3t} = \lambda_2 d_1 - \lambda_1 d_2 + v^{-1}\left(\lambda_{3s} - v_t d_3\right).$$

So, we can say that these equations represent the motion of the polynomial curve with the Flc frame.

3. RULED SURFACES CONSTRUCTED BY THE EVOLUTION OF A POLYNOMIAL SPACE CURVE

In this section, we study the evolution of the tangent ruled surface, normal-like ruled surface and binormal-like ruled surface using the Flc frame along a polynomial space curve. The parameter equation of the ruled surface is given

$$X(s,u)=\alpha(s)+uw(s), \tag{5}$$

where $\alpha(s)$ is called the base curve and $w(s)$ is the director curve of the ruled surface. By moving with time t curves $\alpha(s)$ and $w(s)$, the evolution equation of this ruled surface can be written as:

$$X(s,u,t)=\alpha(s,t)+uw(s,t).$$

Now let's give the following definition:

Definition 3.1. Let $X(s,u,t)$ be a surface evolution of the surface $X(s,u)$ given by Eq. (5) in E^3. Then the surface evolution $X(s,u,t)$ is inextensible if the coefficients E,F and G of the first fundemandal form of $X(s,u,t)$ satisfy:

$$\frac{\partial E}{\partial t}=\frac{\partial F}{\partial t}=\frac{\partial G}{\partial t}=0,$$

for all time t [1].

3.1. EVOLUTION OF TANGENT RULED SURFACE

Let X_T be a tangent ruled surface which is generated by the motion of the tangent vector of the curve α with the Flc frame, then the evolution equation of this surface is represented by

$$X_T(s,u,t)=\alpha(s,t)+uT(s,t). \tag{6}$$

So, the subscripts s and u represent partial derivatives of the surface X_T are easily calculated in the following form:

$$\begin{gathered}\frac{\partial X_T(s,u,t)}{\partial s}=v\left(\mathrm{T}(s,t)+u\left(d_1D_2(s,t)+d_2D_1(s,t)\right)\right),\\ \frac{\partial X_T(s,u,t)}{\partial u}=T(s,t),\end{gathered} \tag{7}$$

where the curvatures d_1 and d_2 depend on the parameters s and t. By considering the partial derivatives of the surface X_T and Eq. (1), one can get the normal vector field of the surface X_T as:

$$U_T(s,u,t)=\frac{d_2D_2(s,t)-d_1D_1(s,t)}{\sqrt{(d_1{}^2+d_2{}^2)}},\ d_1{}^2+d_2{}^2=0.$$

On the other hand, by considering Eqs. (2) and (7), the coefficients E_T, F_T and G_T of the first fundamental form of the surface X_T are obtained as follow:

$$E_T(s,u,t)=v^2\left(1+u^2\left(d_1{}^2+d_2{}^2\right)\right), F_T(s,u,t)=v, G_T(s,u,t)=1. \tag{8}$$

Theorem 3.2. Let $X_T(s,u,t)$ be a tangent ruled surface evolution of X_T with the Flc frame $\{T, D_2, D_1\}$. The tangent ruled surface evolution $X_T(s,u,t)$ is inextensible if

$$\frac{\partial v}{\partial t}=0,\ \frac{\partial d_1}{\partial t}=0 \text{ and } \frac{\partial d_2}{\partial t}=0,$$

for all time t.

Proof: Let $X_T(s,u,t)$ be a tangent ruled surface evolution with the Flc frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Eq. (8) with respect to t, we have

$$\frac{\partial E_T}{\partial t}=2v\left(\frac{\partial v}{\partial t}\left(1+u^2\left(d_1{}^2+d_2{}^2\right)\right)+v\left(u^2\left(d_1\frac{\partial d_1}{\partial t}+d_2\frac{\partial d_2}{\partial t}\right)\right)\right),$$
$$\frac{\partial F_T}{\partial t}=\frac{\partial v}{\partial t},\ \frac{\partial G_T}{\partial t}=0.$$

If there exist $\frac{\partial v}{\partial t}=0$, $\frac{\partial d_1}{\partial t}=0$ and $\frac{\partial d_2}{\partial t}=0$, the conditions given in Definition 3.1 are satisfied. Thus, the proof is completed. By differentiating Eq. (7), one can get

$$\frac{\partial^2 X_T(s,u,t)}{\partial s^2}=\left(\frac{\partial v}{\partial s}-uv^2\left(d_1{}^2+d_2{}^2\right)\right)T(s,t)+\left(v^2\left(d_1-ud_2d_3\right)+u\frac{\partial(vd_1)}{\partial s}\right)D_2(s,t)$$
$$+\left(v^2\left(d_2+ud_1d_3\right)+u\frac{\partial(vd_2)}{\partial s}\right)D_1(s,t),$$
$$\frac{\partial^2 X_T(s,u,t)}{\partial s\partial u}=v\left(d_1D_2(s,t)+d_2D_1(s,t)\right),$$
$$\frac{\partial^2 X_T(s,u,t)}{\partial u^2}=0.$$

Considering these last equations together with the normal vector field of X_T, the coefficients of the second fundamental form of X_T are given by

$$e_T(s,u,t) = -\frac{uv\left(d_3 v\left(d_1^2 + d_2^2\right) + \frac{\partial}{\partial s}\left(\frac{d_2}{d_1}\right)d_1^2\right)}{\sqrt{d_1^2 + d_2^2}},\; f_T(s,u,t) = 0,\; g_T(s,u,t) = 0. \tag{9}$$

Proposition 3.3. The Gaussian curvature K_T and the mean curvature H_T of X_T are

$$K_T(s,u,t) = 0 \text{ and } H_T(s,u,t) = -\frac{d_3 v\left(d_1^2 + d_2^2\right) + \frac{\partial}{\partial s}\left(\frac{d_2}{d_1}\right)d_1^2}{2uv\left(d_1^2 + d_2^2\right)^{3/2}},$$

respectively.

Proof: By substituting Eqs. (8) and (9) into Eq. (4), one can obtain the Gaussian curvature and the mean curvature of the surface X_T as above.

Corollary 3.4. Let X_T be a tangent ruled surface with the Flc frame, then the surface X_T is

- developable,
- minimal surface if $\frac{d_2}{d_1}$ is constant and d_3 vanishes.

Example 3.5. Let $\alpha(s,t)$ be a polynomial space curve evolution expressed by

$$\alpha(s,t) = \left(s+t, \frac{s^2+t^2}{2}, \frac{s^3+t^3}{6}\right).$$

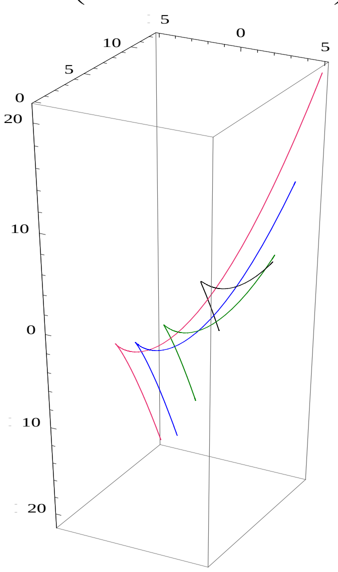

Figure 1. The evolution of a polynomial curve with $u \in (-5,5)$ and $t \in \{0,1,2,3\}$.

Moreover, the parametrical equation of tangent ruled surface with Flc frame is respresented by

$$X_T = \left(1+s+\frac{2u}{2+s^2}, \frac{1+s^2}{2}+\frac{2su}{2+s^2}, \frac{1+s^3}{6}+\frac{s^2u}{2+s^2}\right), \text{ for } t=1$$

and

$$X_T = \left(3+s+\frac{2u}{2+s^2}, \frac{9+s^2}{2}+\frac{2su}{2+s^2}, \frac{27+s^3}{6}+\frac{s^2u}{2+s^2}\right), \text{ for } t=3.$$

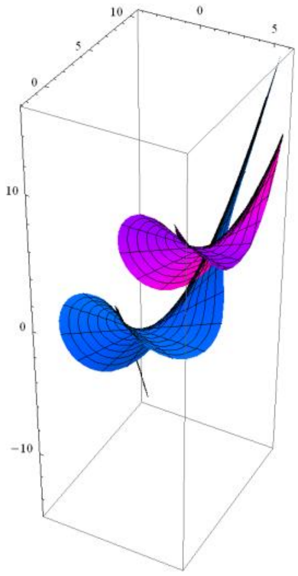

Figure 2. The graph of the tangent ruled surface (in red) for $t=3$ and (in blue) $t=1$ with $s\in(-4,4)$ and $u\in(-5,5)$.

3.2. EVOLUTION OF NORMAL-LIKE RULED SURFACE

Let X_{D_2} be a normal-like ruled surface which is generated by the motion of the normal-like vector of the curve α with the Flc frame, then the evolution equation of this surface is represented by

$$X_{D_2}(s,u,t)=\alpha(s,t)+uD_2(s,t). \tag{10}$$

So, the partial derivatives of the normal-like ruled surface X_{D_2} with respect to s and u are easily calculated in the following form:

$$\frac{\partial X_{D_2}(s,u,t)}{\partial s}=v\big((1-ud_1)\mathrm{T}(s,t)+ud_3D_1(s,t)\big),$$
$$\frac{\partial X_{D_2}(s,u,t)}{\partial u}=D_2(s,t), \tag{11}$$

where the curvatures d_1 and d_3 depend on the parameters s and t. By considering the partial derivatives of the surface X_{D_2} given by Eq (10) and Eq. (1), one can get the normal vector field of the normal-like ruled surface as:

$$U_{D_2}(s,t)=\frac{-ud_3T(s,t)+(1-ud_1)D_1(s,t)}{\sqrt{\left((-1+ud_1)^2+u^2d_3{}^2\right)}},\ (-1+ud_1)^2+u^2d_3{}^2\neq 0.$$

On the other hand, by considering Eqs. (2) and (11), the coefficients E_{D_2}, F_{D_2} and G_{D_2} of the first fundamental form of the normal-like ruled surface X_{D_2} are obtained as follow:

$$E_{D_2}=v^2\left((-1+ud_1)^2+u^2d_3{}^2\right),\ F_{D_2}=0,\ G_{D_2}=1. \tag{12}$$

Theorem 3.6. Let $X_{D_2}(s,u,t)$ be a normal-like ruled surface evolution of X_{D_2} with the Flc frame $\{T,D_2,D_1\}$. The normal-like ruled surface evolution $X_{D_2}(s,u,t)$ is inextesible if and only if

$$\frac{\partial v}{\partial t}=0,\ \frac{\partial d_1}{\partial t}=0 \text{ and } \frac{\partial d_3}{\partial t}=0.$$

Proof: Let $X_{D_2}(s,u,t)$ be a normal-like ruled surface evolution with the Flc frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Eq. (12) with respect to t, we have

$$\frac{\partial E_T}{\partial t}=2v\left(\frac{\partial v}{\partial t}\left((1-ud_1)^2+(ud_3)^2\right)+vu\left((-1+ud_1)\frac{\partial d_1}{\partial t}+ud_3\frac{\partial d_3}{\partial t}\right)\right),$$

$$\frac{\partial F_T}{\partial t}=0,\ \frac{\partial G_T}{\partial t}=0.$$

If there exist $\frac{\partial v}{\partial t}=0,\ \frac{\partial d_1}{\partial t}=0$ and $\frac{\partial d_3}{\partial t}=0$, the conditions given in Definition 3.1 are satisfied. Thus, the proof is completed. By differentiating Eq. (11), one can get

$$\frac{\partial^2 X_{D_2}(s,u,t)}{\partial s^2}=\left(\frac{\partial v}{\partial s}(1-ud_1)-uv\left(\frac{\partial d_1}{\partial s}+vd_2d_3\right)\right)T(s,t)+v^2\left(d_1-u\left(d_1{}^2+d_3{}^2\right)\right)D_2(s,t)$$
$$+\left(v^2d_2(1-ud_1)+u\frac{\partial(d_3v)}{\partial s}\right)D_1(s,t),$$

$$\frac{\partial^2 X_{D_2}(s,u,t)}{\partial s\partial u}=v\left(-d_1T(s,t)+d_3D_1(s,t)\right)$$

$$\frac{\partial^2 X_{D_2}(s,u,t)}{\partial u^2}=0.$$

Considering these last equations together with the normal vector field of X_{D_2}, the coefficients of the second fundamental form of X_{D_2} are calculated by

$$e_{D_2}=\frac{v\left(d_2\left((-1+ud_1)^2+u^2{d_3}^2\right)v+u\left(ud_3\frac{\partial d_1}{\partial s}+(1-ud_1)\frac{\partial d_3}{\partial s}\right)\right)}{\sqrt{(-1+ud_1)^2+u^2{d_3}^2}}$$
$$f_{D_2}=\frac{d_3v}{\sqrt{(-1+ud_1)^2+u^2{d_3}^2}},\ g_{D_2}=0. \tag{13}$$

Proposition 3.7. The Gaussian curvature and the mean curvature of the surface X_{D_2} are

$$K_{D_2}=-\frac{{d_3}^2}{\left((-1+ud_1)^2+u^2{d_3}^2\right)^2}$$

and

$$H_{D_2}=\frac{vd_2\left((-1+ud_1)^2+u^2{d_3}^2\right)+u\left(ud_3\frac{\partial d_1}{\partial s}+(1-ud_1)\frac{\partial d_3}{\partial s}\right)}{2v\left((-1+ud_1)^2+u^2{d_3}^2\right)^{3/2}},$$

respectively.

Proof: By substituting Eqs. (12) and (13) into Eq. (3), the Gaussian curvature and the mean curvature of the normal-like ruled surface are obtained as above.

Corollary 3.8. Let X_{D_2} be a normal-like ruled surface with the Flc frame, then the surface X_{D_2} is

- developable if d_3 vanishes,
- minimal surface if d_1 and d_3 are constant and d_2 vanishes.

Example 3.9. Let's take the polynomial space curve given in Example 3.5 the parametrical equation of normal-like ruled surface is respresented by

$$X_{D_2}=\left(1+s-\frac{s^2u}{\sqrt{1+s^2}\left(2+s^2\right)},\frac{1}{2}\left(1+s^2-\frac{2s^3u}{\sqrt{1+s^2}\left(2+s^2\right)}\right),\frac{1}{6}\left(1+s^3\right)+\frac{2\sqrt{1+s^2}u}{2+s^2}\right),\text{ for } t=1$$

and

$$X_{D_2}=\left(3+s-\frac{s^2u}{\sqrt{1+s^2}\left(2+s^2\right)},\frac{9+s^2}{2}-\frac{s^3u}{\sqrt{1+s^2}\left(2+s^2\right)},\frac{s^3+27}{6}+\frac{2\sqrt{1+s^2}u}{2+s^2}\right),\text{ for } t=3.$$

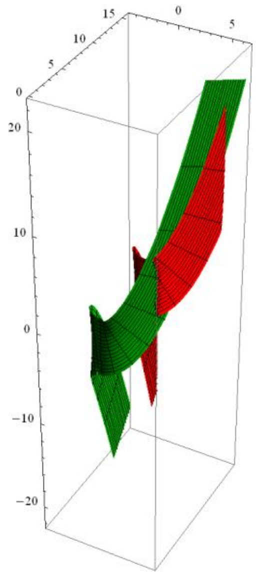

Figure 3. The graph of the normal-like ruled surface (in red) for $t=3$ and (in green) $t=1$ with $s\in(-5,5)$ and $u\in(-3,3)$

3.3. EVOLUTION OF BINORMAL-LIKE RULED SURFACE

Let X_{D_1} be a binormal-like ruled surface which is generated by the motion of the binormal-like vector of the curve α with the Flc frame, then the evolution equation of this surface is represented by

$$X_{D_1}(s,u,t)=\alpha(s,t)+uD_1(s,t). \tag{14}$$

So, the partial derivatives of the binormal-like ruled surface X_{D_1} with respect to s and u are easily determinated in the following form:

$$\begin{aligned}\frac{\partial X_{D_1}(s,u,t)}{\partial s}&=v\left((1-ud_2)\mathrm{T}(s,t)-ud_3D_2(s,t)\right),\\ \frac{\partial X_{D_1}(s,u,t)}{\partial u}&=D_1(s,t),\end{aligned} \tag{15}$$

where the curvatures d_2 and d_3 depend on the parameters s and t. By considering the partial derivatives of the surface X_{D_1} and Eq. (1), the normal vector field of the binormal-like ruled surface is calculated as:

$$U_{D_1}(s,t)=\frac{-ud_3T(s,t)+(-1+ud_2)D_2(s,t)}{\sqrt{\left((-1+ud_2)^2+u^2d_3^{\,2}\right)}},\ (-1+ud_2)^2+u^2d_3^{\,2}\neq 0.$$

On the other hand, by considering Eq. (15), the coefficients E_{D_1}, F_{D_1} and G_{D_1} of the first fundamental form of the binormal-like ruled surface X_{D_1} given by Eq. (14) are obtained as follow:

$$E_{D_1} = v^2\left(\left(-1+ud_2\right)^2 + u^2 {d_3}^2\right),\ F_{D_1} = 0,\ G_{D_1} = 1. \tag{16}$$

Theorem 3.10. Let $X_{D_1}(s,u,t)$ be a binormal-like ruled surface evolution of X_{D_1} with the Flc frame $\{T, D_2, D_1\}$. The binormal-like ruled surface evolution $X_{D_1}(s,u,t)$ is inextensible if and only if

$$\frac{\partial v}{\partial t} = 0,\ \frac{\partial d_2}{\partial t} = 0 \text{ and } \frac{\partial d_3}{\partial t} = 0.$$

Proof: Let $X_{D_1}(s,u,t)$ be a binormal-like ruled surface evolution with the Flc frame. Taking the partial derivatives of the coefficients of the first fundamental form given by Eq. (16) with respect to t, we have

$$\frac{\partial E_{D_2}}{\partial t} = 2v\left(\frac{\partial v}{\partial t}\left(\left(1-ud_2\right)^2 + \left(ud_3\right)^2\right) + vu\left(\left(-1+ud_2\right)\frac{\partial d_2}{\partial t} + ud_3\frac{\partial d_3}{\partial t}\right)\right),$$

$$\frac{\partial F_{D_2}}{\partial t} = 0,\ \frac{\partial G_{D_2}}{\partial t} = 0.$$

If there exist $\frac{\partial v}{\partial t} = 0,\ \frac{\partial d_2}{\partial t} = 0$ and $\frac{\partial d_3}{\partial t} = 0$, the conditions given in Definition 3.1 are satisfied. Thus, the proof is completed. By differentiating Eq. (15), one can get

$$\frac{\partial^2 X_{D_1}(s,u,t)}{\partial s^2} = \left(\frac{\partial v}{\partial s}\left(1-ud_2\right) + uv\left(vd_1d_3 - \frac{\partial d_2}{\partial s}\right)\right)T(s,t) + \left(v^2 d_1\left(1-ud_2\right) - u\frac{\partial (vc)}{\partial s}\right)D_2(s,t)$$
$$+v^2\left(d_2 - u\left({d_2}^2 + {d_3}^2\right)\right)D_1(s,t),$$
$$\frac{\partial^2 X_{D_1}(s,u,t)}{\partial s \partial u} = -v\left(d_2 T(s,t) + d_3 D_2(s,t)\right)$$
$$\frac{\partial^2 X_{D_1}(s,u,t)}{\partial u^2} = 0.$$

Considering these last equations together with the normal vector field of X_{D_1}, the coefficients of the second fundamental form of X_{D_1} are calculated by

$$e_{D_1}=-\frac{v\left(vd_1\left(\left(-1+ud_2\right)^2+u^2d_3{}^2\right)+u\left(-ud_3\frac{\partial d_2}{\partial s}+\left(-1+ud_2\right)\frac{\partial d_3}{\partial s}\right)\right)}{\sqrt{\left(-1+ud_2\right)^2+u^2d_3{}^2}},$$

$$f_{D_1}=\frac{d_3v}{\sqrt{\left(-1+ud_2\right)^2+u^2d_3{}^2}},\ g_{D_1}=0. \tag{17}$$

Proposition 3.11. The Gaussian curvature and the mean curvature of the binormal-like ruled surface are

$$K_{D_1}=-\frac{d_3{}^2}{\left(\left(-1+ud_2\right)^2+u^2d_3{}^2\right)^2}$$

and

$$H_{D_1}=\frac{-vd_1\left(\left(-1+ud_2\right)^2+u^2d_3{}^2\right)+u\left(ud_3\frac{\partial d_2}{\partial s}+\left(1-ud_2\right)\frac{\partial d_3}{\partial s}\right)}{2v\left(\left(-1+ud_2\right)^2+u^2d_3{}^2\right)^{3/2}},$$

respectively.

Proof: By substituting Eqs. (16) and (17) into Eq. (4), the Gaussian curvature and the mean curvature of the binormal-like ruled surface is obtained as above.

Corollary 3.12. Let X_{D_1} be a binormal-like ruled surface with the Flc frame, then the surface X_{D_1} is

- developable if d_3 vanishes,
- minimal surface if d_2 and d_3 are constant and d_1 vanishes.

Example 3.13. Let's take the polynomial curve given in Example 3.5 the parametrical equation of binormal-like ruled surface with Flc frame is respresented by

$$X_{D_1}=\left(1+s+\frac{su}{\sqrt{1+s^2}},\frac{1+s^2}{2}-\frac{u}{\sqrt{1+s^2}},\frac{1+s^3}{6}\right),\text{ for } t=1$$

and

$$X_{D_1}=\left(3+s+\frac{su}{\sqrt{1+s^2}},\frac{9+s^2}{2}-\frac{u}{\sqrt{1+s^2}},\frac{s^3+27}{6}\right),\text{ for } t=3.$$

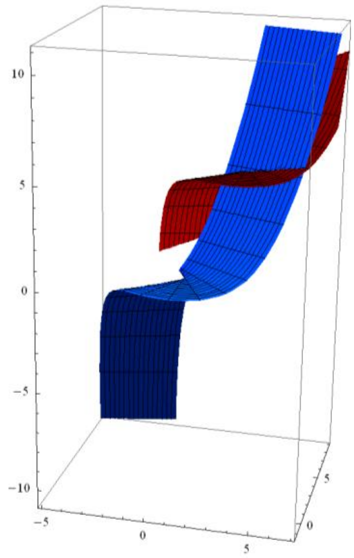

Figure 4. The graph of the binormal-like surface (in red) for $t=3$ and (in blue) $t=1$ with $s\in(-4,4)$ and $u\in(-2,2)$

4. CONCLUSION

In this paper, evolutions of ruled surfaces formed by tangent, normal-like, and binormal-like vector fields of a polynomial space curve are presented. These evolutions of the surfaces are found using the evolutions of their directions of the Flc frame along a polynomial space curve. Some geometric properties of the surfaces are examined, examples of these surfaces are given, and their graphics are drawn using the Mathematica 9 program. Thus, this research has created a new resource on surfaces and is the basis for future studies.

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